1. Maxwellian Integrals:

Plasma physics makes frequent use integrals of integrals over the Maxwell distribution.
(a) Show

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx = \sqrt{\pi}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx &= \left[\left(\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx\right)\left(\int_{-\infty}^{\infty} \exp\left(-y^2\right) dy\right)\right]^{1/2} \\
&= \left[\int_{-\infty}^{\infty} \exp\left(-x^2 - y^2\right) dxdy\right]^{1/2} \\
&= \left[\int_{0}^{2\pi}\int_{0}^{\infty} r \exp\left(-r^2\right) drd\varphi\right]^{1/2} \\
&= \left[2\pi\left[-\frac{1}{2}\exp\left(-r^2\right)\right]_0^{\infty}\right]^{1/2} \\
&= \sqrt{\pi}
\end{aligned}$$

(b) Prove that

$$\int_0^{\infty} x^m \exp\left(-cx^2\right) dx = \begin{cases} \Gamma\left(l + 1/2\right)/2c^{l+1/2} & \text{for} \quad m = 2l \\ l!/2c^{l+1} & \text{for} \quad m = 2l + 1 \end{cases}$$

for m integer and > 0. Hint: $\Gamma\left(1/2\right) = \sqrt{\pi}$ and use recursion.

Integration by parts

$$\begin{aligned}
I_m\left(c\right) &= \int_0^{\infty} x^m \exp\left(-cx^2\right) dx \\
&\quad -\frac{1}{2c}\int_0^{\infty} x^{m-1}\left(-2cx\right)\exp\left(-cx^2\right) dx \\
&= -\frac{1}{2c}\left[x^{m-1}\exp\left(-cx^2\right)\right]_0^{\infty} + \frac{m-1}{2c}\int_0^{\infty} x^{m-2}\exp\left(-cx^2\right) dx \\
&= \frac{m-1}{2c}\int_0^{\infty} x^{m-2}\exp\left(-cx^2\right) dx \\
I_m\left(c\right) &= \frac{m-1}{2c} I_{m-2}
\end{aligned}$$

i) Consider $m = 2l$: Result in (a) demonstrates $I_0 = \sqrt{\pi}/\left(2c^{1/2}\right)$ and the integration by parts yields

$$I_m\left(c\right) = \frac{m-1}{2c} I_{m-2} = \frac{l - 1/2}{c} I_{m-2}$$

Now consider the expression given to prove: $I_m = \Gamma\left(l+1/2\right)/2c^{l+1/2}$

For $m=0$ we get: $I_0\left(c\right) = \Gamma\left(1/2\right)/\left(2c^{1/2}\right) = \sqrt{\pi}/\left(2c^{1/2}\right)$

For $m=2l>0$ we obtain

$$I_m\left(c\right) = \frac{\Gamma\left(l+1/2\right)}{2c^{l+1/2}} = \frac{l-1/2}{c}\frac{\Gamma\left(l-1/2\right)}{2c^{l-1/2}} = \frac{m-1}{2c}I_{m-2}$$

such that the expression defined in (b) for m even has the same value I_0 and the same recursion as the result from the integration by parts.

(ii) Consider $m=2l+1$: In this case we need to compute first the value for I_1:

$$I_1\left(c\right) = \int_0^{\infty} x\exp\left(-cx^2\right)dx = -\frac{1}{2c}\left[\exp\left(-cx^2\right)\right]_0^{\infty} = \frac{1}{2c}$$

Evaluating I_1 from the expression in (b) yields $I_1\left(c\right) = 1/\left(2c\right)$ and for $m=2l+1$

$$I_m\left(c\right) = \frac{l!}{2c^{l+1}} = \frac{l}{c}\frac{\left(l-1\right)!}{2c^{l}} = \frac{m-1}{2c}I_{m-2}$$

Again the element I_1 and the recursion for I_m determined by our integration are the same as for the expression in (b) which demonstrate that this expression provides the correct value for I_m.

2. Magnetic gradient drift:

Evaluate the time average for the gradient drift velocity

$$\mathbf{v}_\nabla = \frac{1}{B_0}\left\langle \mathbf{v}_g x \frac{dB_0}{dx} \right\rangle$$

by using the general motion of a single particle with charge q and mass m in a magnetic field along the z direction with a gradient in the the x direction (without any electric field in the Lorentz equation). Show that the resulting drift is in the y direction and can be written as

$$\mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}\mathbf{B} \times \nabla B$$

Solution: Particle motion for $\mathbf{B}_0 = B_0\mathbf{e}_z$ (from class):

$$\begin{aligned} v_x &= v_\perp \sin(\omega_g t + \phi) \\ v_y &= v_\perp \cos(\omega_g t + \phi) \\ v_z &= v_\parallel \end{aligned} \tag{1}$$

with $v_\perp^2 = v_x^2 + v_y^2$ and the trajectory

$$\begin{aligned} x - x_0 &= -\frac{v_\perp}{\omega_g}\cos(\omega_g t + \phi) \\ y - y_0 &= \frac{v_\perp}{\omega_g}\sin(\omega_g t + \phi) \\ z - z_0 &= v_\parallel t \end{aligned} \tag{2}$$

X component of the gradient drift;

$$\begin{aligned} \mathbf{v}_{\nabla x} &= -\frac{1}{B_0}\left\langle v_x x \frac{dB_0}{dx}\right\rangle \\ &= \frac{1}{B_0}\frac{dB_0}{dx}\left\langle v_\perp \sin(\omega_g t + \phi)\frac{v_\perp}{\omega_g}\cos(\omega_g t + \phi)\right\rangle \\ &= \frac{1}{B_0}\frac{dB_0}{dx}\frac{v_\perp^2}{\omega_g}\frac{\omega_g}{2\pi}\int_0^{\frac{\omega_g}{2\pi}} \sin(\omega_g t + \phi)\cos(\omega_g t + \phi)\, dt \\ &= \frac{1}{B_0}\frac{dB_0}{dx}\frac{v_\perp^2}{\omega_g}\frac{\omega_g}{2\pi}\int_0^{\frac{\omega_g}{2\pi}} \frac{1}{2}\sin 2(\omega_g t + \phi)\, dt = 0 \end{aligned}$$

In the above derivation we have omitted the term x_0 because it is constant and the time average of a constant multiplied with a sin or cos is 0.

Y component of the gradient drift:

$$
\begin{aligned}
\mathbf{v}_{\nabla y} &= -\frac{1}{B_0}\left\langle v_y x \frac{dB_0}{dx}\right\rangle \\
&= \frac{1}{B_0}\frac{dB_0}{dx}\left\langle v_\perp \cos\left(\omega_g t+\phi\right)\frac{v_\perp}{\omega_g}\cos\left(\omega_g t+\phi\right)\right\rangle \\
&= \frac{1}{B_0}\frac{dB_0}{dx}\frac{v_\perp^2}{\omega_g}\frac{\omega_g}{2\pi}\int_0^{\frac{\omega_g}{2\pi}}\cos^2\left(\omega_g t+\phi\right)dt \\
&= \frac{1}{B_0}\frac{dB_0}{dx}\frac{v_\perp^2}{\omega_g}\frac{\omega_g}{2\pi}\int_0^{\frac{\omega_g}{2\pi}}\frac{1}{2}\left(1+\cos 2\left(\omega_g t+\phi\right)\right)dt \\
&= \frac{1}{2B_0}\frac{dB_0}{dx}\frac{v_\perp^2}{\omega_g} = \frac{mv_\perp^2}{2qB_0^2}\frac{dB_0}{dx}
\end{aligned}
$$

The z component is not considered because it just results in motion parallel to the magnetic field $v_\parallel$.

$$
\begin{aligned}
\mathbf{v}_\nabla &= \frac{mv_\perp^2}{2qB^3}\mathbf{B}\times\nabla B \\
&= \frac{mv_\perp^2}{2qB^3}B_0\mathrm{sign}\left(B_0\right)\,\mathbf{e}_z\times\nabla B_0 \\
&= \frac{mv_\perp^2}{2qB^3}B_0\mathrm{sign}\left(B_0\right)\,\mathbf{e}_z\times\nabla B_0 \\
&= \frac{mv_\perp^2}{2qB^2}\mathbf{e}_z\times\left(\frac{dB_0}{dx}\mathbf{e}_x\right) \\
&= \frac{mv_\perp^2}{2qB^2}\frac{dB_0}{dx}\mathbf{e}_y
\end{aligned}
$$

3. Drift currents:

Assume a plasma consisting of electrons and ions with equal number density n and temperature T. If electrons and ions have (in the average) different drift velocities, the relative particle drift sets up a current. Determine this current for
(a) the electric field ($\mathbf{E} \times \mathbf{B}$) drift,
(b) the gradient B drift,
(c) the curvature B drift, and
(d) the polarization drift,
using the expressions derived in section 1.4.1. In order to express the average parallel and perpendicular particle velocities (squared) for the gradient B and curvature drifts assume that the particles are in thermal equilibrium, i.e., Maxwell distributed. **Note** that the drift expressions are derived for individual particles, however, the current density is a property of the total particle distribution.

Solution:

The 4 different drifts are

$$\begin{aligned}
\text{Electric Field :} \quad & \mathbf{v}_E = \frac{1}{B^2}\mathbf{E} \times \mathbf{B} \\
\text{Gradient :} \quad & \mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}\left(\mathbf{B} \times \nabla B\right) \\
\text{Curvature :} \quad & \mathbf{v}_C = \frac{mv_\parallel^2}{qB^4}\mathbf{B} \times \left[\left(\mathbf{B} \cdot \nabla\right)\mathbf{B}\right] \\
\text{Polarisation :} \quad & \mathbf{v}_P = \frac{m}{qB^2}\frac{d\mathbf{E}_\perp}{dt}
\end{aligned}$$

(a) The $\mathbf{E} \times \mathbf{B}$ drift is charge independent.

The current density for a species s associated with the drift is

$$\mathbf{j}_{Es} = q_s \int\int\int d^3v \mathbf{v}_E f\left(\mathbf{v}, t\right) = q_s n_s \frac{1}{B^2}\mathbf{E} \times \mathbf{B}$$

However, the drift velocity is the same for all particles irrespective of particle charge and velocity. Therefore the net current density is always 0. This can easily be seen for the case of electrons and a single ion species which satisfy the neutrality condition. However, more fundamental: The $\mathbf{E} \times \mathbf{B}$ drift implies that the particles are all at rest in the frame of reference where the electric field is 0. Therefore there is no current associated with this drift.

(b) The gradient B drift is both charge and energy dependent and the associated current density for species s is

$$\begin{aligned}
\mathbf{j}_{\nabla s} &= q_s \int\int\int d^3v \mathbf{v}_{\nabla s} f_s\left(\mathbf{v}, t\right) = q_s \frac{m_s}{2q_s B^3}\left(\mathbf{B} \times \nabla B\right) \int\int\int d^3v v_\perp^2 f_s\left(\mathbf{v}, t\right) \\
&= \frac{m_s}{2B^3}\left(\mathbf{B} \times \nabla B\right) I_{\perp s}
\end{aligned}$$

Assume a Maxwell distribution and the magnetic field $\mathbf{B}$ into the z direction

$$f_{s0} = \frac{n_0}{(2\pi k_B T_s/m_s)^{3/2}} \exp\left[-\frac{m_s v^2}{2k_B T_s}\right]$$

such that

$$I_{\perp s} = \int\int\int_{-\infty}^{\infty} d^3v v_\perp^2 f_s(\mathbf{v}, t) = \frac{n_{0s}}{(2\pi k_B T_s/m_s)^{3/2}} \int\int\int_{-\infty}^{\infty} d^3v \left(v_x^2 + v_y^2\right) \exp\left[-\frac{m_s v^2}{2k_B T_s}\right]$$

Using the substitution $\tilde{v}_i = v_i / (2k_B T_s/m_s)^{1/2}$ and noting that

$$\int\int\int_{-\infty}^{\infty} d^3v v_x^2 \exp\left[-\frac{m_s v^2}{2k_B T_s}\right] = \int\int\int_{-\infty}^{\infty} d^3v v_y^2 \exp\left[-\frac{m_s v^2}{2k_B T_s}\right]$$

we obtain

$$\begin{aligned}
I_{\perp s} &= \frac{2n_{0s}}{(2\pi k_B T_s/m_s)^{3/2}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z v_x^2 \exp\left[-\frac{m_s v^2}{2k_B T_s}\right] \\
&= \frac{2n_{0s}(2k_B T_s/m_s)^{5/2}}{(2\pi k_B T_s/m_s)^{3/2}} \int_{-\infty}^{\infty} d\tilde{v}_x \tilde{v}_x^2 \exp\left[-\tilde{v}_x^2\right] \int_{-\infty}^{\infty} d\tilde{v}_y \exp\left[-\tilde{v}_y^2\right] \int_{-\infty}^{\infty} d\tilde{v}_z \exp\left[-\tilde{v}_z^2\right] \\
&= \frac{4n_{0s} k_B T_s/m_s}{\pi^{3/2}} I_2(1) I_0(1) I_0(1) \\
&= \frac{4n_{0s} k_B T_s/m_s}{\pi^{3/2}} 8\frac{1}{2}\left[I_0(1)\right]^3 = \frac{4n_{0s} k_B T_s/m_s}{\pi^{3/2}} 8\frac{1}{2}\frac{1}{8}\pi^{3/2} = 2n_{0s} k_B T_s/m_s
\end{aligned}$$

Defining the thermal velocity for species s as $u_s^2 = k_B T_s/m_s$ with the result of $I_{\perp s} = 2n_{0s}u_s^2$ and the gradient drift current is

$$\begin{aligned}
\mathbf{j}_\nabla &= \sum_s \frac{m_s}{2B^3} (\mathbf{B} \times \nabla B) I_{\perp s} = n_0 \frac{m_i u_i^2 + m_e u_e^2}{B^3} (\mathbf{B} \times \nabla B) \\
or \qquad & \mathbf{j}_\nabla = n_0 \frac{k_B T_i + k_B T_e}{B^3} (\mathbf{B} \times \nabla B) = \frac{p_i + p_e}{B^3} (\mathbf{B} \times \nabla B)
\end{aligned}$$

Note, if we were to distinguish between parallel and perpendicular temperature and pressure, the temperature or pressure in the result would have to be replaced by the perpendicular temperature or pressure!

(c) The result for the curvature drift current density is similar to the computation in (b) because the drift is mass and charge dependent. However, since the curvature drift is due to the particle velocity parallel to the magnetic field we only need to consider the integral over v_z^2.

$$\begin{aligned}
\mathbf{j}_{Cs} &= q_s \int\int\int d^3v \mathbf{v}_{Cs} f_s(\mathbf{v}, t) = q_s \frac{m_s}{q_s B^4} (\mathbf{B} \times [(\mathbf{B} \cdot \nabla) \mathbf{B}]) \int\int\int d^3v v_\parallel^2 f_s(\mathbf{v}, t) \\
&= \frac{m_s}{B^4} (\mathbf{B} \times [(\mathbf{B} \cdot \nabla) \mathbf{B}]) I_{\parallel s}
\end{aligned}$$

with

$$I_{\|s} = \int\int\int_{-\infty}^{\infty} d^3v v_\|^2 f_s(\mathbf{v},t) = \frac{n_{0s}}{(2\pi k_B T_s/m_s)^{3/2}} \int\int\int_{-\infty}^{\infty} d^3v v_z^2 \exp\left[-\frac{m_s v^2}{2k_B T_s}\right]$$

Making the same substitutions

$$\begin{aligned} I_{\|s} &= \frac{n_{0s}}{(2\pi k_B T_s/m_s)^{3/2}} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z v_z^2 \exp\left[-\frac{m_s v^2}{2k_B T_s}\right] \\ &= n_{0s} k_B T_s / m_s \end{aligned}$$

yields the total curvature drift current density

$$\mathbf{j}_C = n_0 \frac{k_B T_i + k_B T_e}{B^4} \left(\mathbf{B} \times [(\mathbf{B}\cdot\nabla)\mathbf{B}]\right) = \frac{p_i + p_e}{B^4} \left(\mathbf{B} \times [(\mathbf{B}\cdot\nabla)\mathbf{B}]\right)$$

Note that between the gradient and the curvature drifts the additional factor of $1/2$ in the gradient drift is cancelled by the factor of 2 between the perpendicular and parallel kinetic energy density.

(d) The polarization drift current is relatively straightforward. The drift is only charge dependent and does not depend on particle velocity:

$$\begin{aligned} \mathbf{j}_{Ps} &= q_s \int\int\int d^3v \mathbf{v}_{Ps} f_s(\mathbf{v},t) = q_s \frac{m_s}{q_s B^2} \frac{d\mathbf{E}_\perp}{dt} \int\int\int d^3v f_s(\mathbf{v},t) \\ &= \frac{m_s}{B^2} \frac{d\mathbf{E}_\perp}{dt} n_{0s} \end{aligned}$$

where we have used the definition $n_{0s} = \int\int\int d^3v f_s(\mathbf{v},t)$. Summation over ions and electrons yields

$$\mathbf{j}_{Ps} = n_0 \frac{m_i + m_e}{B^2} \frac{d\mathbf{E}_\perp}{dt}$$

In summary the $\mathbf{E}\times\mathbf{B}$ drift does not generate any current because it is explained by a Galilei transformation into a different frame of reference. The gradient and the curvature drifts provide and electron and ion current density into the same direction where the magnitude scales directly with the respective ion and electron pressure. For a gyrotropic distribution the gradient drift current density scales with the perpendicular pressure and the curvature drift current density with the parallel pressure. Finally the polarization drift current is proportional to density and mass such that the corresponding electron current density is much smaller than the ion current density.