**13. Electrostatic Dispersion relation:**

The dispersion relation for electrostatic waves

$$1 + \frac{\omega_e^2}{k^2} \int_{-\infty}^{\infty} du \frac{d_u g(u)}{\omega/k - u} = 0$$

must be solved using the Landau contour; alternatively, the integral can be evaluated for $\omega_i > 0$ and the result analytically continued to $\omega_i < 0$. Evaluate the dispersion relation and find the normal modes $\omega(k)$ for the following distributions $g(u)$.
(a) Cold plasma, $g(u) = \delta(u)$.
(b) Cold beam, $g(u) = \delta(u - u_0)$.
(c) Square distribution, $g(u) = (2c)^{-1}$ for $|u| < c$, $g(u) = 0$ for $|u| > c$, with $c$ real and positive.
(d) Cauchy distribution, $g(u) = (c/\pi)(u^2 + c^2)^{-1}$.

**Solution:**

(a) Cold plasma, $g(u) = \delta(u)$:

$$\begin{aligned}
\epsilon &= 1 + \frac{\omega_e^2}{k^2} \int_{-\infty}^{\infty} du \frac{d_u g(u)}{\omega/k - u} \\
0 &= 1 - \frac{\omega_e^2}{k^2} \int_{-\infty}^{\infty} du \frac{\delta(u)}{(\omega/k - u)^2} = 1 - \frac{\omega_e^2}{k^2} \frac{k^2}{\omega^2} \\
\text{or} \quad & \omega^2 = \omega_e^2
\end{aligned}$$

**(b)** Cold beam, $g(u) = \delta(u - u_0)$

$$\begin{aligned}
0 &= 1 - \frac{\omega_e^2}{k^2} \int_{-\infty}^{\infty} du \frac{\delta(u - u_0)}{(\omega/k - u)^2} = 1 - \frac{\omega_e^2}{k^2} \frac{k^2}{(\omega/k - u_0)^2} \\
\omega &= u_0 k \pm \omega_e
\end{aligned}$$

**(c)** Square distribution, $g(u) = (2c)^{-1}$ for $|u| < c$, $g(u) = 0$ for $|u| > c$:

Derivative of the square distribution: $d_u g(u) = (2c)^{-1} [\delta(u + c) - \delta(u - c)]$

$$\begin{aligned}
\epsilon &= 1 + \frac{\omega_e^2}{k^2} \int_{-\infty}^{\infty} du \frac{d_u g(u)}{\omega/k - u} \\
0 &= 1 + \frac{\omega_e^2}{k^2} \frac{1}{2c} \int_{-\infty}^{\infty} du \frac{\delta(u + c) - \delta(u - c)}{\omega/k - u} = 1 + \frac{\omega_e^2}{k^2} \frac{1}{2c} \left[ \frac{1}{\omega/k + c} - \frac{1}{\omega/k - c} \right] \\
&= 1 + \frac{\omega_e^2}{k^2} \frac{1}{2c} \frac{\omega/k - c - (\omega/k + c)}{\omega^2/k^2 - c^2} = 1 - \frac{\omega_e^2}{k^2} \frac{1}{2c} \frac{2c}{\omega^2/k^2 - c^2} \\
&= 1 - \frac{\omega_e^2}{k^2} \frac{k^2}{\omega^2 - k^2 c^2} \\
\text{or} \quad & \omega^2 = c^2 k^2 + \omega_e^2
\end{aligned}$$

**(d)** Cauchy distribution, $g(u) = (c/\pi)\left(u^2+c^2\right)^{-1}$. Derivative $d_u g(u) = -2u\,(c/\pi)\left(u^2+c^2\right)^{-2}$

$$
\begin{aligned}
\epsilon &= 1 - 2\frac{\omega_e^2}{k^2}\frac{c}{\pi}\int_{-\infty}^{\infty} du \frac{u}{(\omega/k - u)\left(u^2+c^2\right)^2} \\
0 &= 1 - \frac{\omega_e^2}{k^2}\frac{c}{\pi}\int_{-\infty}^{\infty} du \frac{1}{(\omega/k - u)^2\left(u^2+c^2\right)}
\end{aligned}
$$

**Solution 1:** Evaluate $\epsilon$ through contour integral over the upper half of the complex $u$ plane (note that integrant $\to\infty$ for $R\to\infty$) using the residue theorem with 2nd order poles at $u=\omega/k$ on the real axis and 1rst order pole at $u=\pm ic$. We abbreviate $v=\omega/k$:

$$
\begin{aligned}
I_0 &= \int_{-\infty}^{\infty} du \frac{1}{(v-u)^2\left(u^2+c^2\right)} = \int_D du \frac{1}{(v-u)^2\left(u^2+c^2\right)} \\
&= \pi i \mathrm{Res}\left(\frac{1}{(v-u)^2\left(u^2+c^2\right)}, v\right) + 2\pi i \mathrm{Res}\left(\frac{1}{(v-u)^2\left(u^2+c^2\right)}, iu\right) \\
&= \pi i \frac{1}{(2-1)!}\lim_{u\to v}\frac{d}{du}\left[(u-v)^2\frac{1}{(v-u)^2\left(u^2+c^2\right)}\right] \\
&\quad + 2\pi i \lim_{u\to ic}\left[(u-ic)\frac{1}{(v-u)^2\left(u^2+c^2\right)}\right] \\
&= \pi i\left[\frac{-2v}{\left(v^2+c^2\right)^2}\right] + 2\pi i\left[\frac{1}{(v-ic)^2 2ic}\right] \\
&= \frac{-2\pi i v}{\left(v^2+c^2\right)^2} + \frac{\pi}{(v-ic)^2 c} = \frac{-2\pi i v}{\left(v+c^2\right)^2} + \frac{\pi (v+ic)^2}{\left(v^2+c^2\right)^2 c} \\
&= \frac{-2\pi i v}{\left(v+c^2\right)^2} + \frac{\pi\left(v^2+2ivc-c^2\right)}{\left(v^2+c^2\right)^2 c} = \frac{\pi}{c}\frac{v^2-c^2}{\left(v^2+c^2\right)^2 c}
\end{aligned}
$$

Solution of the dispersion relation:

$$
\begin{aligned}
0 &= 1 - \frac{\omega_e^2}{k^2}\frac{c}{\pi}\int_{-\infty}^{\infty} du \frac{1}{(v-u)^2\left(u^2+c^2\right)} du \frac{1}{(\omega/k-u)^2\left(u^2+c^2\right)} \\
&= 1 - \frac{\omega_e^2}{k^2}\frac{c}{\pi}\frac{\pi}{c}\frac{v^2-c^2}{\left(v^2+c^2\right)^2} = 1 - \frac{\omega_e^2}{k^2}\frac{v^2-c^2}{\left(v^2+c^2\right)^2} \\
\left(v^2+c^2\right)^2 &= v_e^2\left(v^2-c^2\right) \\
v^4 - 2v^2\left(\frac{1}{2}v_e^2 - c^2\right) + \left(c^2 - \frac{1}{2}v_e^2\right)^2 &= -\left(v_e^2+c^2\right)c^2 + \left(c^2 - \frac{1}{2}v_e^2\right)^2 \\
&= \frac{1}{4}v_e^4\left(1 - 8c^2/v_e^2\right) \\
(\omega/k)^2 &= \frac{1}{2}(\omega_e/k)^2 - c^2 \pm \frac{1}{2}(\omega_e/k)^2\sqrt{1 - 8c^2/(\omega_e/k)^2}
\end{aligned}
$$

With $\lambda_D^2 = u_e^2/\omega_{pe}^2$

**Solution 2:** Direct evaluation of the integral in the dielectric function: Decompose the integrant in $I_0$ into basic rational fractions which can easily be integrated (note we can use the same approach for $I_0$):

$$I_0 = \int h_0 du = \int du \frac{u}{(v-u)^2 (u^2+c^2)}$$

with:

$$\begin{aligned}
h_0 &= \frac{1}{(v-u)^2 (u^2+c^2)} = \frac{A}{(v-u)} + \frac{B}{(v-u)^2} + \frac{C+Du}{(u^2+c^2)} \\
&= \frac{A(v-u)(u^2+c^2) + B(u^2+c^2) + (C+Du)(v-u)^2}{(v-u)^2 (u^2+c^2)} \\
&= \frac{(-A+D)u^3 + (Av+B+C-2Dv)u^2 + (-Ac^2-2Cv+Dv^2)u + (Avc^2+Bc^2+Cv^2)}{(\omega/k-u)^2 (u^2+c^2)}
\end{aligned}$$

Coefficients:

$$\begin{aligned}
-A+D &= 0 \\
Av+B+C-2Dv &= 0 \\
-Ac^2-2Cv+Dv^2 &= 0 \\
Avc^2+Bc^2+Cv^2 &= 1
\end{aligned}$$

Elimination of $D$:

$$\begin{aligned}
-Av+B+C &= 0 \\
A\left(v^2-c^2\right) - 2Cv &= 0 \\
Avc^2+Bc^2+Cv^2 &= 1
\end{aligned}$$

Elimination of $B$:

$$\begin{aligned}
A\left(v^2-c^2\right) - 2Cv &= 0 \\
2Avc^2 + C\left(v^2-c^2\right) &= 1
\end{aligned}$$

Elimination of $C$ and Solution:

$$\begin{aligned}
4Av^2c^2 + A\left(v^2-c^2\right)^2 &= A\left(v^2+c^2\right)^2 = 2v \\
A = D &= 2v/\left(v^2+c^2\right)^2 \\
C &= \left(v^2-c^2\right)/\left(v^2+c^2\right)^2 \\
B &= 1/\left(v^2+c^2\right)
\end{aligned}$$

Evaluation of the integral in the dielectric function:

$$\begin{aligned}
I_0 &= \int_{-\infty}^{\infty} du \frac{1}{(\omega/k-u)^2 (u^2+c^2)} = \int_{-\infty}^{\infty} du \left\{ \frac{A}{(v-u)} + \frac{B}{(v-u)^2} + \frac{C+Du}{u^2+c^2} \right\} \\
&= \int_{-\infty}^{\infty} \frac{Adu}{(v-u)} + \int_{-\infty}^{\infty} \frac{Bdu}{(v-u)^2} + \int_{-\infty}^{\infty} \frac{Du}{u^2+c^2} du + \int_{-\infty}^{\infty} \frac{C}{u^2+c^2} du \\
&= \int_{-\infty}^{\infty} \frac{Adu}{(v-u)} + \left[\frac{B}{(v-u)}\right]_{-\infty}^{\infty} + \frac{D}{2}\left[\ln\left(u^2+c^2\right)\right]_{-\infty}^{\infty} + \frac{C}{c}\left[\arctan\frac{u}{c}\right]_{-\infty}^{\infty} \\
&= C\frac{\pi}{c} = \frac{\pi}{c}\frac{v^2-c^2}{(v^2+c^2)^2}
\end{aligned}$$

This evaluation made use of the following integral and recursion (see problem 14 for the recursion):

$$\begin{aligned}
\int^x \left[1+x^2\right]^{-1} dx &= \arctan x \\
\int^x \left[1+x^2\right]^{-(n+1)} dx &= \frac{x}{2n}\left[1+x^2\right]^{-n} + \frac{2n-1}{2n}\int^x \left[1+x^2\right]^{-n} dx
\end{aligned}$$

**Solution 3:** Direct evaluation of the integral in the dielectric function: Decompose the integrant in $I_1$ into basic rational fractions which can easily be integrated (note we can use the same approach for $I_0$):

$$I_1 = \int h_1 du = \int du \frac{u}{(v-u)\left(u^2+c^2\right)^2}$$

with

$$\begin{aligned}
h_1 &= \frac{1}{(v-u)\left(u^2+c^2\right)^2} \\
&= \frac{A}{(v-u)} + \frac{B+Cu}{(u^2+c^2)} + \frac{D+Eu}{(u^2+c^2)^2} \\
&= \frac{A\left(u^2+c^2\right)^2 + (B+Cu)(v-u)\left(u^2+c^2\right) + (D+Eu)(v-u)}{(v-u)\left(u^2+c^2\right)^2} \\
&= \frac{(A-C)u^4 + (-B+Cv)u^3 + \left(2Ac^2+Bv-Cc^2-E\right)u^2}{(\omega/k-u)\left(u^2+c^2\right)^2} \\
&\qquad + \frac{\left(-Bc^2+Cvc^2-D+Ev\right)u + \left(Ac^4+Bvc^2+Dv\right)}{(\omega/k-u)\left(u^2+c^2\right)^2}
\end{aligned}$$

Coefficients:

$$\begin{aligned}
A - C &= 0 \\
-B + Cv &= 0 \\
2Ac^2 + Bv - Cc^2 - E &= 0 \\
-Bc^2 + Cvc^2 - D + Ev &= 1 \\
Ac^4 + Bvc^2 + Dv &= 0
\end{aligned}$$

Elimination of $A$ and $B$:

$$\begin{aligned}
C\left(c^2+v^2\right) - E &= 0 \\
-D + Ev &= 1 \\
Cc^2\left(c^2+v^2\right) + Dv &= 0
\end{aligned}$$

Elimination of $C$ and solution:

$$\begin{aligned}
-D + Ev &= 1 \\
Dv + Ec^2 &= 0 \\
E\left(c^2+v^2\right) &= v \\
E &= v/\left(c^2+v^2\right) \\
D &= -c^2/\left(c^2+v^2\right) \\
C &= v/\left(c^2+v^2\right)^2 \\
B &= v^2/\left(c^2+v^2\right)^2 \\
A &= v/\left(c^2+v^2\right)^2
\end{aligned}$$

Evaluation of the resulting integrals:

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} du \frac{u}{(\omega/k-u)\left(u^2+c^2\right)^2} = \int_{-\infty}^{\infty} du \left\{ \frac{A}{(v-u)} + \frac{B+Cu}{u^2+c^2} + \frac{D+Eu}{\left(u^2+c^2\right)^2} \right\} \\
&= \int_{-\infty}^{\infty} \frac{A du}{(v-u)} + \frac{B}{c}\left[\arctan\frac{u}{c}\right]_{-\infty}^{\infty} + \frac{C}{2}\left[\ln\left(u^2+c^2\right)\right]_{-\infty}^{\infty} - \frac{E}{2}\left[\frac{1}{u^2+c^2}\right]_{-\infty}^{\infty} + \frac{D}{c^3}\int_{-\infty}^{\infty} \frac{du/c}{\left(u^2/c^2+1\right)^2} \\
&= B\frac{\pi}{c} + \frac{D}{c^3}\left[\frac{u/c}{2\left(1+u^2/c^2\right)}\right]_{-\infty}^{\infty} + \frac{D}{2c^3}\int_{-\infty}^{\infty} \frac{du/c}{\left(u^2/c^2+1\right)} = \frac{\pi}{c}B + \frac{D}{2c^3}\left[\arctan\frac{u}{c}\right]_{-\infty}^{\infty} \\
&= \frac{\pi}{c}\left(B + \frac{D}{2c^2}\right) = \frac{\pi}{c}\left(\frac{v^2}{\left(c^2+v^2\right)^2} - \frac{1}{2}\frac{c^2+v^2}{2\left(c^2+v^2\right)}\right) = \frac{\pi}{2c}\frac{v^2-c^2}{\left(c^2+v^2\right)^2}
\end{aligned}$$

**14. Kappa Distribution:**

A Kappa distribution function is defined by

$$f_\kappa(\mathbf{v}) = c_N[1 + \frac{m\mathbf{v}^2}{2\kappa W_0}]^{-(\kappa+1)}$$

with the normalization constant $c_N$ and the typical energy $W_0$.
(a) Describe qualitatively the properties of this distribution function in comparison with a Maxwellian. Why is it necessary to assume $\kappa > 1.5$?
(b) Determine the normalization constant $c_N$ for $\kappa = 2$ if the plasma density is given by $n_0$ for one species.
(c) Determine the average kinetic energy for $\kappa = 2$.

**Solution:** With the substitution $x^2 = \frac{mv^2}{2\kappa W_0}$ in the kappa distribution the evaluation of the moments involves integrals of the type $I_n = \int^x [1+x^2]^{-n} dx$ with $\int^x [1+x^2]^{-1} dx = \arctan x$. These integrals can be evaluated using a recursion

$$I_{n+1} = \int^x \left[1+x^2\right]^{-(n+1)} dx = \frac{x}{2n}\left[1+x^2\right]^{-n} + \frac{2n-1}{2n}\int^x \left[1+x^2\right]^{-n} dx = \frac{x}{2n}\left[1+x^2\right]^{-n} + \frac{2n-1}{2n} I_n$$

This can be demonstrated by invoking the derivative of the integrant

$$\begin{aligned} h_n(x) &= \left[1+x^2\right]^{-n} \\ \frac{x}{2n}h'_n(x) &= -x^2\left[1+x^2\right]^{-n-1} = -\left(x^2+1-1\right)\left[1+x^2\right]^{-n-1} = -h_n + h_{n+1} \\ h_{n+1} &= \left[1+x^2\right]^{-n-1} = h_n + \frac{x}{2n}h'_n(x) \end{aligned}$$

and using integration by parts for the 2nd term in the expression (alternatively an integral table is acceptable for the evaluation).

**(a)** Properties of this distribution function in comparison with a Maxwellian. Why is it necessary to assume $\kappa > 1$?

In the limit to $\infty$ the distribution approaches 0 with a power law $f \sim \mathbf{v}^{-2(\kappa+1)}$ whereas the Maxwellian decreases exponentially, i.e., much faster. For $\kappa = 1.5$ the limit to $\infty$ is $f \sim \mathbf{v}^{-4}$ but energy density $\sim v^2 f$ and the integral over enrgy density in spherical coordinates is $\sim v^4 f = v^{-1}$. This integral does not converge such that the distribution has an infinite thermal energy which is not physical.

**(b)** Plasma density:

$$\begin{aligned} n_0 &= \int\int\int_{-\infty}^{\infty} f(v) d^3v \\ &= c_\kappa 4\pi \int_0^\infty \left[1 + \frac{mv^2}{2\kappa W_0}\right]^{-(1+\kappa)} v^2 dv \\ \text{with} \qquad x^2 &= \frac{mv^2}{2\kappa W_0} \\ n_0 &= 4\pi c_\kappa \left(\frac{2\kappa W_0}{m}\right)^{3/2} \int_0^\infty \left[1+x^2\right]^{-(1+\kappa)} x^2 dx \end{aligned}$$

$\kappa = 2$:

$$
\begin{aligned}
n_0 &= 32\pi c_2 \left(\frac{W_0}{m}\right)^{3/2} \int_0^\infty \left[1+x^2\right]^{-3} x^2 dx \\
\text{with} \qquad & \Lambda = 32\pi c_2 \left(\frac{W_0}{m}\right)^{3/2} \\
n_0 &= \Lambda \int_0^\infty \left\{\left[1+x^2\right]^{-2} - \left[1+x^2\right]^{-3}\right\} dx \\
&= -\Lambda \frac{x}{4}\left[1+x^2\right]^{-n}\Big|_0^\infty - \Lambda \frac{3}{4}\int_0^\infty \left[1+x^2\right]^{-2} dx + \Lambda \int_0^\infty \left[1+x^2\right]^{-2} dx \\
&= \frac{\Lambda}{4}\int_0^\infty \left[1+x^2\right]^{-2} dx = \frac{\Lambda}{4}\frac{x}{2}\left[1+x^2\right]^{-1}\Big|_0^\infty + \frac{\Lambda}{4}\frac{1}{2}\int_0^\infty \left[1+x^2\right]^{-1} dx \\
&= \frac{\Lambda}{8} \arctan x|_0^\infty = \pi \frac{\Lambda}{16}
\end{aligned}
$$

**(c)** Energy ($\kappa = 2$)

$$
\begin{aligned}
w(r,t) &= 64\pi c_2 m \left(\frac{W_0}{m}\right)^{5/2} \int_0^\infty \left[1+x^2\right]^{-3} x^4 dx \\
\text{with} \qquad & K = 64\pi c_2 m \left(\frac{W_0}{m}\right)^{5/2} \\
w(r,t) &= K \int_0^\infty \left[1+x^2\right]^{-3} x^4 dx \\
&= K \int_0^\infty \left[1+x^2\right]^{-3} \left[\left(x^4+2x^2+1\right) - 2\left(x^2+1\right) + 1\right] dx \\
&= K \int_0^\infty \left\{\left[1+x^2\right]^{-1} - 2\left[1+x^2\right]^{-2} + \left[1+x^2\right]^{-3}\right\} dx \\
&= K \int_0^\infty \left\{\left[1+x^2\right]^{-1} - 2\left[1+x^2\right]^{-2}\right\} dx \\
&\quad + \frac{x}{4} K \left[1+x^2\right]^{-2}\Big|_0^\infty + \frac{3}{4} K \int_0^\infty \left[1+x^2\right]^{-2} dx \\
&= K \int_0^\infty \left\{\left[1+x^2\right]^{-1} - \frac{5}{4}\left[1+x^2\right]^{-2}\right\} dx \\
&= K \int_0^\infty \left[1+x^2\right]^{-1} dx - \frac{5}{4} K \frac{1}{2}\int_0^\infty \left[1+x^2\right]^{-1} dx \\
&= \frac{3}{16} K\pi
\end{aligned}
$$

Thus the energy density is equal to

$$
w(r,t) = 12\pi^2 c_2 m \left(\frac{W_0}{m}\right)^{5/2} = 6 n_0 W_0
$$

with $w = \frac{3}{2} n k_B T$ => $k_B T = 4W_0$.

For the kappa distribution with $\kappa = 2$ the temperature is four times the characteristic energy while it is identical to the characteristic energy for a Maxwellian. This documents the high energy tail of the kappa distribution compared to the Maxwellian.

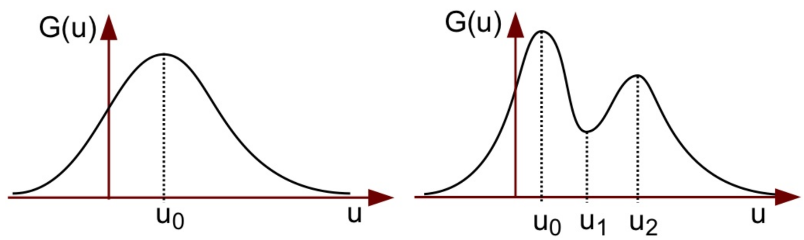
**15. Penrose Criterion:**

Consider the case of a reduced distribution function $g(u)$ with 2 maxima at $u = u_0$ and $u = u_2$, and a minimum at $u = u_1$.
(a) Show that if $g(u_1) = 0$, the Penrose criterion for instability is always satisfied.
(b) Suppose the lower maximum of $g$ is at $u = u_2$. Following the arguments given in class, show that the range of unstable waves is given by

$$\omega_e^2 \int_{-\infty}^{\infty} du \frac{[g(u) - g(u_2)]}{(u - u_2)^2} < k^2 < \omega_e^2 \int_{-\infty}^{\infty} du \frac{[g(u) - g(u_1)]}{(u - u_1)^2}.$$

Why is the the lower limit of $k^2$ in the above expression chosen with the integral at $u_2$ rather than by the equivalent integral for $u_0$?

**Solution:** Considering a double peaked distribution function




and using the arguments provided in class for the case of $\varepsilon_i = 0$ (for which $\partial_v g\,(v = u_s = \omega/k) = 0$) we can express $\varepsilon_r$ through

$$\begin{aligned} \varepsilon_r\,(\omega_r = k u_s) &= 1 - \frac{\omega_{pe}^2}{k^2} \mathsf{P} \int_{-\infty}^{\infty} dv \frac{\partial_v g\,(v)}{v - \omega/k} = 1 - \frac{\omega_{pe}^2}{k^2} \mathsf{P} \int_{-\infty}^{\infty} dv \frac{\partial_v\,[g\,(v) - g\,(u_s)]}{v - u_s} \\ &= 1 + \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} dv \frac{[g\,(u_s) - g\,(v)]}{(v - u_s)^2} \end{aligned}$$

where the index $s$ refers to the index of the respective maximum (at $u_0$ or $u_2$) or minimum (at $u_1$) and the respective values of $\varepsilon_r$represent the points on the real $\varepsilon$ axis where the $\varepsilon$ contour (that corresponds to the $\omega$ contour) crosses the axis. It further follows that $\varepsilon_r\,(\omega_r = k u_1) < \varepsilon_r\,(\omega_r = k u_2) < \varepsilon_r\,(\omega_r = k u_0)$ because of $g\,(u_1) < g\,(u_2) < g\,(u_0)$. Further considering the contour in the omega plane $\omega$ plane, $\omega_r$ passes first through $k u_0$, then $k u_1$, and last through $k u_2$.

**(a)** Noting that for $g\,(u_1) = 0 \quad \Rightarrow g\,(u_1) - g\,(v) < 0$ because $g\,(v) > 0$ by definition of a distribution function. Therfore

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} dv \frac{[g\,(u_1) - g\,(v)]}{(v - u_s)^2} < 0 \\ \textit{such that} &\qquad \varepsilon_r\,(\omega_r = k u_1) = 1 + \frac{\omega_{pe}^2}{k^2} I_1 < 0 \\ \textit{for} &\qquad k^2 < -\omega_{pe}^2 I_1 \end{aligned}$$

We refer to this as condition 1.

**(a)** The solution to (a) represents the right side of the inequality. The left side correponds to the condition

$$\begin{aligned}\frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty} du \frac{[g(u)-g(u_2)]}{(u-u_2)^2} &< 1 \\ or \\ \frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty} du \frac{[g(u_2)-g(u)]}{(u-u_2)^2} &> -1 \\ or \\ \varepsilon_r(\omega_r = ku_2) = 1 + \frac{\omega_e^2}{k^2}\int_{-\infty}^{\infty} du \frac{[g(u_2)-g(u)]}{(u-u_2)^2} &> 0\end{aligned}$$

Provided that condition 1 is satisfied, meaning that the $\varepsilon_r(\omega_r = ku_1) < 0$ the second condition for instability is that the $\epsilon$ contour incircles the origin. However, this is only the case if $\varepsilon_r(\omega_r = ku_2) > 0$. Otherwise the $\epsilon$ contour does not incircle the origin and therefor there is no pole in the upper half of the complex $\omega$ plane as illustrated by the 3 cases in the following figure where all cases satisfy condition 1 but case C does not satisfy condition 2 and indeed does not incircle the origin.