15. Anisotropic distribution function

An anisotropic distribution function for a two-dimensional equilibrium ($\partial/\partial y = 0$) can be written as $F(H,\mu) = G(A)\exp\left(-\beta(A)H - \eta(A)\mu B_0\right)$ with the Hamiltonian $H = (m/2)\left(v_\perp^2 + v_\parallel^2\right) + q\phi$, the adiabatic moment $\mu = (m/2)v_\perp^2/B$, the magnetic field strength B, and the y component of the vector potential A, where $G(A)$, $\beta(A)$, and $\eta(A)$ are arbitrary functions.

(a) Relate β and η to the temperatures $T_\parallel$ and $T_\perp$ for a bi-Maxwellian distribution.

(b) Calculate the number density n by integrating F over velocity space.

(c) Show that the parallel and perpendicular pressures are $p_\parallel = n(A,B)k_BT_\parallel$ and $p_\perp = n(A,B)k_BT_\perp$

(d) Determine the pressure anisotropy $D = \frac{p_\parallel - p_\perp}{p_\parallel}$ as a function of B, B_0, β, and η.

Solution:

(a) Distribution function:

$$\begin{aligned}
F(H,\mu) &= G(A)\exp\left(-\beta(A)H - \eta(A)\mu B_0\right) \\
&= G(A)\exp\left(-\beta\frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right) - \beta q\phi - \eta\frac{B_0 v_\perp^2}{B}\right) \\
&= G(A)\exp\left(-\beta\frac{m}{2}v_\parallel^2 - \beta q\phi - \left(\eta\frac{B_0}{B} + \beta\right)\frac{m}{2}v_\perp^2\right)
\end{aligned}$$

Such that $k_BT_\parallel = 1/\beta\,(A)$ and $k_BT_\perp = 1/\lambda$ with

$$\begin{aligned}
\frac{1}{k_BT_\perp(A,B)} &= \eta(A)\frac{B_0}{B} + \frac{1}{k_BT_\parallel(A)} \\
&\quad or \\
k_BT_\perp(A,B) &= \frac{k_BT_\parallel(A)}{1 + \eta(A)\frac{B_0}{B}k_BT_\parallel}
\end{aligned}$$

(b) Calculate the number density n by integrating F over velocity space.

$$\begin{aligned}
n(A,\phi) &= G(A)\exp(-\beta q\phi)\int\int\int d^3v\,\exp\left(-\beta\frac{m}{2}v_\parallel^2 - \lambda\frac{m}{2}v_\perp^2\right) \\
&= 2\pi G(A)\exp(-\beta q\phi)\int_{-\infty}^{\infty}dv_\parallel\exp\left(-\beta\frac{m}{2}v_\parallel^2\right)\int_0^\infty dv_\perp v_\perp\exp\left(-\lambda\frac{m}{2}v_\perp^2\right) \\
&= 2\pi G(A)\exp(-\beta q\phi)\sqrt{\frac{2\pi}{\beta m}}\left(-\frac{1}{\lambda m}\exp\left(-\lambda\frac{m}{2}v_\perp^2\right)\right)_0^\infty \\
n(A,\phi) &= \sqrt{\frac{2\pi}{\beta m}}\frac{2\pi}{\lambda m}G(A)\exp(-\beta q\phi)
\end{aligned}$$

Normalization:

$$\Longrightarrow \qquad G(A)\exp(-\beta q\phi) = \frac{m}{2\pi k_BT_\perp}\sqrt{\frac{m}{2\pi k_BT_\parallel}}\,n(A,\phi)$$

(c) Show that the parallel and perpendicular pressures are $p_\parallel = n(A,B)k_BT_\parallel$ and $p_\perp = n(A,B)k_BT_\perp$

$$
\begin{aligned}
p_\parallel &= mG(A)\exp\left(-\beta q\phi\right)\int\int\int d^3v v_\parallel^2 \exp\left(-\beta\frac{m}{2}v_\parallel^2 - \lambda\frac{m}{2}v_\perp^2\right) \\
&= m\frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)\int_{-\infty}^{\infty} dv_\parallel v_\parallel^2 \exp\left(-\beta\frac{m}{2}v_\parallel^2\right) \\
&= m\frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)\frac{\sqrt{\pi}}{2}\left(\frac{2}{\beta m}\right)^{3/2} \\
&= \frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)\sqrt{\frac{2\pi}{\beta m}}k_BT_\parallel \\
&= n\left(A,\phi\right)k_BT_\parallel
\end{aligned}
$$

$$
\begin{aligned}
p_\perp &= mG(A)\exp\left(-\beta q\phi\right)\int\int\int d^3v v_\perp^2 \exp\left(-\beta\frac{m}{2}v_\parallel^2 - \lambda\frac{m}{2}v_\perp^2\right) \\
&= 2\pi m\sqrt{\frac{2\pi}{\beta m}}G(A)\exp\left(-\beta q\phi\right)\int_0^{\infty} dv_\perp \frac{1}{2}v_\perp^3 \exp\left(-\lambda\frac{m}{2}v_\perp^2\right) \\
&= \pi m\sqrt{\frac{2\pi}{\beta m}}G(A)\exp\left(-\beta q\phi\right)\left[\left(-\frac{v_\perp^2}{\lambda m} - \frac{2}{\lambda^2 m^2}\right)\exp\left(-\lambda\frac{m}{2}v_\perp^2\right)\right]_0^\infty \\
&= \pi\sqrt{\frac{2\pi}{\beta m}}G(A)\exp\left(-\beta q\phi\right)\frac{2}{\lambda m}k_BT_\perp = n\left(A,\phi\right)k_BT_\perp
\end{aligned}
$$

(d) Determine the pressure anisotropy $D = \frac{p_\parallel - p_\perp}{p_\parallel}$ as a function of B, B_0, β, and η.

$$
\begin{aligned}
D &= \frac{T_\parallel - T_\perp}{T_\parallel} = \frac{1/\beta - 1/\lambda}{1/\beta} \\
&= 1 - \frac{\beta}{\lambda} = 1 - \frac{1}{\frac{\eta B_0}{\beta B} + 1} \\
&= 1 - \left(\eta\frac{B_0}{B}k_BT_\parallel + 1\right)^{-1}
\end{aligned}
$$

16. Magnetotail equilibrium

An asymptotic two-dimensional magnetotail equilibrium is determined by the y component of the vector potential

$$A(x_1, z) = A_0 \ln\left[l(x_1)\cosh(z/l(x_1))\right] \qquad \textit{with} \qquad l(x_1) = (2p_0(x_1))^{-1/2}$$

with $x_1 = \varepsilon x$ and $\varepsilon \ll 1$.

(a) Determine the magnetic field components, the current density, and the pressure resulting from this choice of the vector potential. Describe the magnetic field configuration.

(b) Examine the force balance equation and determine the deviation from exact force balance along the x and z direction. What are the conditions that this error remains small?

Solution:

(a) Magnetic field components, the current density, and the pressure resulting from this choice of the vector potential.

With $\tilde{z} = z/l$ and $x_1 = \varepsilon x$ the **pressure** is straightforward and follows from

$$\begin{aligned}
p(A) &= \frac{1}{2}\exp(-2A) \\
&= \frac{1}{2}\exp\left\{-2\ln\left[l(x_1)\cosh\tilde{z}\right]\right\} \\
&= \frac{1}{2}\left(\exp\left\{\ln\left[l(x_1)\cosh\tilde{z}\right]\right\}\right)^{-2} = \frac{1}{2}\left[(2p_0(x_1))^{-1/2}\cosh\tilde{z}\right]^{-2} \\
&= \frac{1}{2}(2p_0(x_1))\cosh^{-2}\tilde{z} = \frac{1}{2l^2}\cosh^{-2}\tilde{z} \\
&= p_0(x_1)\cosh^{-2}\tilde{z}
\end{aligned}$$

Magnetic field components with $\partial_{x_1}\ln l = \dot{\ln l} = \dot{l}/l$ and $\partial_{x_1} z/l = -\dot{l}z/l^2 = -\tilde{z}\dot{\ln l}$ and $\partial_x = \varepsilon\partial_{x_1}$:

$$\begin{aligned}
B_x(x_1, z) &= -\partial_z A_y = -\partial_z\left\{A_0\ln\left[l(x_1)\cosh\tilde{z}\right]\right\} \\
&= -\frac{A_0 l}{l\cosh(\tilde{z})}\partial_z\cosh\tilde{z} \\
&= -\frac{A_0}{l(x_1)}\tanh\tilde{z} \\
B_z(x_1, z) &= \partial_x A_y = \partial_x\left\{A_0\ln\left[l(x_1)\cosh\tilde{z}\right]\right\} \\
&= \frac{A_0\partial_x\left[l(x_1)\cosh\tilde{z}\right]}{l(x_1)\cosh(\tilde{z})} = \varepsilon\frac{A_0\left[\dot{l}(x_1)\cosh\tilde{z} - \tilde{z}l\dot{\ln l}\sinh\tilde{z}\right]}{l(x_1)\cosh(\tilde{z})} \\
&= \varepsilon A_0\dot{\ln l}\left(1 - \tilde{z}\tanh\tilde{z}\right)
\end{aligned}$$

Current density:

$$\begin{aligned}
j_y &= -\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2}\right) = -\partial_x B_z + \partial_z B_x \\
&= -\varepsilon A_0\partial_x\left[\dot{\ln l}\left(1 - \tilde{z}\tanh\tilde{z}\right)\right] - A_0\partial_z\left[\frac{1}{l(x_1)}\tanh\tilde{z}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon^2 A_0 \left[\ddot{\ln} l \left(1-\tilde{z}\tanh\tilde{z}\right) - \left(\dot{\ln} l\right)\partial_{x_1}\left(\tilde{z}\tanh\tilde{z}\right)\right] - A_0 l^{-2}\cosh^{-2}\tilde{z} \\
&= -\varepsilon^2 A_0 \left[\ddot{\ln} l \left(1-\tilde{z}\tanh\tilde{z}\right) - \left(\dot{\ln} l\right)\left(-\tilde{z}\dot{\ln} l\tanh\tilde{z} - \tilde{z}^2\dot{\ln} l\cosh^{-2}\tilde{z}\right)\right] - A_0 l^{-2}\cosh^{-2}\tilde{z} \\
&= -\varepsilon^2 A_0 \left[\ddot{\ln} l \left(1-\tilde{z}\tanh\tilde{z}\right) + \left(\dot{\ln} l\right)^2\left(\tilde{z}\tanh\tilde{z} + \tilde{z}^2\cosh^{-2}\tilde{z}\right)\right] - A_0 l^{-2}\cosh^{-2}\tilde{z}
\end{aligned}
$$

(b) To examine force balance we need the pressure gradients:

$$
\begin{aligned}
\partial_x p &= \partial_x \frac{1}{2l^2(x_1)}\cosh^{-2}\tilde{z} = -\varepsilon\frac{\dot{\ln} l}{l^2(x_1)}\cosh^{-2}\tilde{z}\left(1-\tilde{z}\tanh\tilde{z}\right) \\
\partial_z p &= \partial_z \frac{1}{2l^2(x_1)}\cosh^{-2}\tilde{z} = -\frac{1}{l^3(x_1)}\cosh^{-2}\tilde{z}\tanh\tilde{z}
\end{aligned}
$$

Force balance:

$$
\begin{aligned}
-\partial_x p + j_y b_z &= \varepsilon\frac{\dot{\ln} l}{l^2(x_1)}\cosh^{-2}\tilde{z}\left(1-\tilde{z}\tanh\tilde{z}\right) + \varepsilon\dot{\ln} l\left(1-\tilde{z}\tanh\tilde{z}\right)\cdot \\
&\quad \left\{-\varepsilon^2\left[\ddot{\ln} l\left(1-\tilde{z}\tanh\tilde{z}\right) + \left(\dot{\ln} l\right)^2\left(\tilde{z}\tanh\tilde{z} + \tilde{z}^2\cosh^{-2}\tilde{z}\right)\right] - l^{-2}(x_1)\cosh^{-2}\tilde{z}\right\} \\
&= -\varepsilon^3\dot{\ln} l\left(1-\tilde{z}\tanh\tilde{z}\right)\left[\ddot{\ln} l\left(1-\tilde{z}\tanh\tilde{z}\right) + \left(\dot{\ln} l\right)^2\left(\tilde{z}\tanh\tilde{z} + \tilde{z}^2\cosh^{-2}\tilde{z}\right)\right] \\
-\partial_z p - j_y b_x &= \frac{1}{l^3(x_1)}\cosh^{-2}\tilde{z}\tanh\tilde{z} + \frac{1}{l(x_1)}\tanh\tilde{z}\cdot \\
&\quad \left\{-\varepsilon^2\left[\ddot{\ln} l\left(1-\tilde{z}\tanh\tilde{z}\right) + \left(\dot{\ln} l\right)^2\left(\tilde{z}\tanh\tilde{z} + \tilde{z}^2\cosh^{-2}\tilde{z}\right)\right] - l^{-2}(x_1)\cosh^{-2}\tilde{z}\right\} \\
&= \frac{-\varepsilon^2}{l(x_1)}\tanh\tilde{z}\left[\ddot{\ln} l\left(1-\tilde{z}\tanh\tilde{z}\right) + \left(\dot{\ln} l\right)^2\left(\tilde{z}\tanh\tilde{z} + \tilde{z}^2\cosh^{-2}\tilde{z}\right)\right]
\end{aligned}
$$

Force balance along the x direction ($z = 0$):

$$
-\partial_x p + j_y b_z = -\varepsilon^3\dot{\ln} l\ddot{\ln} l
$$

Force balance along the z direction ($x = 0$, $z \gg 1$, $\tanh\tilde{z} \approx 1$):

$$
-\partial_z p - j_y b_x = \frac{-\varepsilon^2}{l(x_1)}\left[\ddot{\ln} l\left(1-\tilde{z}\right) + \left(\dot{\ln} l\right)^2\tilde{z}\right]
$$

=> error $= O(1)$ for $\tilde{z} = \varepsilon^{-2}$