1. Three-Point correlation (Coins):

In deriving the equation for single point distribution f_1 and $g\,(1,2,t)$ we defined the joint probability f_3 in terms of the one-point probability f_1, the two-point correlation function g, and the three-point correlation function h. We apply this to the case of three coins, each of which can come up with heads (+) or tails (-). What is the meaning of f_3 in this case? Write out f_2 and f_3 in the form

$$\begin{aligned} f_2\,(1,2) &= f_1\,(1)\,f_1\,(2) + g\,(1,2) \\ f_3\,(1,2,3) &= f_1\,(1)\,f_1\,(2)\,f_1\,(3) + f_1\,(1)\,g\,(2,3) + f_1\,(2)\,g\,(1,3) + f_1\,(3)\,g\,(1,2) + h\,(1,2,3) \end{aligned}$$

and evaluate f_3, f_1, g, and h in each of the following cases.

(a) All three coins are "honest", that is, each coin is equally likely to come up heads or tails, and each coin is unaffected by any other coin.

(b) Because the coins are mysteriously locked together, in any one throw all three are heads.

(c) The first coins two are honest but 3 is manipulated to show heads if 1 or 2 are heads. How is your chance changing if you bet on ttt and can you see this looking at the two- or thee-point correlation?

Hint: Characterize the states of the random single point distribution with the Kronecker δ as $f_1\,(1) = 0.5\,(\delta_{1h} + \delta_{1t})$. Make use of the relations $\delta_{1h} + \delta_{1t} = 1$ (because there are only two states) to manipulate/simplify your results.

2. Fokker-Planck Collision Terms:

A simple approximation of the Fokker Planck collision terms is:

$$\left.\frac{\partial f}{\partial t}\right|_c = \nu_s \nabla_{\mathbf{v}} \cdot \left[(\mathbf{v} - \mathbf{v}_0)\, f + v_e^2 \nabla_{\mathbf{v}} f \right]$$

Compute the following zero, first, and second moments for the collision term in the fluid eqautions:

$$\begin{aligned} I_{c0} &= \int_{-\infty}^{\infty} d^3v \; \left.\frac{\partial f}{\partial t}\right|_c \\ I_{c1} &= \int_{-\infty}^{\infty} d^3v \; v_x \left.\frac{\partial f}{\partial t}\right|_c \\ I_{c2} &= \int_{-\infty}^{\infty} d^3v \; v_x^2 \left.\frac{\partial f}{\partial t}\right|_c \end{aligned}$$

For each of these integrals discuss the influence and meaning of the parameters $\mathbf{v}_0$ and v_e. Under what conditions is mass, momentum and energy conserved if the terms resulting from the above moments are used as source terms in the fluid equations.

Hint: Make use of the definitions:

$$n u_x = \int_{-\infty}^{\infty} d^3v v v_x f \qquad \text{and} \qquad p_{xx} = m \int_{-\infty}^{\infty} d^3v\, (v_x - u_x)^2\, f$$

3. Two-electron beam instability:

A plasma is described by two cold electron beams $f_{e0}(\mathbf{v}) = c_e \left[\delta(v_x - v_0) + \delta(v_x + v_0)\right] \exp\left[-\left(v_y^2 + v_z^2\right)/2u_e^2\right]$ and a neutralizing ion background of density n_0 .

(a) Show that the normalization constant satisfies $2c_e = n_0 \left(2\pi u_e^2\right)^{-1}$.

(b) Calculate the reduced distribution function (ignore ion contributions) and show that it is

$$g(v_x) = \frac{1}{n_0} \int_{-\infty}^{\infty} dv_y dv_z \left[f_{e0}(v)\right] = \frac{1}{2}\left[\delta(v_x - v_0) + \delta(v_x + v_0)\right]$$

(c) Evaluate the dielectric function

$$\epsilon = 1 + \frac{\omega_e^2}{k^2} \int du \frac{d_u g(u)}{\omega/k - u} = 0$$

and show that the dispersion relation becomes: $2 - \omega_e^2/(\omega - kv_0)^2 - \omega_e^2/(\omega + kv_0)^2 = 0$.

(d) Evaluate and discuss the dispersion relation for ω^2. What is the condition for maximum growth and what is the maximum growth rate?

4. Harris sheet:

The distribution functions for the Harris sheet (for invariance along y or $\partial/\partial_y = 0$) is given by

$$f_s(\mathbf{v}, \mathbf{x}) = n_0 \left(\frac{m_s}{2\pi k_B T}\right)^{3/2} \exp\left[\frac{q_s u_s}{k_B T} A(\mathbf{x})\right] \exp\left[-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + (v_y - u_s)^2\right)\right]$$

with the vector potential $\mathbf{A} = A\mathbf{e}_y$ (using exact neutrality and $u_e = -u_i = u$).

(a) Show by explicit substitution that f_s solves the collisionless Boltzmann equation.

(b) Compute pressure and current density and demonstrate $j_y(A) = dp(A)/dA$

General Hint: Integration over velocity space: Use substitutions such as $s_x^2 = \frac{m_s v_x^2}{2k_B T}$. Integrals:

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^2 \exp\left(-x^2\right) dx = \frac{\sqrt{\pi}}{2}$$

This final exam is 3 hours and open book (any plasma physics book, and plasma or math formulary excluding homework). Select 3 of the 4 problems. Each problem is $33\frac{1}{3}\%$.