18. Force balance equation in cylindrical coordinates:

Assume $\partial/\partial\theta = \partial/\partial z$ and $B_r = 0$ and show that the force balance equation in cylindrical coordinates (r, θ, z) takes the form

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

Solution:

Force balance equation:

$$-\nabla p + \mathbf{j} \times \mathbf{B} = 0$$

in cylindrical coordinates with $\partial/\partial\theta = \partial/\partial z = 0$

$$\begin{aligned} \nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r \\ \mu_0 \mathbf{j} &= -\frac{\partial B_z}{\partial r}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial r B_\theta}{\partial r}\mathbf{e}_z \end{aligned}$$

such that

$$\begin{aligned} \mathbf{j} \times \mathbf{B} &= \tfrac{1}{\mu_0} \left(-\frac{\partial B_z}{\partial r}B_z - \frac{1}{r}\frac{\partial r B_\theta}{\partial r}B_\theta\right)\mathbf{e}_r \\ &= \left[-\frac{1}{2\mu_0}\frac{d}{dr}\left(B_z^2 + B_\theta^2\right) - \frac{B_\theta^2}{\mu_0}\right]\mathbf{e}_r \end{aligned}$$

yielding the force balance equation in the radial direction:

$$\frac{dp}{dr} + \frac{1}{2\mu_0}\frac{d}{dr}\left(B_z^2 + B_\theta^2\right) + \frac{B_\theta^2}{\mu_0} = 0$$

19. θ **pinch:**

Assume a constant current j_0 in the z direction in a cylindrical coordinate system.
a) Compute the magnetic field $B_\theta(r)$ and integrate the force balance equation to obtain $p(r)$. The pressure at $r = 0$ is p_0.
b) Determine the critical radius for which the pressure decreases to 0.
c) Show that the equilibrium condition for the θ pinch

$$\frac{dp_0}{dr} = \frac{B_0}{\mu_0 r}\frac{d\,(r B_0)}{dr}$$

can be expressed as

$$\frac{d\ln B_0}{d\ln r} = \frac{\beta}{2}\frac{d\ln p_0}{d\ln r} - 1$$

Solution:

(a) Magnetic field $B_\theta(r)$:

$$\begin{aligned} \frac{1}{r}\frac{\partial r B_\theta}{\partial r} &= \mu_0 j_0 \\ B_\theta &= \frac{\mu_0 j_0}{r}\int_0^r r dr = \frac{\mu_0 j_0}{2} r \end{aligned}$$

Note $B_\theta = 0$ at $r = 0$ because $r = 0$ is a singular point for B_θ.

Pressure:

$$\frac{dp}{dr} = -j_z B_\theta$$

which yields

$$p(r) = -\frac{\mu_0 j_0^2}{2}\int^r r dr = -\frac{\mu_0 j_0^2}{4} r^2 + p_0$$

such that $p(0) = p_0$.

(b) The pressure assumes 0 for

$$-\frac{\mu_0 j_0^2}{4} r_c^2 + p_0 = 0$$

or

$$r_c^2 = \frac{4 p_0}{\mu_0 j_0^2}$$

(c) Show that

$$\frac{dp_0}{dr} = \frac{B_0}{\mu_0 r}\frac{d(rB_0)}{dr}$$

is eqivalent to

$$\begin{aligned} \frac{d\ln B_0}{d\ln r} &= \frac{\beta}{2}\frac{d\ln p_0}{d\ln r} - 1 \\ \text{with} \quad & \frac{d}{d\ln r} = \left(\frac{d\ln r}{dr}\right)^{-1}\frac{d}{dr} = r\frac{d}{dr} \\ \frac{r}{B_0}\frac{dB_0}{dr} &= \frac{2\mu_0 p_0}{2B_0^2}\frac{r}{p_0}\frac{dp_0}{dr} - 1 = \frac{\mu_0 r}{B_0^2}\frac{dp_0}{dr} - 1 \end{aligned}$$

Multiplication with B_0^2/μ_0:

$$\frac{dp_0}{dr} = \frac{B_0}{\mu_0}\frac{dB_0}{dr} + \frac{B_0^2}{\mu_0 r} = \frac{B_0}{\mu_0 r}\frac{d(rB_0)}{dr}$$