4. Klimontovich equation:

The derivation of the Klimontovich equation requires (a) to replace the particle velocities $\mathbf{V}_i = d\mathbf{R}_i/dt$ with phase space velocity coordinates

$$\dot{\mathbf{R}}_i(t) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) = \mathbf{v} \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t))$$

and (b) to replace the particle acceleration $d\mathbf{V}_i/dt$ with the Lorentz Force term at phase space coordinates $\mathbf{r}, \mathbf{v}$, i.e.,

$$\begin{aligned} d\mathbf{V}_i/dt &= \frac{q}{m}\{\mathbf{E}(\mathbf{R}_i(t), t) + \mathbf{V}_i(t) \times \mathbf{B}(\mathbf{R}_i(t), t)\} \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &= \frac{q}{m}\{\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)\} \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

Demonstrate rigorusly that this is justified for (a) and (b).

Solution:

(a) For the δ function we can make use of the property: $\int f(a)\,\delta(a-b)\,da = f(b) = \int f(b)\,\delta(a-b)\,da$ or short $f(a)\,\delta(a-b) = f(b)\,\delta(a-b)$. Further we can ignore the δ function over the velocity in (a)

$$\begin{aligned} \dot{\mathbf{R}}_i(t) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) &= \delta(\mathbf{v} - \mathbf{V}_i(t))\, \mathbf{V}_i \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t)) \\ &= \mathbf{v} \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

where we have used $\delta(\mathbf{v} - \mathbf{V}_i(t))$ to replace $\mathbf{V}_i$ with $\mathbf{v}$.

(b) For the Lorentz force term we consider first at the Electric field contribution:

$$\begin{aligned} \mathbf{E}(\mathbf{R}_i(t), t) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) &= \delta(\mathbf{r} - \mathbf{R}_i(t))\, \mathbf{E}(\mathbf{R}_i(t), t) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &= \mathbf{E}(\mathbf{r}, t) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

Now we consider the magnetic force term:

$$\begin{aligned} &\{\mathbf{V}_i(t) \times \mathbf{B}(\mathbf{R}_i(t), t)\} \\ &\cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{R}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t)) = \delta(\mathbf{r} - \mathbf{R}_i(t))\, \mathbf{V}_i(t) \times \mathbf{B}(\mathbf{R}_i(t), t) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{v} - \mathbf{V}_i(t)) \\ &\qquad = \mathbf{V}_i(t) \times \mathbf{B}(\mathbf{r}, t) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{v} - \mathbf{V}_i(t))\, \delta(\mathbf{r} - \mathbf{R}_i(t)) \end{aligned}$$

It remains to be shown that we can also replace $\mathbf{V}_i(t)$ with the corresponding coordinates $\mathbf{v}$. In order to show this we only consider

$$\begin{aligned} [\mathbf{V}_i(t) \times \mathbf{B}(\mathbf{r}, t)]_x \cdot \partial_{v_x} \delta(\mathbf{v} - \mathbf{V}_i(t)) &= (V_{iy} B_z - V_{iz} B_y)\, \delta(v_y - V_{iy})\, \delta(v_z - V_{iz})\, \partial_{v_x} \delta(v_x - V_{ix}) \\ &= (v_y B_z - v_z B_y)\, \delta(v_y - V_{iy})\, \delta(v_z - V_{iz})\, \partial_{v_x}(v_x - V_{ix}) \\ &= [\mathbf{v} \times \mathbf{B}(\mathbf{r}, t)]_x \cdot \partial_{v_x} \delta(\mathbf{v} - \mathbf{V}_i(t)) \end{aligned}$$

The same can be shown for the contributions of the y and z components to the scalar product.combining these results demonstrates

$$\begin{aligned}\{\mathbf{E}(\mathbf{R}_i(t),t)+\mathbf{V}_i(t)\times\mathbf{B}(\mathbf{R}_i(t),t)\}\\ \cdot\nabla_{\mathbf{v}}\delta(\mathbf{r}-\mathbf{R}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t)) &= \{\mathbf{E}(\mathbf{r},t)+\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\}\cdot\nabla_{\mathbf{v}}\delta(\mathbf{r}-\mathbf{R}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))\end{aligned}$$

Note, that we have ommitted an upper index m for the fields to indicate their microscopic character.

5. Liouville Equation:

Prove the equivalence between the convective form of the Liouville equation ($DN/Dt = 0$) and the continuity equation ($\partial N/\partial t + \sum_{i=1}^{N_0} \nabla_{\mathbf{r}_i} \cdot (\mathbf{v}_i N) + \sum_{i=1}^{N_0} \nabla_{\mathbf{v}_i} \cdot \left(\dot{\mathbf{V}}_i N\right) = 0$).

Solution: The two equations are equivalent if $\nabla_{\mathbf{r}_i} \cdot \mathbf{v}_i = 0$ and $\nabla_{\mathbf{v}_i} \cdot \dot{\mathbf{V}}_i = 0$

i) $\nabla_{\mathbf{r}_i} \cdot \mathbf{v}_i = 0$ is trivial because both $\mathbf{r}_i$ and $\mathbf{v}_i$ are independent phase space coordinates.

ii) With $\nabla_{\mathbf{v}_i} \cdot \dot{\mathbf{V}}_i = \nabla_{\mathbf{v}_i} \cdot \left\{ \frac{q_s}{m_s} \left[\mathbf{E}\left(\mathbf{r}_i, t\right) + \mathbf{v}_i \times \mathbf{E}\left(\mathbf{r}_i, t\right)\right] \right\}$ we need to demonstrate that the velocity derivatives are 0. We consider first the v_x derivative

$$
\begin{aligned}
\partial_{v_{ix}} \left[\mathbf{E}\left(\mathbf{r}_i, t\right) + \mathbf{v}_i \times \mathbf{E}\left(\mathbf{r}_i, t\right)\right]_x &= \partial_{v_{ix}} E_x\left(\mathbf{r}_i, t\right) + \partial_{v_{ix}} \left(v_{iy} E_z\left(\mathbf{r}_i, t\right) - v_{iz} E_y\left(\mathbf{r}_i, t\right)\right) \\
&= 0 + 0
\end{aligned}
$$

Here the first term is 0 because $E_x\left(\mathbf{r}_i, t\right)$ does not depend on v_{ix} and the second term because it contains only the v_{iy} and v_{iz} velocity components (because of the crossproduct) such the the v_{ix} derivative is 0.

6. BBGKY Hierarchy:

Demonstrate that the recurrent relation

$$\begin{aligned}\frac{\partial f_k}{\partial t} + \sum_{i=1}^{k} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_k + \sum_{i=1}^{k}\sum_{j=1}^{k} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_k & \\ + \frac{N_0 - k}{V} \sum_{i=1}^{k} \int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1} \mathbf{a}_{i,k+1} \cdot \nabla_{\mathbf{v}_i} f_{k+1} &= 0 \end{aligned} \quad (1)$$

for the equations in the BBGKY hierarchy is correct. You can do this by induction, i.e., take the equation for f_k and integrate it with respect to $d\mathbf{r}_k d\mathbf{v}_k$ and show that the resulting equation for f_{k-1} is consistent with the general form for equation for f_k.

Solution: The reduced distribution function is defined as

$$\begin{aligned} f_k(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, ...\mathbf{r}_k, \mathbf{v}_k, t) &\equiv V^k \int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1} d\mathbf{r}_{k+2} d\mathbf{v}_{k+2} ... d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} f_{N_0} \\ & \text{such that} \\ f_{k-1} &= V^{-1} \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k f_k \end{aligned}$$

If the differential equation for f_k (1) is correct the integral of over $\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k$ should produce the same relation only with $k$replaced by $k-1$:

$$\begin{aligned}\frac{\partial f_{k-1}}{\partial t} + \sum_{i=1}^{k-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{k-1} + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{k-1} & \\ + \frac{N_0 - k + 1}{V} \sum_{i=1}^{k-1} \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{i,k} \cdot \nabla_{\mathbf{v}_i} f_k &= 0 \end{aligned}$$

Integrating each of the 4 terms in (1) yields for term 1:

$$T_1 = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k f_k = V \frac{\partial f_{k-1}}{\partial t}$$

Term 2:

$$\begin{aligned} T_2 &= \sum_{i=1}^{k-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k f_k + \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{v}_k \cdot \nabla_{\mathbf{r}_k} f_k \\ &= V \sum_{i=1}^{k-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{k-1} + \int_{-\infty}^{\infty} dy_k dz_k d\mathbf{v}_k v_{xk} \cdot f_k |_{x_k=-\infty}^{x_k=\infty} + ... \\ &= V \sum_{i=1}^{k-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{k-1} \end{aligned}$$

Term 3:

$$
\begin{aligned}
T_3 &= \sum_{i=1}^{k}\sum_{j=1}^{k}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{ij}\cdot\nabla_{\mathbf{v}_i} f_k \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{ij}\cdot\nabla_{\mathbf{v}_i} f_k + \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{kk}\cdot\nabla_{\mathbf{v}_i} f_k \\
&\quad + \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \sum_{j=1}^{k-1}\mathbf{a}_{kj}\cdot\nabla_{\mathbf{v}_k} f_k + \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \sum_{i=1}^{k-1}\mathbf{a}_{ik}\cdot\nabla_{\mathbf{v}_i} f_k \\
&= V\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathbf{a}_{ij}\cdot\nabla_{\mathbf{v}_i} f_{k-1} + \sum_{i=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{ik}\cdot\nabla_{\mathbf{v}_i} f_k
\end{aligned}
$$

Term 4:

$$
\begin{aligned}
T_4 &= \frac{N_0-k}{V}\sum_{i=1}^{k}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k d\mathbf{r}_{k+1} d\mathbf{v}_{k+1}\mathbf{a}_{i,k+1}\cdot\nabla_{\mathbf{v}_i} f_{k+1} \\
&= \frac{N_0-k}{V}\sum_{i=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k d\mathbf{r}_{k+1} d\mathbf{v}_{k+1}\mathbf{a}_{i,k+1}\cdot\nabla_{\mathbf{v}_i} f_{k+1} \\
&\quad + \frac{N_0-k}{V}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k d\mathbf{r}_{k+1} d\mathbf{v}_{k+1}\mathbf{a}_{k,k+1}\cdot\nabla_{\mathbf{v}_k} f_{k+1} \\
&= \frac{N_0-k}{V}\sum_{i=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1}\mathbf{a}_{i,k+1}\cdot\nabla_{\mathbf{v}_i}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k f_{k+1}
\end{aligned}
$$

where the $i=k$ term vanishes becuse of integration over the component of v_k. In the last expression we can make use of the symmetry and switch labels of k and $k+1$ in f_{k+1} such that

$$
\begin{aligned}
T_4 &= \frac{N_0-k}{V}\sum_{i=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{i,k}\cdot\nabla_{\mathbf{v}_i}\int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1} f_{k+1} \\
&= (N_0-k)\sum_{i=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{i,k}\cdot\nabla_{\mathbf{v}_i} f_k
\end{aligned}
$$

This term can be combined with the last term in T_3 such that adding all terms up and dividing by V yields

$$
\begin{aligned}
&\frac{\partial f_{k-1}}{\partial t} + \sum_{i=1}^{k-1}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i} f_{k-1} + \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathbf{a}_{ij}\cdot\nabla_{\mathbf{v}_i} f_{k-1} \\
&\qquad + \frac{N_0-k+1}{V}\sum_{i=1}^{k-1}\int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k \mathbf{a}_{i,k}\cdot\nabla_{\mathbf{v}_i} f_k = 0
\end{aligned}
$$

which is indeed the expression we expected for the recursion.