21. Stability of the Z pinch: The pressure and magnetic field profile of the Z pinch is

$$B_\theta = \frac{\mu_0 j_0}{2} r \qquad and \qquad p = -\frac{\mu_0 j_0^2}{4} r^2 + p_0 \qquad for \qquad r \le r_c = \left(\frac{4p_0}{\mu_0 j_0^2}\right)^{1/2}$$

Assume perturbations with $\partial/\partial\theta = 0$ and apply the stability criterion derived in class to the equilibrium to determine it equilibrium properties. How do these change if a constant pressure p_c is superimposed.

1. Solution

Magnetic field and pressure (homework problem 18) are given by

$$\begin{aligned} B_\theta &= \frac{\mu_0 j_0}{r} \int^r r dr = \frac{\mu_0 j_0}{2} r \\ p &= -\frac{\mu_0 j_0^2}{2} \int^r r dr = -\frac{\mu_0 j_0^2}{4} r^2 + p_0 = \frac{\mu_0 j_0^2}{4} \left(r_c^2 - r^2\right) \\ with \qquad & r_c^2 = \frac{4p_0}{\mu_0 j_0^2} \end{aligned}$$

Constant current j_0 in the z direction in a cylindrical coordinate system.

Stability criterion (from class)

$$-\frac{r}{p}\frac{dp}{dr} < \frac{4\gamma}{2+\gamma\beta}$$

for stability. With

$$\begin{aligned} -\frac{r}{p}\frac{dp}{dr} &= \frac{r}{p_0 - \frac{\mu_0 j_0^2}{4} r^2} \frac{\mu_0 j_0^2}{2} r \\ &= \frac{2r^2/r_c^2}{(1 - r^2/r_c^2)} \end{aligned}$$

and

$$\begin{aligned} \beta = \frac{2\mu_0 p}{B^2} &= \frac{2\mu_0 \left(p_0 - \frac{\mu_0 j_0^2}{4} r^2\right)}{\frac{\mu_0^2 j_0^2}{4} r^2} \\ &= \frac{2\left(1 - r^2/r_c^2\right)}{r^2/r_c^2} \end{aligned}$$

Substitution in the stability condition:

$$\begin{aligned} \frac{2r^2/r_c^2}{(1 - r^2/r_c^2)} &< \frac{4\gamma}{2+\gamma\beta} = \frac{2\gamma}{1 + \gamma\frac{(1 - r^2/r_c^2)}{r^2/r_c^2}} = \frac{2\gamma r^2/r_c^2}{r^2/r_c^2 + \gamma\left(1 - r^2/r_c^2\right)} \\ r^2/r_c^2 + \gamma\left(1 - r^2/r_c^2\right) &< \gamma\left(1 - r^2/r_c^2\right) \end{aligned}$$

or

$$r^2/r_c^2 < 0$$

Thus this configuration is everywhere unstable. Note that this is caused by the assumed current density profile. Adding a constant pressure does not change this result because it means we can just increase p_0 by the amount of the constant pressure

Often β is assumed to be constant or small. Assuming that β is constant the stability condition is

$$-\frac{r}{p}\frac{dp}{dr} = \frac{2r^2/r_c^2}{(1-r^2/r_c^2)} < \frac{4\gamma}{2+\gamma\beta} = \lambda$$

or

$$\begin{aligned} r^2 &< \frac{\lambda}{2+\lambda} r_c^2 \\ &= \frac{\gamma}{\gamma+1} r_c^2 \qquad \text{for } \beta \ll 1 \end{aligned}$$

for stability.

Thus the small β approximation yields the correct result for of instability for sufficiently large radii but fails for smaller values of r/r_c where the system is actually unstable

2nd solution:

Starting from the coefficients

$$a_{11} = \gamma\beta + 2 \qquad \beta = \frac{2\mu_0 p}{B^2}$$

$$\begin{aligned} a_{12} &= 2\left(\frac{d\ln B}{d\ln r} + \frac{\beta}{2}\frac{d\ln p}{d\ln r} - 1\right) \\ a_{22} &= \left(\frac{d\ln B}{d\ln r} - 1\right) a_{12} \end{aligned}$$

we note that the force balance (equilibrium condition requires

$$\frac{d\ln B}{d\ln r} = -\frac{\beta}{2}\frac{d\ln p}{d\ln r} - 1$$

which implies $a_{12} = -4$. For the special case of $B = \mu_0 j_0 r/2$ we can see immidiately

$$\frac{d\ln B}{d\ln r} = \frac{r}{B}\frac{d\ln B}{d\ln r} = 1 \hookrightarrow a_{22} = 0$$

Therefore the stability condition becomes

$$a_{11}a_{22} - a_{12}^2 = -16 > 0$$

Which is alway violated and therefore we find unconditional instability. Adding a constant pressure does not change the result because the forcebalance condition does not change and the magnetic field only depends on the pressure gradient such that it also does not change. The only possibilities to generate a more stable configuration are (a) to use a different current density profile or (b) to employ also a B_z component.

21. Stability for the Harris sheet:

Determine the stability criterion for the Harris sheet equilibrium using the energy principle. Assume that the perturbation on the boundary is 0

Harris sheet with

$$\begin{aligned} \mathbf{B} = B\mathbf{e}_y &= B_0 \tanh \frac{x}{L} \mathbf{e}_y \\ p &= p_0 \cosh^{-2} \frac{x}{L} \\ \mathbf{j} &= \frac{B_0}{\mu_0 L} \cosh^{-2} \frac{x}{L} \mathbf{e}_z \end{aligned}$$

Solution:

The energy principle for a displacement $\boldsymbol{\xi}$ without considering surface terms is

$$\begin{aligned} U &= \frac{1}{2} \int_V \left[\gamma p_0 \left(\nabla \cdot \boldsymbol{\xi}\right)^2 + \frac{1}{\mu_0} \left(\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\right)^2 \right. \\ &\qquad \left. + \boldsymbol{\xi} \cdot \nabla p_0 \nabla \cdot \boldsymbol{\xi} - \frac{1}{\mu_0} \left(\boldsymbol{\xi} \times (\nabla \times \mathbf{B}_0)\right) \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \right] d\mathbf{x} \end{aligned}$$

(a) Show that

$$\begin{aligned} \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) &= B \left[\frac{\partial \xi_x}{\partial y} \mathbf{e}_x - \left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B} \frac{\partial \xi_x B}{\partial x} \right) \mathbf{e}_y + \frac{\partial \xi_z}{\partial y} \mathbf{e}_z \right] \\ \boldsymbol{\xi} \times (\nabla \times \mathbf{B}) &= \frac{\partial B}{\partial x} \left(\xi_y \mathbf{e}_x - \xi_x \mathbf{e}_y \right) \end{aligned}$$

Noting that the equilibrium magnetic field has only an y component and the current density only a z component and both depend on x only we get

$$\begin{aligned} \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) &= \nabla \times \left(-\xi_z B \mathbf{e}_x + \xi_x B \mathbf{e}_z \right) = -\partial_z \xi_z B \mathbf{e}_y + \partial_y \xi_z B \mathbf{e}_z + \partial_y \xi_x B \mathbf{e}_x - \partial_x \xi_x B \mathbf{e}_y \\ &= B \partial_y \xi_x \mathbf{e}_x - \left(B \partial_z \xi_z + \partial_x \xi_x B \right) \mathbf{e}_y + B \partial_y \xi_z \mathbf{e}_z \\ &= B \left[\frac{\partial \xi_x}{\partial y} \mathbf{e}_x - \left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B} \frac{\partial \xi_x B}{\partial x} \right) \mathbf{e}_y + \frac{\partial \xi_z}{\partial y} \mathbf{e}_z \right] \\ &\mathit{and} \\ \boldsymbol{\xi} \times (\nabla \times \mathbf{B}) &= \boldsymbol{\xi} \times \left(\partial_x B \mathbf{e}_z \right) = \frac{\partial B}{\partial x} \left(\xi_y \mathbf{e}_x - \xi_x \mathbf{e}_y \right) \end{aligned}$$

(b) With

$$\begin{aligned} \nabla \cdot \boldsymbol{\xi} &= \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \\ \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) &= B \left[\frac{\partial \xi_x}{\partial y} \mathbf{e}_x - \left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B} \frac{\partial \xi_x B}{\partial x} \right) \mathbf{e}_y + \frac{\partial \xi_z}{\partial y} \mathbf{e}_z \right] \\ \boldsymbol{\xi} \times (\nabla \times \mathbf{B}_0) &= \frac{\partial B}{\partial x} \left(\xi_y \mathbf{e}_x - \xi_x \mathbf{e}_y \right) \end{aligned}$$

the potential becomes

$$
\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p\,(\nabla\cdot\boldsymbol{\xi})^2 + \frac{\partial p}{\partial x}\xi_x \nabla\cdot\boldsymbol{\xi} \right.\\
&\quad + \frac{B^2}{\mu_0}\left[\left(\frac{\partial \xi_x}{\partial y}\right)^2 + \left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B}\frac{\partial \xi_x B}{\partial x}\right)^2 + \left(\frac{\partial \xi_z}{\partial y}\right)^2\right]\\
&\quad \left. - \frac{1}{2\mu_0}\frac{\partial B^2}{\partial x}\left(\xi_y \frac{\partial \xi_x}{\partial y} + \xi_x\left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B}\frac{\partial \xi_x B}{\partial x}\right)\right)\right] d\mathbf{x}
\end{aligned}
$$

For $\partial/\partial y = 0$ and $\nabla\left(p + B^2/2\mu_0\right) = 0$:

$$
\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p\,(\nabla\cdot\boldsymbol{\xi})^2 + \frac{\partial p}{\partial x}\xi_x \nabla\cdot\boldsymbol{\xi} + \frac{B^2}{\mu_0}\left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B}\frac{\partial \xi_x B}{\partial x}\right)^2\right.\\
&\quad \left. + \frac{\partial p}{\partial x}\xi_x\left(\frac{\partial \xi_z}{\partial z} + \frac{1}{B}\frac{\partial \xi_x B}{\partial x}\right)\right] d\mathbf{x}
\end{aligned}
$$

With

$$
\frac{\partial \xi_z}{\partial z} + \frac{1}{B}\frac{\partial \xi_x B}{\partial x} = \nabla\cdot\boldsymbol{\xi} + \xi_x \frac{1}{B}\frac{dB}{dx}
$$

we obtain

$$
\begin{aligned}
U &= \frac{1}{2}\int_V \left[\gamma p_0\,(\nabla\cdot\boldsymbol{\xi})^2 + \frac{\partial p}{\partial x}\xi_x \nabla\cdot\boldsymbol{\xi} + \frac{B^2}{\mu_0}\left(\nabla\cdot\boldsymbol{\xi} + \xi_x \frac{1}{B}\frac{dB}{dx}\right)^2\right.\\
&\quad \left. + \frac{\partial p}{\partial x}\xi_x\left(\nabla\cdot\boldsymbol{\xi} + \xi_x \frac{1}{B}\frac{dB}{dx}\right)\right] d\mathbf{x}\\
&= \frac{1}{2}\int_V \left[a_{11}\,(\nabla\cdot\boldsymbol{\xi})^2 + 2a_{12}\xi_x \nabla\cdot\boldsymbol{\xi} + a_{22}\xi_x^2\right] d\mathbf{x}
\end{aligned}
$$

with

$$
\begin{aligned}
a_{11} &= \gamma p + \frac{B^2}{\mu_0}\\
a_{12} &= \frac{dp}{dx} + \frac{B}{\mu_0}\frac{dB}{dx}\\
a_{22} &= \frac{1}{\mu_0}\left(\frac{dB}{dx}\right)^2 + \frac{dp}{dx}\frac{1}{B}\frac{dB}{dx}
\end{aligned}
$$

(c) Using the equilibrium condition for the Harris sheet:

$$
0 = \frac{d}{dx}\left(p + \frac{B^2}{2\mu_0}\right) = \frac{dp}{dx} + \frac{B}{\mu_0}\frac{dB}{dx}
$$

we obtain $a_{12} = 0$ and

$$
a_{22} = \frac{1}{\mu_0}\frac{1}{B}\frac{dB}{dx}\left(B\frac{dB}{dx}\right) + \frac{\partial p}{\partial x}\frac{1}{B}\frac{dB}{dx} = 0
$$

such that the equilibrium condition reads

$$a_{11}a_{22} - a_{12}^2 = 0 \geq 0$$

This is insufficient to clarify stability uniquely, but:

Compressible perturbations are always stable because $a_{11} > 0$. For incompressible perturbations the result is inconclusive.

This result has actually not made use of the specific Harris equilibrium but applies to any one-dimenional equilibrium where the magnetic field has only a single component.