

10. Ion-acoustic Waves:

The dispersion relation

$$1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} du \frac{d_u g(u)}{u - \omega/k} = 0$$

has a branch describing ion-acoustic waves as well as one describing Langmuir waves. Consider $g(u)$ with electron and ion Maxwellian contributions as introduced in class with $T_i \ll T_e$ and look for a wave with a phase speed such that

$$v_i \ll \omega/k \ll v_e$$

(v_i, v_e are ion and electron thermal velocities). Expand the denominator (for the ion contribution) in the above relation as we did for the Langmuir waves. For the electron contribution approximate the integral by assuming $\omega/k \ll u$. Then solve the dispersion relation $\epsilon(k, \omega) = 0$ to find $\omega(k)$ for ion-acoustic waves.

Solution 1:

From class we define $g(u) = g_e(u) + g_i(u)$ through:

$$g_e(u) \equiv \frac{1}{(2\pi)^{1/2} u_e} \exp\left[-\frac{u^2}{2u_e^2}\right]$$

$$g_i(u) \equiv \frac{m_e}{m_i} \frac{1}{(2\pi)^{1/2} u_i} \exp\left[-\frac{u^2}{2u_i^2}\right]$$

(u_e and u_i are the thermal velocities defined in class, $u_s^2 = k_B T_s / m_s$). Using the real part of the dielectric function

$$1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} du \frac{dg_e(u)}{(u - \omega/k)} - \frac{\omega_{pi}^2}{k^2} \int_{-\infty}^{\infty} du \frac{dg_i(u)}{(u - \omega/k)} = 0 \quad (1)$$

Expansion of the ion contribution for $u_i \ll \omega/k$ is the same as for Langmuir waves:

$$\begin{aligned} \int_{-\infty}^{\infty} du \frac{dg_i(u)}{(u - \omega/k)} &= \int_{-\infty}^{\infty} du \frac{g(u)}{(\omega/k - u)^2} \\ &= \frac{k^2}{\omega^2} \int_{-\infty}^{\infty} du g_i(u) \left(1 + \frac{2uk}{\omega} + \frac{3u^2 k^2}{\omega^2}\right) \\ &= \frac{m_e}{m_i} \frac{k^2}{\omega^2} \left[1 + 3u_i^2 \frac{k^2}{\omega^2}\right] \end{aligned}$$

Expansion of the electron contribution for $\omega/k \ll u_e$:

$$\begin{aligned} I_e &= \int_{-\infty}^{\infty} du \frac{d_u g_e(u)}{(u - \omega/k)} \\ &= -\frac{1}{(2\pi)^{1/2} u_e^3} \int_{-\infty}^{\infty} du \frac{u}{(u - \omega/k)} \exp\left[-\frac{u^2}{2u_e^2}\right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{2^{1/2}u_e}{(2\pi)^{1/2}u_e^3} \int_{-\infty}^{\infty} dz \frac{z+\lambda}{z} \exp[-z^2 - 2\lambda z - \lambda^2] \\
&= -\frac{\exp[-\lambda^2]}{\pi^{1/2}u_e^2} \int_{-\infty}^{\infty} dz \exp(-z^2) \frac{z+\lambda}{z} \left(1 - 2\lambda z + \frac{1}{2}(2\lambda z)^2 - \frac{1}{6}(2\lambda z)^3 + \dots\right) \\
&= -\frac{\exp[-\lambda^2]}{\pi^{1/2}u_e^2} \int_{-\infty}^{\infty} dz \exp(-z^2) \frac{1}{z} \left[\lambda + (1 - 2\lambda^2)z - 2\lambda(1 - \lambda^2)z^2 + 2\lambda^2\left(1 - \frac{2}{3}\lambda^2\right)z^3 + \dots\right] \\
&= -\frac{\exp[-\lambda^2]}{\pi^{1/2}u_e^2} \int_{-\infty}^{\infty} dz \exp(-z^2) \left[\left(1 - 2\lambda^2\right) + 2\lambda^2\left(1 - \frac{2}{3}\lambda^2\right)z^2 + \dots\right] \\
&= -\frac{1}{\pi^{1/2}u_e^2} \left[1 - \lambda^2 + \frac{1}{2}\lambda^4 - \dots\right] \left[\pi^{1/2}(1 - 2\lambda^2) + \frac{\pi^{1/2}}{2}2\lambda^2\left(1 - \frac{2}{3}\lambda^2\right) + \dots\right] \\
&= -\frac{1}{u_e^2} [1 - 2\lambda^2 + \dots] = -\frac{1}{u_e^2} \left[1 - \frac{\omega^2}{u_e^2 k^2} + \dots\right]
\end{aligned}$$

where we used the substitution $z = (u - \omega/k) / (2^{1/2}u_e)$ or $u = 2^{1/2}u_e(z + \lambda)$ and $\lambda = (\omega/k) / (2^{1/2}u_e)$ and $\lambda \ll 1$. Substituting these results into the dispersion relation yields for the real part of the dielectric function

$$1 + \frac{\omega_{pe}^2}{k^2} \frac{1}{u_e^2} \left[1 - \frac{\omega^2}{u_e^2 k^2} + \dots\right] - \frac{\omega_{pe}^2}{k^2} \frac{m_e}{m_i} \frac{k^2}{\omega^2} \left[1 + 3u_i^2 \frac{k^2}{\omega^2}\right] = 0$$

In the following we will make use of some basic relations

$$\begin{aligned}
\lambda_D^2 &= \lambda_{De}^2 = u_e^2 / \omega_{pe}^2 & \lambda_{Di}^2 &= u_i^2 / \omega_{pi}^2 \\
\lambda_{Ds}^2 &= \frac{\epsilon_0 k_B T_s}{n_0 e^2} & \omega_{ps}^2 &= \frac{n_0 e^2}{m_s \epsilon_0}
\end{aligned}$$

We will also measure or normalize k and ω using the definitions (not necessary but provides better insight into the expansions and limitation):

$$\tilde{k} = k \lambda_D \quad \tilde{\omega} = \omega / \omega_{pi}$$

With these definitions the condition for the expansion of the dispersion relations are

$$\delta = \frac{\omega^2}{k^2 u_e^2} = \frac{\tilde{\omega}^2 m_e}{\tilde{k}^2 m_i} \ll 1 \quad \frac{\omega^2}{k^2 u_i^2} = \frac{\tilde{\omega}^2 T_e}{\tilde{k}^2 T_i} \gg 1$$

in other words if $\tilde{\omega}^2 / \tilde{k}^2 = O(1)$ the expansion is valid only for $T_e / T_i \gg 1$. Substitution into the dispersion relation:

$$\begin{aligned}
1 + \frac{1}{\tilde{k}^2} [1 - \delta + \dots] - \frac{1}{\tilde{\omega}^2} \left[1 + 3 \frac{T_i}{T_e} \frac{\tilde{k}^2}{\tilde{\omega}^2}\right] &= 0 \\
&or \\
\left(1 + \frac{1 - \delta}{\tilde{k}^2}\right) \tilde{\omega}^4 - \tilde{\omega}^2 - 3 \frac{T_i}{T_e} \tilde{k}^2 &= 0 \\
\tilde{\omega}^4 - \frac{1}{1 + (1 - \delta) / \tilde{k}^2} \tilde{\omega}^2 - \frac{3}{1 + (1 - \delta) / \tilde{k}^2} \frac{T_i}{T_e} \tilde{k}^2 &= 0
\end{aligned}$$

This structure demonstrates that δ represents only a small correction to the dispersion relation and can be neglected. Solving for ω^2 leads to

$$\begin{aligned}\tilde{\omega}^2 &= \frac{1}{2} \frac{1}{1+1/\tilde{k}^2} \pm \frac{1}{2} \frac{1}{1+1/\tilde{k}^2} \left[1 + 12 \frac{1+\tilde{k}^2}{\tilde{k}^2} \tilde{k}^2 \frac{T_i}{T_e} \right]^{1/2} \\ &= \frac{1}{2} \frac{1}{1+1/\tilde{k}^2} \pm \frac{1}{2} \frac{1}{1+1/\tilde{k}^2} \left[1 + 6 (1+\tilde{k}^2) \frac{T_i}{T_e} \right] \\ &= \frac{\tilde{k}^2}{1+\tilde{k}^2} \left[1 + 3 (1+\tilde{k}^2) \frac{T_i}{T_e} \right] \quad \text{for} \quad 1+\tilde{k}^2 \ll \frac{T_e}{T_i}\end{aligned}$$

Solution 2: (alternate derivation):

$$\begin{aligned}\varepsilon(k, p) &= 1 - \frac{1}{2k^2 \lambda_{De}^2} \frac{dZ(\zeta_e)}{d\zeta_e} - \frac{1}{2k^2 \lambda_{Di}^2} \frac{dZ(\zeta_i)}{d\zeta_i} \\ \zeta_s &= \frac{\omega}{k} \frac{1}{2^{1/2} u_s} \quad \text{and} \quad Z(\zeta) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{\exp(-z^2) dz}{z - \zeta}\end{aligned}$$

Expansion for $u_i \ll \omega/k \ll u_e$ implies $\zeta_e \ll 1$ and $\zeta_i \gg 1$. Using the expansions from the manuscript (Appendix A.4) for $\zeta_e \ll 1$

$$\begin{aligned}Z(\zeta_e) &= i\pi^{1/2} \exp(-\zeta_e^2) - 2\zeta_e \left(1 - \frac{2\zeta_e^2}{3} + \frac{4\zeta_e^4}{15} - \dots \right) \\ \frac{dZ(\zeta_e)}{d\zeta_e} &= -2i\pi^{1/2} \zeta_e \exp(-\zeta_e^2) - 2 \left(1 - 2\zeta_e^2 + \frac{4\zeta_e^4}{3} - \dots \right)\end{aligned}$$

and for $\zeta_i \gg 1$

$$\begin{aligned}Z(\zeta_i) &= -\frac{1}{\zeta_i} \left(1 + \frac{1}{2\zeta_i^2} + \frac{3}{4\zeta_i^4} + \dots \right) + \sigma i\pi^{1/2} \exp(-\zeta_i^2) \\ \frac{dZ(\zeta_i)}{d\zeta_i} &= \frac{1}{\zeta_i^2} \left(1 + \frac{3}{2\zeta_i^2} + \frac{15}{4\zeta_i^4} + \dots \right) - 2\sigma i\pi^{1/2} \zeta_i \exp(-\zeta_i^2)\end{aligned}$$

Here $\sigma = 1$ corresponds to $\text{Im}\zeta_i = 0$. Substitution of the real part into the dispersion relation:

$$\begin{aligned}\varepsilon_r(k, \omega) &= 1 + \frac{1}{k^2 \lambda_{De}^2} (1 - 2\zeta_e^2 + \dots) - \frac{1}{2k^2 \lambda_{Di}^2} \frac{1}{\zeta_i^2} \left(1 + \frac{3}{2\zeta_i^2} + \dots \right) \\ 0 &= 1 + \frac{1}{k^2 \lambda_{De}^2} \left(1 - \frac{\omega^2}{k^2 u_e^2} \right) - \frac{1}{k^2 \lambda_{Di}^2} \frac{k^2 u_i^2}{\omega^2} \left(1 + 3 \frac{k^2 u_i^2}{\omega^2} \right) \\ &= 1 + \frac{1}{k^2 \lambda_{De}^2} - \frac{\omega_{pi}^2}{\omega^2} \left(1 + 3 \frac{k^2 u_i^2}{\omega^2} \right)\end{aligned}$$

This is the same expression that has been derived explicitly in the first solution using the expansion of the integrant. The remainder of the solution is the same. The use of the plasma dispersion function avoids this explicit expansion. For detailed discussion of the short and long wavelength behaviour see problem 13.

11. Contour Integration:

(a) Show that the integral $\oint 1/z dz$ along a closed contour around the origin yields the result $2\pi i$ (choose a closed circle with radius R and represent z in terms of the cylindrical coordinates r and φ).

(b) Prove the residue theorem by generalizing the result from (a) to the case $\oint f(z)/z dz$ and assume the $f(z)$ is analytic, i.e., that you can expand it in a power series of z in the vicinity of the origin. How can it be generalized to the case of a simple pole at $z = a$ instead of the origin?

(c) Find the principal value of $\int_{-\infty}^{\infty} \cos(mx)/(x - a) dx$ with a real, $m > 0$.

Solution:

(a) With $z = R \exp(i\varphi)$ and $dz = iR \exp(i\varphi) d\varphi$ the integral over a closed circle is

$$\oint \frac{dz}{z} = \oint \frac{iR \exp(i\varphi) d\varphi}{R \exp(i\varphi)} = i \oint d\varphi = 2\pi i$$

(b) If f is analytic at the origin it can be expanded into a series expansion in a vicinity of the origin $f(z) = \sum_{l=0}^{\infty} a_l z^l$. the integrals of the lowest order term in the expansion is

$$\oint \frac{a_0 dz}{z} = a_0 \oint \frac{iR \exp(i\varphi) d\varphi}{R \exp(i\varphi)} = ia_0 \oint d\varphi = 2\pi ia_0$$

all other terms in the expansion yield integrals of form (with $l \geq 1$):

$$\begin{aligned} \oint \frac{a_l z^l dz}{z} &= ia_0 \oint \frac{R^{l+1} \exp[i(l+1)\varphi]}{R \exp(i\varphi)} d\varphi \\ &= ia_0 R^l \oint \exp(il\varphi) d\varphi \\ &= ia_0 R^l \int_0^{2\pi} [\cos(il\varphi) + i \sin(il\varphi)] d\varphi = 0 \end{aligned}$$

because the real and imaginary parts of the integral are over 1 periods of the sine and cosine functions. Therefore

$$\oint \frac{f(z) dz}{z} = 2\pi ia_0$$

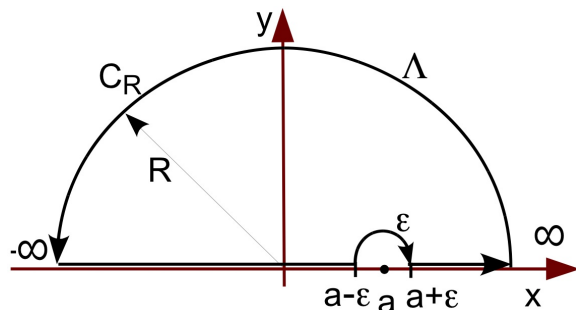
which is equal to the residue of $f(z)$ at $z = 0$. With a pole at $z = a$ we can use a coordinate transformation using $\tilde{z} = z - a$ such that the pole is at $\tilde{z} = 0$ for function $f(\tilde{z})$ and we have the expansion $f(\tilde{z}) = \sum_{l=0}^{\infty} a_l \tilde{z}^l$. This shows

$$\oint \frac{f(\tilde{z}) d\tilde{z}}{\tilde{z}} = \oint \frac{f(z) dz}{z - a} = 2\pi ia_0$$

(c) Principal value of

$$P \int_{-\infty}^{\infty} \frac{\cos(mx)}{x - a} dx$$

Consider closed contour integral along Λ with the substitution $z = a + R \exp(i\varphi)$



$$\begin{aligned}
I_{\Lambda} &= \lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \oint_{\Lambda} \frac{\exp(imz)}{z-a} dz = 0 \\
&= \lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \left[\int_{-R}^{a-\varepsilon} + \int_{\varepsilon} + \int_{a+\varepsilon}^R + \int_{C_R} \right] \\
&= \lim_{\varepsilon \rightarrow 0, R \rightarrow \infty} \left[\int_{-R}^{a-\varepsilon} + \int_{a+\varepsilon}^R \right] + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon} \frac{\exp(imz)}{z-a} dz + \lim_{R \rightarrow \infty} \int_{C_R} \frac{\exp(imz)}{z-a} dz \\
&= P \int_{-\infty}^{\infty} \frac{\exp(mx)}{x-a} dx + i \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon} \exp[im(a + \varepsilon \exp(i\varphi))] d\varphi \\
&\quad + i \lim_{R \rightarrow \infty} \int_{C_R} \exp[im(a + R \exp(i\varphi))] d\varphi \\
&= P \int_{-\infty}^{\infty} \frac{\exp(mx)}{x-a} dx + i \exp(ima) \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon} \exp[i\varepsilon m \exp(i\varphi)] d\varphi \\
&\quad + i \exp(ima) \lim_{R \rightarrow \infty} \int_{C_R} \exp(imR \cos \varphi) \exp(-mR \sin \varphi) d\varphi \\
&= P \int_{-\infty}^{\infty} \frac{\exp(mx)}{x-a} dx - i\pi \exp(ima) = 0
\end{aligned}$$

Considering only the real part of I_{Λ} leads to

$$P \int_{-\infty}^{\infty} \frac{\cos(mx)}{x-a} dx = -\pi \sin(ma)$$

12. Ion Acoustic Wave Properties and Damping:

The real part of the dielectric function for ion acoustic waves is

$$\epsilon_r = 1 + \frac{1}{k^2 \lambda_e^2} - \frac{m_e \omega_e^2}{m_i \omega^2} \left(1 + 3 \frac{v_i^2 k^2}{\omega^2} \right)$$

with $\lambda_e =$ electron Debye length, $\omega_e =$ electron plasma frequency, and $v_i = \sqrt{k_B T_i / m_i} =$ ion thermal velocity. Determine the damping rate for Landau damping. Discuss the dispersion relation (ω_r and ω_i) as a function of wave number under the constraints $\omega/k \ll v_e$ and $\omega/k \gg v_i$. For what conditions (wave number) becomes Landau damping dominant. Is this condition consistent with the constraints?

Solution: The damping/ growth rate is given by

$$\gamma = - \frac{\epsilon_i(k, \omega_r)}{\partial \epsilon_r(k, \omega_r) / \partial \omega_r}$$

For the imaginary part of the dielectric function (see solution 2 in problem 10) we obtain (assuming $\text{Im} \zeta_i = 0$):

$$\begin{aligned} \epsilon_i(k, \omega) &= \frac{i\pi}{k^2 \lambda_{De}^2} \zeta_e \exp(-\zeta_e^2) + \frac{\pi^{1/2}}{k^2 \lambda_{Di}^2} \zeta_i \exp(-\zeta_i^2) \\ &= \frac{(\pi/2)^{1/2}}{k^2 \lambda_{De}^2} \frac{\omega}{k} \frac{1}{u_e} \exp\left(-\frac{\omega^2}{2k^2 u_e^2}\right) + \frac{(\pi/2)^{1/2}}{k^2 \lambda_{Di}^2} \frac{\omega}{k} \frac{1}{u_i} \exp\left(-\frac{\omega^2}{2k^2 u_i^2}\right) \\ &\approx \frac{(\pi/2)^{1/2}}{k^3 \lambda_{De}^3} \frac{\omega}{\omega_{pe}} + \frac{(\pi/2)^{1/2}}{k^3 \lambda_{Di}^3} \frac{\omega}{\omega_{pi}} \exp\left(-\frac{\omega^2}{2k^2 u_i^2}\right) \\ &= \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{k^3 \lambda_{De}^3} \frac{\omega}{\omega_{pi}} \left[\frac{\omega_{pi}}{\omega_{pe}} + \frac{\lambda_{De}^3}{\lambda_{Di}^3} \exp\left(-\frac{\omega^2}{2k^2 u_i^2}\right) \right] \end{aligned}$$

and

$$\frac{\partial \epsilon_r(k, \omega_r)}{\partial \omega_r} = 2 \frac{\omega_{pi}^2}{\omega^3} \left(1 + 12 \frac{k^2 u_i^2}{\omega^2} \right)$$

Substitution into the expression for the damping rate:

$$\gamma = - \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{2k^3 \lambda_{De}^3} \frac{\omega^4}{\omega_{pi}^3} \left(1 + 12 \frac{k^2 u_i^2}{\omega^2} \right)^{-1} \left[\frac{\omega_{pi}}{\omega_{pe}} + \frac{\lambda_{De}^3}{\lambda_{Di}^3} \exp\left(-\frac{\omega^2}{2k^2 u_i^2}\right) \right]$$

To examine the damping rate we apply the same normalization as in Problem 11, use the normalized $\tilde{\gamma} = \gamma / \omega_{pi}$:

$$\tilde{\gamma} = - \left(\frac{\pi}{2}\right)^{1/2} \frac{\tilde{\omega}^4}{2\tilde{k}^3} \left(1 + 12 \frac{\tilde{k}^2 T_i}{\tilde{\omega}^2 T_e} \right)^{-1} \left[\left(\frac{m_e}{m_i}\right)^{1/2} + \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{\tilde{\omega}^2 T_e}{\tilde{k}^2 2T_i}\right) \right]$$

and for the general solution we need to use the solution of the real part of the dispersion relation.

$$\tilde{\omega}^2 = \frac{\tilde{k}^2}{1 + \tilde{k}^2} \left[1 + 3 \left(1 + \tilde{k}^2 \right) \frac{T_i}{T_e} \right] \quad \text{for} \quad 1 + \tilde{k}^2 \ll \frac{T_e}{T_i}$$

Let us now consider the short and long wavelength limits of the wave frequency and damping rate. In these limits the resulting expression simplify considerably.

(i) In the short wavelength limit $\tilde{k}^2 > O(1)$ this is approximately:

$$\tilde{\omega}^2 = \left[1 + 3 \left(1 + \tilde{k}^2 \right) \frac{T_i}{T_e} \right] \approx 1$$

Validity:

$$\frac{\omega^2}{k^2 u_i^2} \approx \frac{\omega_{pi}^2}{k^2 u_i^2} \frac{\lambda_D^2}{\lambda_{Di}^2} = \frac{1}{\tilde{k}^2} \frac{\lambda_D^2}{\lambda_{Di}^2} = \frac{1}{\tilde{k}^2} \frac{T_e}{T_i} \gg 1$$

Damping rate:

$$\begin{aligned} \tilde{\gamma} &= - \left(\frac{\pi}{8} \right)^{1/2} \frac{1}{\tilde{k}^3} \left(1 + 12 \tilde{k}^2 \frac{T_i}{T_e} \right)^{-1} \left[\left(\frac{m_e}{m_i} \right)^{1/2} + \left(\frac{T_e}{T_i} \right)^{3/2} \exp \left(- \frac{1}{\tilde{k}^2} \frac{T_e}{2T_i} \right) \right] \\ &\approx - \left(\frac{\pi}{8} \right)^{1/2} \frac{1}{\tilde{k}^3} \left[\left(\frac{m_e}{m_i} \right)^{1/2} + \left(\frac{T_e}{T_i} \right)^{3/2} \exp \left(- \frac{1}{\tilde{k}^2} \frac{T_e}{2T_i} \right) \right] \end{aligned}$$

(ii) In the long wavelength limit $\tilde{k}^2 \ll 1$ this is approximately

$$\tilde{\omega}^2 = \tilde{k}^2 \left[1 + 3 \frac{T_i}{T_e} \right] \quad \text{or} \quad \frac{\omega^2}{k^2} = c_{ia}^2 \left[1 + 3 \frac{T_i}{T_e} \right] = \frac{k_B T_e}{m_i} \left[1 + 3 \frac{T_i}{T_e} \right]$$

Validity:

$$\frac{\omega^2}{k^2 u_i^2} = \frac{\tilde{\omega}^2 T_e}{\tilde{k}^2 T_i} = \frac{T_e}{T_i} \left[1 + 3 \frac{T_i}{T_e} \right] \gg 1 \quad \text{for} \quad \frac{T_e}{T_i} \gg 1$$

Damping rate:

$$\begin{aligned} \tilde{\gamma} &= - \left(\frac{\pi}{8} \right)^{1/2} \tilde{k} \left[1 + 3 \frac{T_i}{T_e} \right]^2 \left(1 + \frac{12}{T_e/T_i + 3} \right)^{-1} \left[\left(\frac{m_e}{m_i} \right)^{1/2} + \left(\frac{T_e}{T_i} \right)^{3/2} \exp \left(- \frac{T_e}{2T_i} - \frac{3}{2} \right) \right] \\ &\approx - \left(\frac{\pi}{8} \right)^{1/2} \tilde{k} \left[\left(\frac{m_e}{m_i} \right)^{1/2} + \left(\frac{T_e}{T_i} \right)^{3/2} \exp \left(- \frac{T_e}{2T_i} - \frac{3}{2} \right) \right] \end{aligned}$$

such that damping is small compared to the long wave length limit and decreases further linearly with decreasing k .

Used basic relations (from Problem 11):

$$\begin{aligned} \lambda_D^2 &= \lambda_{De}^2 = u_e^2 / \omega_{pe}^2 & \lambda_{Di}^2 &= u_i^2 / \omega_{pi}^2 \\ \lambda_{Ds}^2 &= \frac{\epsilon_0 k_B T_s}{n_0 e^2} & \omega_{ps}^2 &= \frac{n_0 e^2}{m_s \epsilon_0} \end{aligned}$$

Normalization of k and ω :

$$\tilde{k} = k \lambda_D \quad \tilde{\omega} = \omega / \omega_{pi}$$

Condition for the expansion of the dispersion relations:

$$\delta = \frac{\omega^2}{k^2 u_e^2} = \frac{\tilde{\omega}^2 m_e}{\tilde{k}^2 m_i} \ll 1 \quad \frac{\omega^2}{k^2 u_i^2} = \frac{\tilde{\omega}^2 T_e}{\tilde{k}^2 T_i} \gg 1$$