7. Three-Point correlation (Coins):

In deriving the equation for f_1 and $g(1,2,t)$ we defined the joint probability f_3 in terms of the one-point probability f_1, the two-point correlation function g, and the three-point correlation function h. Suppose we apply this to the case of three coins, each of which can come up with heads (+) or tails (-). What is the meaning of f_3 in this case? Write out f_3 in the form

$f_3(1,2,3) = f_1(1)\, f_1(2)\, f_1(3) + f_1(1)\, g(2,3) + f_1(2)\, g(1,3) + f_1(3)\, g(1,2) + h(1,2,3),$

and evaluate f_3, f_1, g, and h in each of the following cases.
(a) All three coins are "honest", that is, each coin is equally likely to come up heads or tails, and each coin is unaffected by any other coin.
(b) Because the coins are mysteriously locked together, in any one throw all three are heads or tails, the result changing randomly from throw to throw.
(c) The first two coins always come up heads, while the third is honest. Note that here the probability functions are not symmetric, so that for example, $f_1(1)$ is not the same as $f_1(3)$.

(d) The first coins two are honest but 3 is manipulated to show heads if 1 and 2 are heads. How is your chance improving if you bet on +++ and can you see this looking at the two- or thee-point correlation?

Solution:

(a) All three coins are "honest": Single coin: $f_1(1) = 1/2\delta_{s_1 h} + 1/2\delta_{s_1 t} = 1/2$ where s_1 represents the 2 possible states heads and tails and since there are only two possible states $\delta_{s_1 h} = 1 - \delta_{s_1 h}$.

Consider a system of 2 coins with $f_2(1,2)$. If all results (combinations) are equally likely, $f_2(1,2) = 1/4$ for any one combination and $f_2(1,2) = f_1(1)\, f_1(2) + g(1,2)$ such that $g(1,2) = 0$

For 3 coins and all combinations equally likely we have $f_3(1,2,3) = 1/8$. Since $g = 0$ the 3 point probability is $f_3(1,2,3) = f_1(1)\, f_1(2)\, f_1(3) + h(1,2,3) = 1/8$. Since $f_1(s_i)$ is 1/2 for all s_i it follows that $h(1,2,3) = 0$.

(b) All three are heads or tails: Consider first the first coin with a random result such that $f_1(1) = 1/2$ for both values of state s_1.

Using symmetric functions for $f_1(2)$ and $f_1(3)$: Symmetry requires to treat $f_1(2)$ like $f_1(1)$ that is independent and honest, i.e., $f_1(2) = 1/2$. We know that the overall outcome is only hh or tt (heads heads or tails tails) such that $f_2(1,2) = 0.5\delta_{s_1 s_2}$ that is the likelihood for the states s_1 and s_2 to be different is 0 and the likelihood to have hh or tt is 0.5 for each of these. Using $f_2(1,2) = f_1(1)\, f_1(2) + g(1,2)$ we obtain $g(1,2) = f_2(1,2) - f_1(1)\, f_1(2) = 0.5\delta_{s_1 s_2} - 1/4$.

Considering 3 coins with the same approach $f_3(1,2,3) = 0.5\delta_{s_1 s_2}\delta_{s_1 s_3}$. Using f_1 and g symmetric for all combinations

$$\begin{aligned}
f_3(1,2,3) &= f_1(1)\, f_1(2)\, f_1(3) + f_1(1)\, g(2,3) + f_1(2)\, g(1,3) + f_1(3)\, g(1,2) + h(1,2,3) \\
h(1,2,3) &= f_3(1,2,3) - f_1(1)\, f_1(2)\, f_1(3) - f_1(1)\, g(2,3) - f_1(2)\, g(1,3) - f_1(3)\, g(1,2) \\
&= 0.5\delta_{s_1 s_2}\delta_{s_1 s_3} - 1/8 - 0.5\left[0.5\delta_{s_1 s_2} - 1/4 + 0.5\delta_{s_1 s_3} - 1/4 + 0.5\delta_{s_2 s_3} - 1/4\right] \\
&= 0.5\delta_{s_1 s_2}\delta_{s_1 s_3} + 0.25 - 0.25\left[\delta_{s_1 s_2} + \delta_{s_1 s_3} + \delta_{s_2 s_3}\right]
\end{aligned}$$

One could be attempted to use nonsymmetric functions that is for instance, $f_1(2) = \delta_{s_1 s_2}$, however, this has already build in a correlation to a different system such that it shouldn't be considered a single coin distribution, which is required for f_1.

(c) First two coins always come up heads and the third is honest: In this case f_1 is not symmetric $f_1(1) = \delta_{1h}$, $f_1(2) = \delta_{2h}$ and $f_1(3) = 1/2$. Since f_1 is not symmetric we anticipate that g may be not symmetric and we need to compute g for all state combinations, i.e., $g(1,2)$, $g(1,3)$, $g(2,3)$, :

Coins 1 and 2: $f_2(1,2) = \delta_{1h}\delta_{2h}$ (likelihood for hh is 1 and all others are 0) such that from $f_2(1,2) = f_1(1) f_1(2) + g(1,2)$ we conclude that $g(1,2) = 0$.

Coins1 and 3: $f_2(1,3) = 0.5\delta_{1h}$ such that from $f_2(1,3) = f_1(1) f_1(3) + g(1,3)$ we conclude that $g(1,3) = 0$.

Coins2 and 3: $f_2(2,3) = 0.5\delta_{2h}$ such that from $f_2(2,3) = f_1(2) f_1(3) + g(1,3)$ we conclude that $g(2,3) = 0$.

All three coins probability: $f_3(1,2,3) = 0.5\delta_{1h}\delta_{2h}$ such that from $f_3(1,2,3) = f_1(1) f_1(2) f_1(3) + f_1(1) g(2,3) + f_1(2) g(1,3) + f_1(3) g(1,2) + h(1,2,3)$ and all $g = 0$ it follows that $h(1,2,3) = 0$

(d) First two are honest but 3 is heads if 1 and 2 are heads: Assume symmetric probability $f_1(1) = f_1(2) = f_1(3) = 1/2$.

Coins 1 and 2: $f_2(1,2) = 1/4$ and from $f_2(1,2) = f_1(1) f_1(2) + g(1,2)$ such that $g(1,2) = 0$.

Coins 1 and 3: $f_2(1,3) = 1/4$ and from $f_2(1,3) = f_1(1) f_1(3) + g(1,3)$ such that $g(1,3) = 0$.

Coins 2 and 3: $f_2(2,3) = 1/4$ and from $f_2(2,3) = f_1(2) f_1(3) + g(2,3)$ such that $g(2,3) = 0$.

Coins 1, 2 and 3:

$$
\begin{aligned}
f_3(1,2,3) &= 0.25\delta_{1h}\delta_{2h}\delta_{3h} + 0.25\delta_{1h}\delta_{2t} + 0.25\delta_{1t}\delta_{2h} + 0.25\delta_{1t}\delta_{2t} \\
&= 0.125 + 0.125\delta_{1h}\delta_{2h}\delta_{3h} - 0.125\delta_{1h}\delta_{2h}\delta_{3t}
\end{aligned}
$$

because the sum over all combinations is a likelihood of 1/8 except for the outcome off hhh which has the probability 1/4 and the outcome hht which has the probability of 0. From

$$
\begin{aligned}
f_3(1,2,3) &= f_1(1) f_1(2) f_1(3) + f_1(1) g(2,3) + f_1(2) g(1,3) + f_1(3) g(1,2) + h(1,2,3) \\
h(1,2,3) &= f_3(1,2,3) - f_1(1) f_1(2) f_1(3) \\
&= 0.25\delta_{1h}\delta_{2h}\delta_{3h} + 0.25\delta_{1h}\delta_{2t} + 0.25\delta_{1t}\delta_{2h} + 0.25\delta_{1t}\delta_{2t} - 0.125 \\
&= 0.125\left(\delta_{1h}\delta_{2h}\delta_{3h} - \delta_{1h}\delta_{2h}\delta_{3t}\right)
\end{aligned}
$$

It turns out that h is exactly the correction that we applied already in f_3 to account for the outcomes hhh and hht.

8. Fourier Transform:

Show that the Fourier Transforms of the electrostatic potential $\varphi(\mathbf{r}) = q/\left(4\pi\epsilon_0 |\mathbf{r}|\right)$ and of the electrostatic force $\mathbf{a_{12}}(\mathbf{r}) = e^2\mathbf{r}/\left(4\pi\epsilon_0 m_e |\mathbf{r}|^3\right)$ are $\tilde{\varphi}(\mathbf{k}) = e^2/(\epsilon_0 (2\pi)^{3/2} |\mathbf{k}|^2)$ and $\mathbf{a_{12}}(\mathbf{k}) = -i\mathbf{k}\tilde{\varphi}(\mathbf{k})/m_e$ respectively. (Hint: To obtain a derive the result consider first the transform of $\varphi'(\mathbf{r}) = \varphi(\mathbf{r})\exp(-\nu r)$ with $\nu > 0$ and after you obtained the result take the limit of $\nu \to 0$).

Solution:

i) Fourier transform of $\varphi_\nu(\mathbf{x}) = e^2 \exp(-\nu r)/\left(4\pi\epsilon_0 |\mathbf{x}|\right)$

$$\begin{aligned}\tilde{\varphi}_\nu(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3x\, \varphi(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}) \exp(-\nu r) \\ &= \frac{e^2}{4\pi\epsilon_0 (2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3x\, \frac{1}{|\mathbf{x}|} \exp(-i\mathbf{k}\cdot\mathbf{x}) \exp(-\nu r)\end{aligned}$$

with $\mathbf{k} = k\mathbf{e}_z$ and

$$x = r\sin\vartheta\cos\varphi, \qquad y = r\sin\vartheta\sin\varphi, \qquad z = r\cos\vartheta.$$

we obtain

$$\begin{aligned}\tilde{\varphi}_\nu(\mathbf{k}) &= \frac{e^2}{4\pi\epsilon_0 (2\pi)^{3/2}} \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \int_0^{\infty} dr\, r^2 \sin\vartheta \frac{\exp(-ikr\cos\vartheta)}{r} \exp(-\nu r) \\ &= \frac{e^2}{2\epsilon_0 (2\pi)^{3/2}} \int_0^{\infty} dr \int_0^{\pi} d\vartheta\, r\sin\vartheta \exp(-ikr\cos\vartheta)\exp(-\nu r) \\ &= \frac{e^2}{2\epsilon_0 (2\pi)^{3/2}} \int_0^{\infty} dr\, \frac{1}{ik} \exp(-ikr\cos\vartheta)\Big|_0^{\pi} \exp(-\nu r) \\ &= \frac{e^2}{2i\epsilon_0 (2\pi)^{3/2} k} \int_0^{\infty} dr\, \left(\exp(-\nu r + ikr) - \exp(-\nu r - ikr)\right) \\ &= \frac{e^2}{2i\epsilon_0 (2\pi)^{3/2} k} \left[\frac{\exp(-\nu r + ikr)}{-\nu + ik} - \frac{\exp(-\nu r - ikr)}{-\nu - ik}\right]_{r=0}^{r=\infty} \\ &= \frac{e^2}{2i\epsilon_0 (2\pi)^{3/2} k} \left[\frac{1}{\nu - ik} - \frac{1}{\nu + ik}\right] = \frac{e^2}{2i\epsilon_0 (2\pi)^{3/2} k} \left[\frac{\nu + ik}{\nu^2 + k^2} - \frac{\nu - ik}{\nu^2 + k^2}\right] \\ &= \frac{e^2}{\epsilon_0 (2\pi)^{3/2}} \frac{1}{\nu^2 + k^2}\end{aligned}$$

And using the limit of $\nu \to 0$ we obtain

$$\tilde{\varphi}(\mathbf{k}) = \lim_{\nu\to 0} \tilde{\varphi}_\nu(\mathbf{k}) = -\frac{e^2}{\epsilon_0 (2\pi)^{3/2} k^2}$$

ii) x component of the acceleration

$$\begin{aligned}\tilde{a}_x(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \frac{e^2}{4\pi\epsilon_0 m_e} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3x\, \frac{x}{|\mathbf{x}|^3} \exp(-i\mathbf{k}\cdot\mathbf{x}) \\ &= \frac{1}{(2\pi)^{3/2}} \frac{e^2}{4\pi\epsilon_0 m_e} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3x\, \frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{x}|}\right) \exp(-ik_x x)\exp(-ik_y y - ik_z z)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{(2\pi)^{3/2}}\frac{e^2}{4\pi\epsilon_0 m_e}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dydz\,\left[\frac{1}{|\mathbf{x}|}\exp\left(-ik_x\cdot x\right)\right]_{x=-\infty}^{x=\infty}\right. \\
&\qquad\qquad\left. +ik_x\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3x\,\frac{1}{|\mathbf{x}|}\exp\left(-i\mathbf{k}\cdot\mathbf{x}\right)\right\} \\
&= -ik_x\frac{\tilde{\varphi}\left(\mathbf{k}\right)}{m_e}
\end{aligned}
$$

$$
=> \qquad \tilde{\mathbf{a}}\left(\mathbf{k}\right) = -i\mathbf{k}\frac{\tilde{\varphi}\left(\mathbf{k}\right)}{m_e}
$$

9. Lenard Balescu Equation:

Demonstrate that a Maxwellian ($f(\mathbf{v}) = const \cdot \exp(mv^2/2T)$) is an exact time independent solution of the Lenard Balescu (LB) equation.

Lenard-Balescu equation

$$\begin{aligned}
\frac{\partial f(\mathbf{v},t)}{\partial t} &= -\frac{8\pi^4 n_0}{m_e^2}\nabla_{\mathbf{v}} \cdot \int d^3k d^3v' \mathbf{k}\mathbf{k} \cdot \frac{\varphi^2(\mathbf{k})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \\
&\qquad \delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))\left[(\nabla_{\mathbf{v}'} - \nabla_{\mathbf{v}}) f(\mathbf{v}) f(\mathbf{v}')\right]
\end{aligned}$$

with the dielectric function

$$\begin{aligned}
\epsilon(\mathbf{k},\omega) &= 1 + \frac{\omega_e^2}{\mathbf{k}^2}\int d^3v \frac{\mathbf{k}\cdot\nabla_{\mathbf{v}} f(\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v}} \\
\varphi(\mathbf{k}) &= \frac{e^2}{8\pi^3\epsilon_0\mathbf{k}^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f(\mathbf{v},t)}{\partial t} &= -\frac{8\pi^4 n_0}{m_e^2}\nabla_{\mathbf{v}} \cdot \int d^3k d^3v' \mathbf{k}\mathbf{k} \cdot \frac{\varphi^2(\mathbf{k})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \\
&\qquad \delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))\left[(\mathbf{v}' - \mathbf{v}) f(\mathbf{v}) f(\mathbf{v}')\right] \\
&= -\frac{8\pi^4 n_0}{m_e^2}\nabla_{\mathbf{v}} \cdot \int d^3k d^3v' \mathbf{k} \cdot \frac{\varphi^2(\mathbf{k})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \\
&\qquad \delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))\left[\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v}) f(\mathbf{v}) f(\mathbf{v}')\right] \\
&= 0
\end{aligned}$$

because we can choose the k_x direction parallel to $\mathbf{v} - \mathbf{v}'$ such that $\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v}) = 0$ because of the delta function $\delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))$.