1. Anisotropic distribution function: Consider a magnetic field along the z direction and a distribution function

$$f(\mathbf{v}) = c_0 \exp\left(-\frac{mv_z^2}{2k_BT_\parallel} - \frac{m\left(v_x^2+v_y^2\right)}{2k_BT_\perp}\right)$$

a) Show that normalization of the density to n_0 yields

$$c_0 = n_0\left(\frac{m}{2\pi k_BT_\parallel}\right)^{1/2}\left(\frac{m}{2\pi k_BT_\perp}\right)$$

b) Compute the average energy parallel and perpendicular to the magnetic field.

Hint:

$$\begin{aligned}\int_{-\infty}^{\infty}\exp\left(-x^2\right)dx &= \sqrt{\pi}\\ \int_{-\infty}^{\infty}x^2\exp\left(-x^2\right)dx &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

Solution:

a) Normalization to the density to n_0 and using the substitutions

$s_x^2 = \frac{mv_x^2}{2k_BT_\perp}, ds_x = \left(\frac{m}{2k_BT_\perp}\right)^{1/2}dv_x, s_y^2 = \frac{mv_y^2}{2k_BT_\perp}, ds_y = \left(\frac{m}{2k_BT_\perp}\right)^{1/2}dv_y, s_z^2 = \frac{mv_z^2}{2k_BT_\parallel}, ds_z = \left(\frac{m}{2k_BT_\parallel}\right)^{1/2}dv_z$ yields

$$\begin{aligned}n_0 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(v)dv_xdv_ydv_z\\ &= c_0\int_{-\infty}^{\infty}\exp\left(-\frac{mv_x^2}{2k_BT_\perp}\right)dv_x\int_{-\infty}^{\infty}\exp\left(-\frac{mv_y^2}{2k_BT_\perp}\right)dv_y\int_{-\infty}^{\infty}\exp\left(-\frac{mv_z^2}{2k_BT_\parallel}\right)dv_z\\ &= c_0\left(\frac{2k_BT_\perp}{m}\right)\left(\frac{2k_BT_\parallel}{m}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left(-s_x^2\right)ds_x\int_{-\infty}^{\infty}\exp\left(-s_y^2\right)ds_y\int_{-\infty}^{\infty}\exp\left(-s_z^2\right)ds_z\\ &= c_0\left(\frac{2k_BT_\perp}{m}\right)\left(\frac{2k_BT_\parallel}{m}\right)^{1/2}\pi^{3/2}\end{aligned}$$

where we have used

$$\int_{-\infty}^{\infty}\exp\left(-x^2\right)dx = \sqrt{\pi}$$

Solving for c_0 yields

$$c_0 = n_0\left(\frac{m}{2\pi k_BT_\parallel}\right)^{1/2}\left(\frac{m}{2\pi k_BT_\perp}\right)$$

b) Compute the average energy parallel and perpendicular to the magnetic field.

Average energy parallel to the magnetic field:

$$
\begin{aligned}
e_{\parallel} &= \left\langle \frac{m}{2} v_{\parallel}^2 \right\rangle = \frac{1}{n_0} \frac{m}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_z^2 f(v) dv_x dv_y dv_z \\
&= \frac{m}{2n_0} c_0 \left(\frac{2\pi k_B T_{\perp}}{m} \right) \int_{-\infty}^{\infty} v_z^2 \exp\left(-\frac{m v_z^2}{2k_B T_{\parallel}} \right) dv_z \\
&= \frac{m}{2} \left(\frac{m}{2\pi k_B T_{\parallel}} \right)^{1/2} \left(\frac{2k_B T_{\parallel}}{m} \right)^{3/2} \int_{-\infty}^{\infty} s_z^2 \exp\left(-s_z^2 \right) ds_z \\
&= \frac{m}{2\sqrt{\pi}} \left(\frac{2k_B T_{\parallel}}{m} \right) \frac{\sqrt{\pi}}{2} \\
&= \frac{1}{2} k_B T_{\parallel}
\end{aligned}
$$

with

$$
\int_{-\infty}^{\infty} x^2 \exp\left(-x^2 \right) dx = \frac{\sqrt{\pi}}{2}
$$

Average energy perpendicular to the magnetic field:

$$
\begin{aligned}
e_{\perp} &= \frac{m}{2} \left\langle v_x^2 + v_y^2 \right\rangle = \frac{1}{n_0} \frac{m}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(v_x^2 + v_y^2 \right) f(v) dv_x dv_y dv_z \\
&= \frac{m}{2n_0} c_0 \left(\frac{2\pi k_B T_{\parallel}}{m} \right)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(v_x^2 + v_y^2 \right) \exp\left(-\frac{m\left(v_x^2 + v_y^2 \right)}{2k_B T_{\perp}} \right) dv_x dv_y \\
&= \frac{m}{2} \left(\frac{m}{2\pi k_B T_{\perp}} \right) \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{m\left(v_x^2 + v_y^2 \right)}{2k_B T_{\perp}} \right) dv_x dv_y \right. \\
&\qquad \left. + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_y^2 \exp\left(-\frac{m\left(v_x^2 + v_y^2 \right)}{2k_B T_{\perp}} \right) dv_x dv_y \right] \\
&= m \left(\frac{m}{2\pi k_B T_{\perp}} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{m\left(v_x^2 + v_y^2 \right)}{2k_B T_{\perp}} \right) dv_x dv_y \\
&= m \left(\frac{m}{2\pi k_B T_{\perp}} \right) \left(\frac{2k_B T_{\perp}}{m} \right)^2 \int_{-\infty}^{\infty} s_x^2 \exp\left(-s_x^2 \right) ds_x \int_{-\infty}^{\infty} \exp\left(-s_y^2 \right) ds_y \\
&= m \left(\frac{m}{2\pi k_B T_{\perp}} \right) \left(\frac{2k_B T_{\perp}}{m} \right)^2 \frac{\sqrt{\pi}}{2} \sqrt{\pi} \\
&= k_B T_{\perp}
\end{aligned}
$$

2. Field line equation: Consider the magnetic field $B_x = B_0 y/L$, $B_y = \epsilon B_0 x/L$.

a) Compute the z component of the vectorpotential
b) Determine the equation for magnetic field lines.
c) Discuss and plot the magnetic field lines.
d) Compute the current density. Is there a value for ϵ for which this magnetic field is a vacuum field?

Solution:

a) z component of the vectorpotential: From $\mathbf{B} = \nabla A_z \times \mathbf{e}_z$

$$\begin{aligned} B_x &= \partial_y A_z \\ B_y &= -\partial_x A_z \end{aligned}$$

Integration:

$$\begin{aligned} A_z &= \frac{B_0}{L} \int^y y dy = \frac{B_0}{2L} y^2 + g(x) \\ A_z &= -\frac{\epsilon B_0}{L} \int^x x dx = -\frac{\epsilon B_0}{2L} y^2 + h(y) \end{aligned}$$

Combining the two expressions:

$$A_z = \frac{B_0}{2L} \left(y^2 - \epsilon x^2 \right) + \mathrm{const}$$

b) Equation for magnetic field lines: From $A_z = \mathrm{const}$

$$y = \pm\sqrt{\epsilon}\sqrt{x^2 + c_0}$$

c) Discuss and plot the magnetic field lines.

For c_0 thefield lines are straight line with the slopes $\pm\sqrt{\epsilon}$. These we call x lines.

For $c_0 < 0$ the lines are hyperbolas which cross the x axis at $\pm\sqrt{c_0}$ and aproach x lines asymptotically for $x \to \pm\infty$.

For $c_0 < 0$ the lines are hyperbolas which cross the y axis at $\pm\sqrt{\epsilon c_0}$ and aproach x lines asymptotically for $x \to \pm\infty$.

d) Current density:

$$\begin{aligned} j_x &= \frac{1}{\mu_0} \left(\partial_y B_z - \partial_z B_y \right) = 0 \\ j_y &= \frac{1}{\mu_0} \left(\partial_z B_x - \partial_x B_z \right) = 0 \\ j_z &= \frac{1}{\mu_0} \left(\partial_x B_y - \partial_y B_x \right) = \frac{1}{\mu_0} \left(\frac{\epsilon B_0}{L} - \frac{B_0}{L} \right) = \frac{B_0}{\mu_0 L} \left(\epsilon - 1 \right) \end{aligned}$$

The current density is exactly 0 for $\epsilon = 1$, i.e., when the x lines are a right angles the field is a vacuum field.

3. Klimontovich Equation: Starting from the Klimontovich distribution, derive the Klimontovich equation. What procedure and conditions are necessary to obtain the collisionless Boltzmann equation?

Klimontovich distribution:

$$N_s(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N_{0s}} \delta(\mathbf{x} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t))$$

with the equation of motion for the individual particles:

$$\begin{aligned} \dot{\mathbf{R}}_i &= \mathbf{V}_i(t) \\ \dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\mathbf{E}^m(\mathbf{R}_i(t), t) + \frac{q_s}{m_s}\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{R}_i(t), t) \end{aligned}$$

With

$$\begin{aligned} \frac{\partial f(a-b)}{\partial a} &= -\frac{\partial f(a-b)}{\partial b} \\ \frac{df(g(t))}{dt} &= \frac{df}{dg}\frac{dg}{dt} \end{aligned}$$

the time derivative of the distribution is

$$\begin{aligned} \frac{\partial N_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \dot{\mathbf{R}}_i \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &\quad -\sum_{i=1}^{N_{0s}} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

Substitute: $\dot{\mathbf{R}}_i$ and $\dot{\mathbf{V}}_i$:

$$\begin{aligned} \frac{\partial N_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \mathbf{V}_i(t) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &\quad -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{R}_i(t), t) + \frac{q_s}{m_s}\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{R}_i(t), t)\right\} \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &= -\sum_{i=1}^{N_{0s}} \mathbf{V}_i(t)\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &\quad -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{R}_i(t), t) + \frac{q_s}{m_s}\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{R}_i(t), t)\right\} \delta(\mathbf{r} - \mathbf{R}_i(t)) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

We can now use $f(a)\delta(a-b) = f(b)\delta(a-b)$ to replace $\mathbf{V}_i(t)$ in the 1st term and $\mathbf{R}_i(t)$ in the 2nd term:

$$\begin{aligned} \frac{\partial N_s(\mathbf{r}, \mathbf{v}, t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \mathbf{v} \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &\quad -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{r}, t) + \frac{q_s}{m_s}\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{r}, t)\right\} \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

Consider only the contribution from the x components in teh scalar product in the Lorentz force term:

$$\begin{aligned}
\left[\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{r},t)\right]_x \cdot \frac{\partial}{\partial v_x}\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right) &= \Big[V_{yi}(t)\,B_z^m(\mathbf{r},t) - V_{zi}(t)\,B_y^m(\mathbf{r},t)\Big] \\
&\quad \cdot\delta\left(v_y - V_{yi}(t)\right)\delta\left(v_z - V_{zi}(t)\right)\frac{\partial}{\partial v_x}\delta\left(v_x - V_{xi}(t)\right) \\
&= \Big[v_y(t)\,B_z^m(\mathbf{r},t) - v_z(t)\,B_y^m(\mathbf{r},t)\Big] \cdot \frac{\partial}{\partial v_x}\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right) \\
&= \left[\mathbf{v}(t) \times \mathbf{B}^m(\mathbf{r},t)\right]_x \cdot \frac{\partial}{\partial v_x}\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right)
\end{aligned}$$

and substituting this result in the $\partial N_s/\partial t$

$$\begin{aligned}
\frac{\partial N_s(\mathbf{r},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \mathbf{v}\cdot\nabla_{\mathbf{r}}\delta\left(\mathbf{r}-\mathbf{R}_i(t)\right)\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right) \\
&\quad -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{r},t) + \frac{q_s}{m_s}\mathbf{v}(t)\times\mathbf{B}^m(\mathbf{r},t)\right\}\cdot\nabla_{\mathbf{v}}\delta\left(\mathbf{r}-\mathbf{R}_i(t)\right)\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right)
\end{aligned}$$

Now the coefficients in the two terms on the right side do not depend on particle index such that they can be taken out of the integral

$$\begin{aligned}
\frac{\partial N_s(\mathbf{r},\mathbf{v},t)}{\partial t} &= -\mathbf{v}\cdot\nabla_{\mathbf{r}}\sum_{i=1}^{N_{0s}}\delta\left(\mathbf{r}-\mathbf{R}_i(t)\right)\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right) \\
&\quad -\left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{r},t) + \frac{q_s}{m_s}\mathbf{v}(t)\times\mathbf{B}^m(\mathbf{r},t)\right\}\cdot\nabla_{\mathbf{v}}\sum_{i=1}^{N_{0s}}\delta\left(\mathbf{r}-\mathbf{R}_i(t)\right)\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right) \\
&= -\mathbf{v}\cdot\nabla_{\mathbf{r}}N_s(\mathbf{r},\mathbf{v},t) - \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{r},t) + \frac{q_s}{m_s}\mathbf{v}(t)\times\mathbf{B}^m(\mathbf{r},t)\right\}\cdot\nabla_{\mathbf{v}}N_s(\mathbf{r},\mathbf{v},t)
\end{aligned}$$

or

$$\frac{\partial N_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}N_s + \frac{q_s}{m_s}\left(\mathbf{E}^m + \mathbf{v}\times\mathbf{B}^m\right)\cdot\nabla_{\mathbf{v}}N_s = 0$$

which is the Limontovich equation. The electric and magnetic fields in this expression are microscopic meaning they are exact on the individual particle level but include fluctuation that produce a net macrospic effect due to particle cillisions. As illustrated in the derivation of the collision frequency in a plasma these collisions depend on the plasma frequency and on the plasma parameter. The limit to a collisionless system requires the limit of $\Lambda \to \infty$. This is for instance achieved in the limit where the particle number density approaches infinity under the constraints of mass, charge, and energy conservations.

4. Harris sheet: The distribution functions for the exact neutral Harris sheet equilibrium are given by

$$\begin{aligned} f_s(\mathbf{v},\mathbf{r}) &= c_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \exp\left(-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + (v_y - u_s)^2\right)\right) \\ \text{with} \quad & c_s = n_0 \left(\frac{m_s}{2\pi k_B T}\right)^{3/2} \\ \text{and} \quad & u_e = -u_i = u_0 \end{aligned}$$

(a) show that the associated current density is: $j_y(A_y) = -2en_0u_0 \exp\left(-\frac{eu_0}{k_BT} A_y(\mathbf{r})\right)$
(b) Show that the pressure $p = p_e + p_i$ is: $p(A_y) = 2n_0 k_B T \exp\left(-\frac{eu_0}{k_BT} A_y(\mathbf{r})\right)$
(c) Show by explicit substitution that this distribution satisfies the collisionless Boltzmann with $\partial/\partial t = 0$ and $\partial/\partial y = 0$.

Solution: (a) Current density for species s:

$$\begin{aligned} \mathbf{j}_s &= q_s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{v} f_s(\mathbf{r},\mathbf{v})\, d^3v \\ &= q_s c_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{v} \exp\left(-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + (v_y - u_s)^2\right)\right) d^3v \end{aligned}$$

x compoent:

$$j_{xs} = q_s c_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x \exp\left(-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + (v_y - u_s)^2\right)\right) d^3v = 0$$

because the integral is antisymmetric in v_x and therefore 0. The same applies to the z component. For the y component we use th esubstitution $v_y = \tilde{v}_y + u_s$:

$$\begin{aligned} j_{ys} &= q_s c_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\tilde{v}_y + u_s) \exp\left(-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + \tilde{v}_y^2\right)\right) d^3v \\ &= q_s u_s c_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + \tilde{v}_y^2\right)\right) dv_x d\tilde{v}_y dv_z \\ &= q_s u_s \left(\frac{2k_B T}{m_s}\right)^{3/2} \pi^{3/2} n_0 \left(\frac{m_s}{2\pi k_B T}\right)^{3/2} \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \\ j_{ys} &= q_s n_0 u_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \end{aligned}$$

With $u_e = -u_i = u_0$ the total current density in the y direction is

$$\begin{aligned} j_y &= j_{yi} + j_{ye} = en_0 u_i \exp\left(\frac{eu_i}{k_B T} A_y(x,z)\right) - en_0 u_e \exp\left(-\frac{eu_e}{k_B T} A_y(x,z)\right) \\ &= -2en_0 u_0 \exp\left(-\frac{eu_{i0}}{k_B T} A_y(x,z)\right) \end{aligned}$$

(b) Pressure:

$$\begin{aligned} p_s &= \frac{m_s}{3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathbf{v} - \mathbf{u})^2 f_s(\mathbf{r},\mathbf{v})\, d^3v \\ &= \frac{m_s}{3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v_x^2 + \tilde{v}_y^2 + v_z^2\right) f_s(\mathbf{r},\mathbf{v})\, dv_x d\tilde{v}_y dv_z \end{aligned}$$

$$
\begin{aligned}
&= m_s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x^2 f_s(\mathbf{r},\mathbf{v})\, dv_x d\tilde{v}_y dv_z \\
&= m_s c_s \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{m_s}{2k_B T}\left(v_x^2 + v_z^2 + \tilde{v}_y^2\right)\right) dv_x d\tilde{v}_y dv_z \\
&= m_s c_s \left(\frac{2k_B T}{m_s}\right)^{5/2} \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_x^2 \exp\left(-\left(s_x^2 + s_y^2 + s_z^2\right)\right) ds_x ds_y ds_z \\
&= m_s n_0 \left(\frac{m_s}{2\pi k_B T}\right)^{3/2} \left(\frac{2k_B T}{m_s}\right)^{5/2} \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right) \pi \frac{\sqrt{\pi}}{2} \\
&= n_0 k_B T \exp\left(\frac{q_s u_s}{k_B T} A_y(x,z)\right)
\end{aligned}
$$

Pressure of electrons and ions:

$$
\begin{aligned}
p &= p_i + p_e = n_0 k_B T \exp\left(\frac{e u_i}{k_B T} A_y(x,z)\right) + n_0 k_B T \exp\left(-\frac{e u_e}{k_B T} A_y(x,z)\right) \\
&= 2 n_0 k_B T \exp\left(-\frac{e u_0}{k_B T} A_y(x,z)\right)
\end{aligned}
$$

(c) Test of the collisionless Boltzmann equation:

Show by explicit substitution that f_s solves the collisionless Boltzmann equation:

$$
\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} f_s + \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = 0
$$

$$
\begin{aligned}
\text{1st term :}\quad && \frac{\partial f_s}{\partial t} &= \frac{q_s u_s}{k_B T_s}\frac{\partial A_y(\mathbf{r})}{\partial t} f_s = 0 \\
\text{2nd term :}\quad && \nabla_{\mathbf{r}} f_s &= \frac{q_s u_s}{k_B T_s}\left(\nabla_{\mathbf{r}} A_y(\mathbf{r})\right) f_s \\
=> \quad && \mathbf{v}\cdot\nabla_{\mathbf{r}} f_s &= \frac{q_s u_s}{k_B T_s}\left(\mathbf{v}\cdot\nabla_{\mathbf{r}} A_y\right) f_s \\
\text{3rd term :}\quad && \nabla_{\mathbf{v}} f_s &= -\frac{m_s}{k_B T_s}(\mathbf{v} - \mathbf{u}_s) f_s \\
&& \text{with} \quad & \mathbf{u}_s = u_s \mathbf{e}_y
\end{aligned}
$$

with $\mathbf{E} = -\partial A_y/\partial t\, \mathbf{e}_y = 0$:

$$
\begin{aligned}
\frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s &= \frac{q_s}{m_s}(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s \\
&= -\frac{q_s}{m_s}\frac{m_s}{k_B T_s}(\mathbf{v}\times\mathbf{B})\cdot(\mathbf{v} - \mathbf{u}_s) f_s \\
&= \frac{q_s}{k_B T_s}(\mathbf{v}\times\mathbf{B})\cdot u_s \mathbf{e}_y f_s \\
&= \frac{q_s u_s}{k_B T_s}\mathbf{v}\cdot(\mathbf{B}\times\mathbf{e}_y) f_s \\
&= \frac{q_s u_s}{k_B T_s}\mathbf{v}\cdot\left((\nabla A_y \times \mathbf{e}_y)\times\mathbf{e}_y\right) f_s \\
&= -\frac{q_s u_s}{k_B T_s}(\mathbf{v}\cdot\nabla A_y) f_s
\end{aligned}
$$

$$
=> \qquad (2) + (3) = 0
$$