Appendix A

Plasma Parameter and Mathematical Tools

A.1 Tables of Plasma Parameters

The first table gives some fundamental constants:

Property	Symbol	SI	cgs
Speed of light	c	3×10^{8} m s^{-1}	3×10^{10} m s^{-1}
Boltzmann constant	k	1.38×10^{-23} J K^{-1}	1.38 erg K^{-1}
Electron mass	m_e	9.1 $\times10^{-31}$ kg	9.1 $\times10^{-28}$ g
Proton mass	m_i	1836 m_e	1836 m_e
Elementary charge	e	1.6×10^{-19}C	4.8×10^{-10} statcoulomb
Dielectric constant	ϵ_0	8.85×10^{-12} F m^{-1}	-
Permeability of free space	μ_0	$4\pi\times10^{-7}$ H m^{-1}	-

Note that $\epsilon_0\mu_0 = 1/c^2$. The following table presents several basic plasma frequencies, length scales and velocities:

Property	Symbol	SI	cgs
Plasma frequency	ω_{pe}	$\left(\frac{n_e e^2}{\epsilon_0 m_e}\right)^{1/2}$	$\left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2}$
Electron gyro frequency	ω_{ge}	$\frac{eB}{m_e}$	$\frac{eB}{m_e c}$
Coulomb collision frequency	ν_{ei}	$\frac{\omega_{pe}}{4\pi n\lambda_D^3}\ln\Lambda$	$\frac{\omega_{pe}}{4\pi n\lambda_D^3}\ln\Lambda$
Debye length	λ_D	$\left(\frac{\epsilon_0 kT}{ne^2}\right)^{1/2}$	$\left(\frac{kT}{4\pi ne^2}\right)^{1/2}$
Skin depth (electron inertia)	λ_e	$\frac{c}{\omega_{pe}}$	$\frac{c}{\omega_{pe}}$
Electron Gyroradius	r_{ge}	$\frac{v_{the}}{\omega_{pe}}$	$\frac{v_{the}}{\omega_{pe}}$
Electron Thermal velocity	v_{the}	$\left(\frac{kT}{m_e}\right)^{1/2}$	$\left(\frac{kT}{m_e}\right)^{1/2}$
Alfven speed	v_A	$\frac{B}{(\mu_0 n m_i)^{1/2}}$	$\frac{B}{(4\pi n m_i)^{1/2}}$
Number of particles in a Debye sphere	$N_D = \frac{1}{g}$	$\frac{4\pi}{3}n\lambda_D^3$	$\frac{4\pi}{3}n\lambda_D^3$

The corresponding ion properties are $\omega_{pi} = \sqrt{m_e/m_i}\omega_{pe}$; $\omega_{gi} = (m_e/m_i)\omega_{ge}$; $\nu_{ie} = (m_e/m_i)\nu_{ei}$; $r_{gi} = \sqrt{m_i/m_e}r_{ge}$; and $v_{thi} = \sqrt{m_i/m_e}v_{the}$.

The following list provides numerical values for the various plasma parameters in a convenient form. Because it is more common in the field of space physics everything is measured in cgs units in this table, i.e., n in cm^{-3}, B in Gauss, T in eV, and $\ln\Lambda \approx 20$. Note that 1 T = 10^4 Gauss and 1 nT =10^{-5} Gauss. Note also $1eV = 1.16 \times 10^4 K$.

$$\begin{aligned}
\omega_{pe} &= 5.64 \times 10^4\, n^{1/2} \;\; [rad/sec] \\
\omega_{ge} &= 1.76 \times 10^7\, B \;\; [rad/sec] \\
\omega_{gi} &= 9.58 \times 10^3\, B \;\; [rad/sec] \\
\nu_{ei} &= 1.1 \times 10^{-5}\, \ln(\Lambda) n T_e^{3/2} \;\; \left[sec^{-1}\right] \\
\lambda_{De} &= 7.43 \times 10^2\, n^{-1/2} T_e^{1/2} \;\; [cm] \\
\lambda_e &= 5.31 \times 10^5\, n^{-1/2} \;\; [cm] \\
\lambda_i &= 2.28 \times 10^7\, n^{-1/2} \;\; [cm]
\end{aligned}$$

$$\begin{aligned}
r_{ge} &= 2.38 \times 10^0\, T_e^{1/2} B^{-1} \;\; [cm] \\
r_{gi} &= 1.02 \times 10^2\, T_i^{1/2} B^{-1} \;\; [cm] \\
v_{the} &= 4.19 \times 10^7\, T_e^{1/2} \;\; [cm/sec] \\
v_{thi} &= 9.79 \times 10^5\, T_i^{1/2} \;\; [cm/sec] \\
v_A &= 2.18 \times 10^{11}\, B n^{-1/2} \;\; [cm/sec] \\
N_D &= 1.72 \times 10^9 T_e^{3/2} n^{-1/2}
\end{aligned}$$

The last table presents an overview of typical plasma properties in the inner and outer magnetosphere/tail, the solar chromosphere and corona, warm interstallar gas, and a tokamak. L_{ei} is the mean free path and N_D is the number of particles in a Debye cube.

	Inner M'Sphere	Outer M'sphere	Chromosphere	Solar Corona	Interstellar Gas	Fusion Device
n [cm^{-3}]	10^2	1	10^{12}	$3 \cdot 10^8$	1	10^{14}
B [nT]	10^4	40	0.1	10^7	1	$2 \cdot 10^9$
T_e [eV]	10^3	500	0.5	100	0.8	10^3
T_i [eV]	10^3	2×10^3	0.5	100	0.8	10^3
ω_{pe}	5.6×10^5	5.7×10^4	1.8×10^{10}	9.8×10^8	5.6×10^4	5.6×10^{11}
ω_{ge}	1.8×10^6	7.0×10^3	1.8×10^{10}	1.8×10^9	1.8×10^2	3.5×10^{11}
ω_{gi}	9.6×10^2	3.6×10^0	9.6×10^6	9.6×10^5	9.6×10^{-2}	1.9×10^8
ν_{ei}	7.0×10^{-7}	2.0×10^{-8}	2.3×10^7	6.9	3.7×10^{-4}	6.0×10^{11}
L_{ei} [m]	1.9×10^{13}	4.7×10^{14}	1.7×10^{-2}	8.0×10^5	1.3×10^9	2.8×10^{-5}
λ_{De} [m]	2.3×10^1	1.7×10^2	1.5×10^{-5}	4.0×10^{-3}	6.2	2.2×10^{-5}
λ_e [m]	5.3×10^2	5.3×10^3	1.7×10^{-2}	0.31	5.3×10^3	5.3×10^{-4}
λ_i [m]	2.3×10^4	2.3×10^5	0.72	13	2.3×10^5	2.3×10^{-3}
r_{ge} [m]	7.5×10^0	1.3×10^3	2.2×10^{-5}	3.1×10^{-3}	2.8×10^3	4.9×10^{-5}
r_{gi} [m]	3.2×10^2	1.1×10^5	9.6×10^{-4}	0.13	1.2×10^5	2.2×10^{-2}
v_{the} [m/sec]	1.3×10^7	9.4×10^6	3.9×10^5	5.5×10^6	4.9×10^5	1.7×10^7
v_{thi} [m/sec]	3.1×10^5	4.4×10^5	9.1×10^3	1.3×10^5	1.1×10^4	4.1×10^5
v_A [m/sec]	2.18×10^7	8.7×10^5	6.9×10^6	1.3×10^7	2.2×10^4	4.4×10^6
N_D	5.4×10^{12}	1.9×10^{13}	3.6×10^2	1.9×10^7	2.4×10^8	1.0×10^6

Most of these parameters can easily vary by an order of magnitude, however the table demonstrates that plasma conditions can vary over a wide range of values depending on the system.

A.2 Entropy and Adiabatic Convection

In an ordinary gas or fluid the change of heat due to pressure or temperature changes is

$$\Delta Q = C_V dT + pdV$$

Assuming an ideal gas (with 3 degrees of freedom) with $p = nk_BT = Nk_BT/V$:

$$T = pV/R$$

with $R = Nk_B$ and the specific heat and coefficients of specific heat

$$\begin{array}{cc} C_V = \frac{3}{2}R & C_p = \frac{5}{2}R \\ c_V = \frac{3}{2} & c_p = \frac{5}{2} \end{array}$$

and the ratio $\gamma = c_p/c_V = 5/3$.
Thus the change in heat becomes

$$\begin{aligned} \Delta Q &= c_V V dp + c_p p dV \\ &= c_V pV \left(\frac{dp}{p} + \gamma \frac{dV}{V} \right) \\ \Delta Q/T &= c_v R \left(\frac{dp}{p} + \gamma \frac{dV}{V} \right) \end{aligned}$$

Entropy changes are defined as $dS = \Delta Q/T$ or

$$S - S_0 = C_V \ln (pV^\gamma)$$

For adiabatic changes of the state of a system we have $dS = 0$ which can also be expressed as

$$\frac{d\left(p\rho^{-\gamma}\right)}{dt} = \frac{\partial}{\partial t} p\rho^{-\gamma} + \mathbf{u} \cdot \nabla \left(p\rho^{-\gamma}\right) = 0 \qquad \text{(A.1)}$$

with the specific volume $V \propto 1/\rho$. This equation implies that the entropy of a plasma element does not change (moving with this element) in the absence of shocks or ohmic heating.

A.3 Complex Variables, Residue Theorem, and Integration Contours

For the evaluation of Landau damping and dispersion relations of plasma waves it is helpful to introduce some background in complex variables. In the following we assume the complex variable $z = x + iy$ with x and y being real variables. We also assume $f(z)$ to be a complex function.

Laurent series expansion

If $f(z)$ is analytic in a circular ring with $r_1 < |z-a| \leq r_2$ there is a unique power law expansion within this ring called a Laurent series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k + \sum_{k=1}^{\infty} b_k (z-a)^k \tag{A.2}$$

with the coefficients

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{k+1}}, \quad b_k = \frac{1}{2\pi i} \oint_\gamma (\zeta - a)^{k-1} f(\zeta)\, d\zeta$$

in the expansion (A.2) the first part of the series is called the regular and the second part is called the principal part of the series

One can combine the two series in a single expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k \quad with \quad c_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{k+1}}$$

where γ represents an arbitrary closed curve in the counter clockwise (positive) direction as illustrated in Figure A.1.

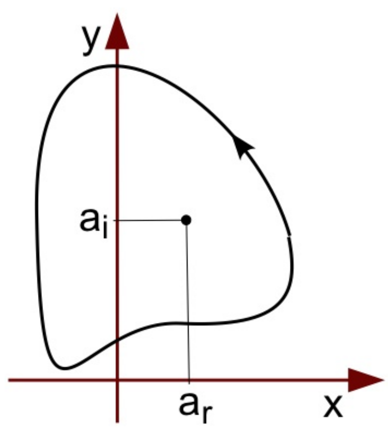


Figure A.1: The contour integral is the counterclockwise (mathematical positive) integral around the singularity.

Residue

If an analytic function $f(z)$ has an isolated singularity at $z = a$ the coefficient c_{-1} of the power $(z-a)^{-1}$ of the Laurent series is called the residue. For $a \neq \infty$ the residue is given by

$$\mathrm{Res}\left[f\left(z\right),a\right]=\frac{1}{2\pi i}\oint_{\gamma}f\left(\zeta\right)d\zeta \tag{A.3}$$

If $f\left(z\right)$ has a pole of order m at $z=a$ the Laurent series is

$$f\left(z\right)=\sum_{k=-m}^{\infty}c_{k}\left(z-a\right)^{k}$$

and the residue can be computed from

$$c_{-1}=\mathrm{Res}\left[f\left(z\right),a\right]=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[\left(z-a\right)^{m}f\left(z\right)\right]$$

Example: $f\left(z\right)=\frac{\exp(iz)}{z^2+a^2}$ has two poles at $z=\pm ia$

$$\mathrm{Res}\left[f\left(z\right),+ia\right]=\lim_{z\to ia}\left[\left(z-ia\right)\frac{\exp\left(iz\right)}{z^{2}+a^{2}}\right]=\lim_{z\to ia}\frac{\exp\left(-a\right)}{z+ia}=\frac{\exp\left(-a\right)}{2ia}$$

Residue Theorem

If a function $f\left(z\right)$ is analytic in a domain D except at a finite number of points $a_1,...,a_k$ and γ is a piecewise smooth curve enclosing all singularities in D then

$$\frac{1}{2\pi i}\oint_{\gamma}f\left(\zeta\right)d\zeta=\sum_{i=1}^{k}\mathrm{Res}\left[f\left(z\right),a_{i}\right] \tag{A.4}$$

Application of the residue theorem: Consider the integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$. The integrand is analytic for $\mathrm{Im}\left(z\right)>0$ except at $z=i$.

$$\mathrm{Res}\left[\frac{1}{1+z^{2}},+i\right]=\lim_{z\to i}\left[\left(z-i\right)\frac{1}{1+z^{2}}\right]=\lim_{z\to i}\frac{1}{z+i}=\frac{1}{2i}$$

such that

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^{2}}=2\pi i\frac{1}{2i}=\pi$$

Fourier and Laplace Transforms

In many cases Fourier and Laplace transformations are helpful to determine the solution of differential equations. Here we use the following conventions for Fourier and its inverse transform in one dimension

$$\tilde{f}(k) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} dx f(x)\exp(-ikx) \quad \text{(A.5)}$$

$$f(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} dk f(k)\exp(ikx) \quad \text{(A.6)}$$

and for the Laplace transform

$$F(p) = \int_{0}^{\infty} dt f(t)\exp(-pt) \quad \text{(A.7)}$$

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp F(p)\exp(pt) \quad \text{(A.8)}$$

If $f(t)$ is exponentially $\sim \exp(\gamma t)$ growing in time with a growth rate (real part of γ) $\gamma_r > 0$ the Laplace transformation requires that p is chosen larger than γ for convergence.

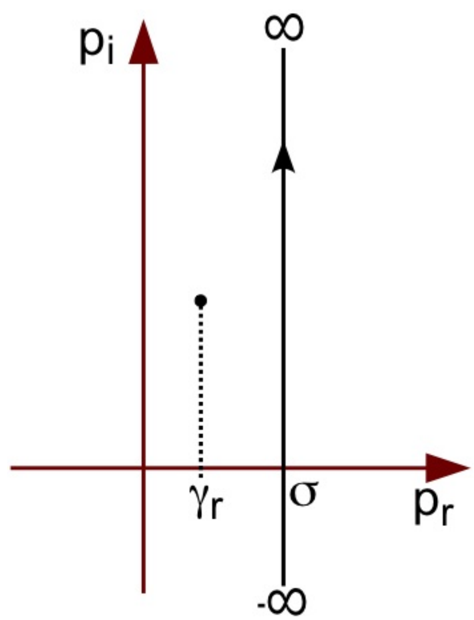


Figure A.2: Integral for the backtransformation of the Laplace transform

For the back transformation of the Laplace transform it must be noted that the integration is parallel to the imaginary axis with σ larger than the real part of all singularities of $F(p)$ as illustrated in Figure A.2. Note that some formulations of the Laplace transform use $-i\omega$ instead of p such that the back transformation path is parallel to the real axis.

Consider the following example of a simple differential equation with a real value of $\gamma > 0$:

$$\frac{df}{dt} = \gamma f$$

with $f(0) = f_0$. The Laplace transform of the equation is optained by multiplication with $\exp(-pt)$ and integration from 0 to ∞. This yields for the left side

$$\begin{aligned}\int_0^\infty dt \frac{df(t)}{dt}\exp(-pt) &= f(t)\exp(-pt)|_0^\infty + p\int_0^\infty dt f(t)\exp(-pt)\\ &= -f(t=0) + pF(p)\end{aligned}$$

provided that p is sufficiently large. Thus the Laplace transform of the equation is

$$-f(t=0) + pF(p) = \gamma F(p)$$

or

$$F(p) = \frac{f(t=0)}{p-\gamma} \tag{A.9}$$

The inverse transform of $F(p)$ is

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp \frac{f(t=0)}{p-\gamma}\exp(pt)$$

where $\sigma > \gamma$ and the Laplace transform assumed $p$sufficiently large and positive! However, the Laplace transform (A.9) can be continued analytically into the entire complex plane and to evaluete the integral we consider the path $cc_1cc'c_2c_3c_2'c'c_1'$ as illustrated in Figure A.3.

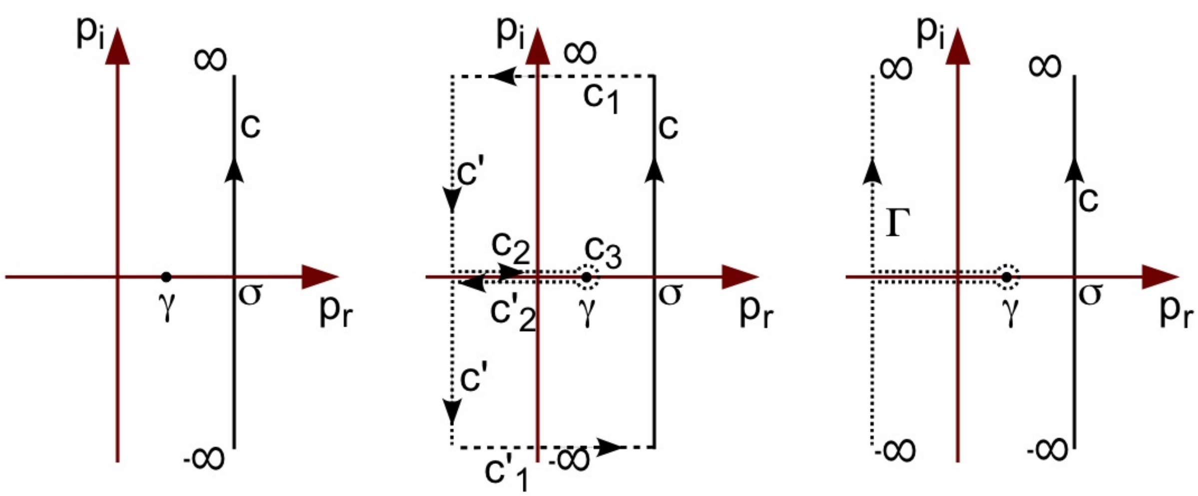


Figure A.3: Illustration of the deformation of the integration path for the inverse Laplace transform.

This integral is a closed integral that excludes the singularity such that the value of the closed integral is 0. The contributions from the path's c_1 and c_1' cancel each other such that $\int_c - \int_\Gamma = 0$ or

$$\int_c = \int_\Gamma$$

However to evaluate the integral over Γ we can close this through a semi circle with infinite radius in the negative half of the real p plane where this semicircle has no contripution when the radius goes to infinity. The new closed integral ahs one sigularity at $p = \gamma$. The residue theorem then yields

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi i}\oint_\Gamma dp \frac{f(t=0)}{p-\gamma}\exp(pt) \\
&= \frac{2\pi i}{2\pi i}\mathrm{Res}\left[\frac{f(t=0)}{p-\gamma}\exp(pt),\gamma\right] \\
&= f_0 \exp(\gamma t)
\end{aligned}
$$

Analytic Continuation

In the case of (A.9) the analytic continuation into the entire complex plane is no issue. However, let us consider the function

$$f(w) = \int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)(z-w)}$$

defined for $w_i = \mathrm{Im} w > 0$ and a real and > 0.

If we close the contour in the lower half plane it encloses only one singularity at $z = -ia$ with the result

$$
\begin{aligned}
f(w) &= -2\pi i \mathrm{Res}\left[\frac{1}{(z^2+a^2)(z-w)}, z=-ia\right] \\
&= -2\pi i \frac{1}{-2ia}\frac{1}{-w-ia} \\
&= -\frac{\pi}{a}\frac{1}{w+ia}
\end{aligned}
$$

The same result is obtained if the integral is computed through a semi-circle in the upper half of the complex plane. However, when $w_i < 0$ the contour closed over the lower half contains 2 singularities and leads to

$$f(w) = -\frac{\pi}{a}\frac{1}{w+ia} - \frac{2\pi i}{w^2+a^2}$$

such that the function $f(w)$ is discontinuous. A proper definition of an analytic function $f(w)$ requires to subtract this contribution such that for negative w_i the proper analytic continuation is

$$f(w) = \int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)(z-w)} + \frac{2\pi i}{w^2+a^2}$$

Another and consistent way to achieve this is to deform the contour as illustrated in Figure A.4. For a positive imagnary part of w the value of the integral along the path c can be obtained by closing the path in the lower hlaf of the complex plane. If the imagninary part of w is negative we need to evaluate the integral through the deformed contour Γ such that

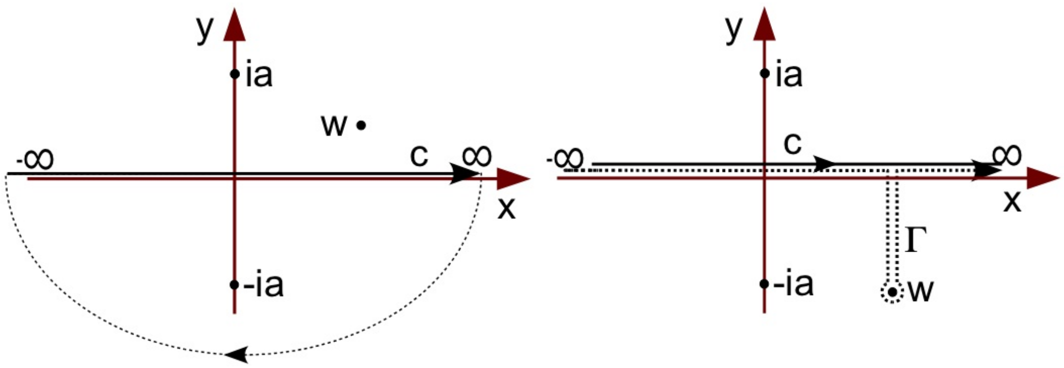


Figure A.4: Deformation of the integration contour for the analytic continuation of $f(w)$.

$$\begin{aligned} f(w) &= \int_\Gamma \frac{dz}{(z^2+a^2)(z-w)} = \int_c \frac{dz}{(z^2+a^2)(z-w)} + 2\pi i \mathrm{Res}\left[\frac{1}{(z^2+a^2)(z-w)}, z=w\right] \\ &= \int_c \frac{dz}{(z^2+a^2)(z-w)} + 2\pi i \frac{1}{(w^2+a^2)} \end{aligned}$$

Plemelj Formula

The prior example omitted the case where the imaginary part of w is $w_i = 0$. Let us consider the case where a singularity approaches the real axis in the form

$$I = \lim_{\delta \to 0} \int_{-\infty}^{\infty} dx \frac{f(x)}{x - a \pm i\delta}$$

with $\delta > 0$.

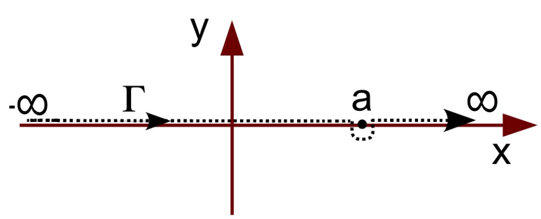


Figure A.5: Illustration of the Plemlj formula for a singularity on the real axis.

In this case the proper evaluation consits of the principal part of the integral plus one half of the residual at the point $z = a$ which can be illustrated through a contour with a semi circle around the singularity as illustrated in which leads to

$$I = \mathsf{P} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-a} + \pi i f(a)$$

with

$$\mathsf{P} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-a} = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{a-\varepsilon} dx \frac{f(x)}{x-a} + \int_{a+\varepsilon}^{\infty} dx \frac{f(x)}{x-a} \right]$$

A.4 Plasma Dispersion Function

$$Z(\zeta) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{\exp\left(-z^2\right) dz}{z - \zeta}$$

Differential equation:

$$\frac{dZ(\zeta)}{d\zeta} = -2\left[1 + \zeta Z(\zeta)\right]$$

Useful relations

$$\begin{aligned} Z(-\zeta) &= -Z(\zeta) + 2\pi^{1/2} i \exp\left(-\zeta^2\right) \quad \text{for Im}\zeta > 0 \\ \widetilde{Z}(\zeta) &= Z(\zeta) - 2\pi^{1/2} \exp\left(-\zeta^2\right) \quad \text{for Im}\zeta < 0 \end{aligned}$$

where $\widetilde{Z}(\zeta)$ is the analytic continuation of $Z(\zeta)$. The complex conjugate of the plasma dispersion function is

$$[Z(\zeta)]^* = Z(\zeta^*) - 2\pi^{1/2} \exp\left(-\zeta^2\right)$$

Expansion for $\zeta < 1$:

$$\begin{aligned} Z(\zeta) &= i\pi^{1/2} \exp\left(-\zeta^2\right) - 2\zeta \left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - ...\right) \\ &= \pi^{1/2} \sum_{n=0}^{\infty} \frac{i^{n+1} \zeta^n}{\Gamma(1 + n/2)} \end{aligned}$$

Expansion for $\zeta \gg 1$:

$$Z(\zeta) = -\frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + ...\right) + \sigma\pi^{1/2} i \exp\left(-\zeta^2\right)$$

with

$$\sigma = \begin{cases} 0, & \text{for Im}\zeta < 0 \\ 1, & \text{for Im}\zeta = 0 \\ 2, & \text{for Im}\zeta > 0 \end{cases}$$

The plasma dispersion function is related to the error function:

$$\begin{aligned} \text{erf}(z) &= \left(1 + \frac{2i}{\pi^{1/2}} \int_0^z e^{t^2} dt\right) e^{-z^2} \\ Z(\zeta) &= i\pi^{1/2} \text{erf}(\zeta) \end{aligned}$$

A.5 Bessel Functions and Modified Bessel Functions

Bessel's differential equation

$$x^2y'' + xy' + \left(x^2 - n^2\right) y = 0$$

Solutions are the Bessel functions of the first kind

$$J_n(x) = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!\Gamma\left(n+v+1\right)} \left(\frac{x}{2}\right)^{n+2v}$$

Modified Bessel functions of the first kind

$$I_n(x) = i^{-n} J_n(x) = \sum_{v=0}^{\infty} \frac{1}{v!\Gamma\left(n+v+1\right)} \left(\frac{x}{2}\right)^{n+2v}$$

are the solution of the differential equation

$$x^2y'' + xy' - \left(x^2 - n^2\right) y = 0$$

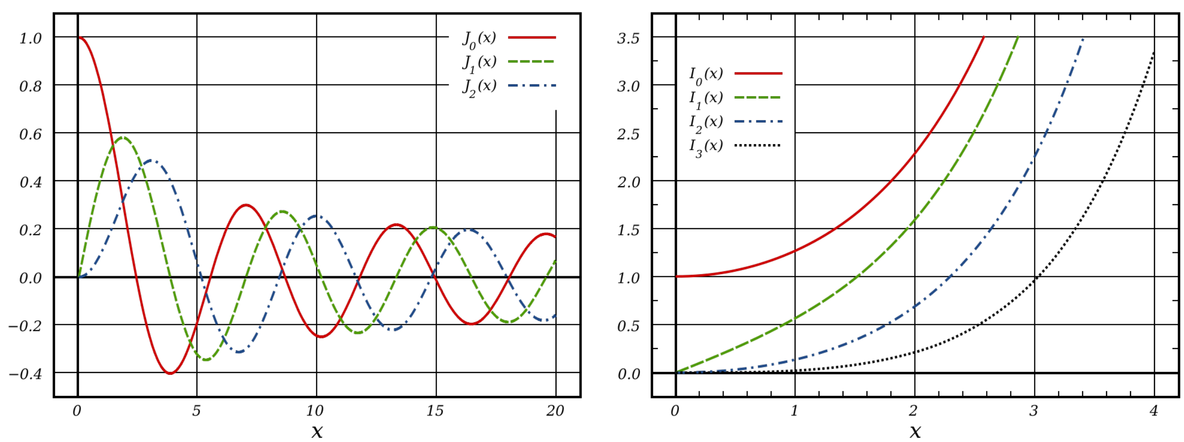


Figure A.6: Bessel functions J_n and modified Bessel functions I_n of the first kind.

Recurrence relations

$$\begin{aligned} \frac{2n}{x} J_n(x) &= J_{n-1}(x) + J_{n+1}(x) \\ \frac{d}{dx}\left(x^n J_n(x)\right) &= x^n J_{n-1}(x) \\ \frac{d}{dx}\left(x^{-n} J_n(x)\right) &= -x^{-n} J_{n+1}(x) \end{aligned}$$

and for the modified Bessel functions

$$\frac{2n}{x}I_n(x) = I_{n-1}(x) - I_{n+1}(x)$$

Asymptotic expansion for $x \gg 1$

$$\begin{aligned} J_n(x) &= \left(\frac{2}{\pi x}\right)^{1/2} [\cos(x - \pi n/2 - \pi/4)] + O(1/x) \\ I_n(x) &= \frac{\exp(x)}{(2\pi x)^{1/2}} [1 + O(1/x)] \end{aligned}$$

Weber Integrals

The following Weber integrals are used to compute the magnetized plasma response function

$$\begin{aligned} \int_0^\infty x dx J_0(px) \exp\left(-q^2x^2\right) &= \left(2q^2\right)^{-1} \exp\left(-\frac{p^2}{4q^4}\right) \\ \int_0^\infty x dx J_l(px) J_l(rx) \exp\left(-q^2x^2\right) &= \left(2q^2\right)^{-1} \exp\left(\frac{-p^2+r^2}{4q^4}\right) I_l\left(\frac{pr}{2q^2}\right) \end{aligned} \tag{A.10}$$

Appendix B

Appendix

B.1 Derivation of the coefficients in the potential energy for the Z pinch

$$\begin{aligned} U_2 &= \frac{1}{2}\int_V \left[\gamma p \left|\nabla \cdot \boldsymbol{\xi}\right|^2 + (\boldsymbol{\xi}\cdot\nabla p)\,\nabla\cdot\boldsymbol{\xi}^* + \frac{1}{\mu_0}\left|\mathbf{B}_1\right|^2 \right. \\ &\qquad \left. -\mathbf{B}_1\cdot(\boldsymbol{\xi}\times\mathbf{j})\right] d\mathbf{x} \end{aligned}$$

with

$$\begin{aligned} \nabla\cdot\boldsymbol{\xi} &= \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \\ \mathbf{B}_1 = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) &= \left(\frac{1}{r}\frac{\partial G_z}{\partial\theta} - \frac{\partial G_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial G_r}{\partial z} - \frac{\partial G_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial r G_\theta}{\partial r} - \frac{\partial G_r}{\partial\theta}\right)\mathbf{e}_z \\ &= \left(\frac{1}{r}\frac{\partial \xi_r B}{\partial\theta}\right)\mathbf{e}_r + \left(-\frac{\partial \xi_z B}{\partial z} - \frac{\partial B\xi_r}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial \xi_z B}{\partial\theta}\right)\mathbf{e}_z \\ &= B\left[\frac{1}{r}\frac{\partial\xi_r}{\partial\theta}\mathbf{e}_r - \left(\frac{1}{B}\frac{\partial}{\partial r}(B\xi_r) + \frac{\partial\xi_z}{\partial z}\right)\mathbf{e}_\theta + \frac{1}{r}\frac{\partial\xi_z}{\partial\theta}\mathbf{e}_z\right] \\ \boldsymbol{\xi}\times(\nabla\times\mathbf{B}) &= \frac{1}{r}\frac{\partial}{\partial r}(rB)\,(\xi_\theta\mathbf{e}_r - \xi_r\mathbf{e}_\theta) \end{aligned}$$

and assuming no azimuthat dependence

$$\begin{aligned}
\frac{dp}{dr} &= -\frac{B}{\mu_0 r}\frac{d}{dr}(rB) \\
\nabla\cdot\boldsymbol{\xi} &= \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r)+\frac{\partial\xi_z}{\partial z}=\frac{\xi_r}{r}+\frac{\partial\xi_r}{\partial r}+\frac{\partial\xi_z}{\partial z} \\
\frac{1}{B}\frac{\partial}{\partial r}(B\xi_r)+\frac{\partial\xi_z}{\partial z} &= \frac{\xi_r}{B}\frac{\partial B}{\partial r}+\frac{\partial\xi_r}{\partial r}+\frac{\partial\xi_z}{\partial z} \\
&= \xi_r\left(\frac{1}{B}\frac{\partial B}{\partial r}-\frac{1}{r}\right)+\nabla\cdot\boldsymbol{\xi} \\
&= \left(\frac{\partial\ln B}{\partial\ln r}-1\right)\frac{\xi_r}{r}+\nabla\cdot\boldsymbol{\xi}
\end{aligned}$$

$$\begin{aligned}
U &= \frac{1}{2}\int_V\frac{B^2}{2\mu_0}\left[\gamma\beta(\nabla\cdot\boldsymbol{\xi})^2+\beta\frac{\partial\ln p}{\partial\ln r}\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi}\right. \\
&\quad +2\left(\left(\left(\frac{d\ln B}{d\ln r}-1\right)\frac{\xi_r}{r}+\nabla\cdot\boldsymbol{\xi}\right)^2\right) \\
&\quad \left.+\beta\frac{d\ln p}{d\ln r}\left(\left(\frac{d\ln B}{d\ln r}-1\right)\frac{\xi_r^2}{r^2}+\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi}\right)\right]d\mathbf{x} \\
&= \frac{1}{2}\int_V\frac{B^2}{2\mu_0}\left[(\gamma\beta+2)(\nabla\cdot\boldsymbol{\xi})^2\right. \\
&\quad +2\left(2\left(\frac{d\ln B}{d\ln r}-1\right)+\beta\frac{d\ln p}{d\ln r}\right)\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi} \\
&\quad \left.+\left(\frac{d\ln B}{d\ln r}-1\right)\left[\beta\frac{d\ln p}{d\ln r}+2\left(\frac{d\ln B}{d\ln r}-1\right)\right]\frac{\xi_r^2}{r^2}\right]d\mathbf{x}
\end{aligned}$$

Collecting the coefficients for $(\nabla\cdot\boldsymbol{\xi})$ and ξ_r/r terms leads to:

$$U=\frac{1}{2}\int_V\left[a_{11}(\nabla\cdot\boldsymbol{\xi})^2+2a_{12}\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi}+a_{22}\frac{\xi_r^2}{r^2}\right]d\mathbf{x}$$

where the coefficients are

$$a_{11}=\gamma\beta+2 \qquad \beta=\frac{2\mu_0 p}{B^2}$$

$$\begin{aligned}
a_{12} &= 2\left(\frac{d\ln B}{d\ln r}+\frac{\beta}{2}\frac{d\ln p}{d\ln r}-1\right) \\
a_{22} &= \left(\frac{d\ln B}{d\ln r}-1\right)a_{12}
\end{aligned}$$

and stability requires

$$a_{11}a_{22}-a_{12}^2>0$$

Equilibrium condition

$$\frac{d\ln B}{d\ln r} = -\frac{\beta}{2}\frac{d\ln p}{d\ln r} - 1$$

such that $a_{12} = -4$

$$\begin{aligned}
a_{11}a_{22}/a_{12} - a_{12} &= (\gamma\beta + 2)\left(\frac{d\ln B}{d\ln r} - 1\right) - 2\left(\frac{d\ln B}{d\ln r} + \frac{\beta}{2}\frac{d\ln p}{d\ln r} - 1\right) \\
&= (\gamma\beta + 2)\left(-\frac{\beta}{2}\frac{d\ln p}{d\ln r} - 2\right) + 4 \\
&= -(\gamma\beta + 2)\frac{\beta}{2}\frac{d\ln p}{d\ln r} - 2\gamma\beta
\end{aligned}$$

such that

Therefore the condition for stability (with a_{12} being negative) is

$$\begin{aligned}
-(\gamma\beta + 2)\frac{\beta}{2}\frac{d\ln p}{d\ln r} - 2\gamma\beta &< 0 \\
&\mathit{or} \\
-\frac{d\ln p}{d\ln r} &< \frac{4\gamma}{\gamma\beta + 2}
\end{aligned}$$

with

$$\begin{aligned}
p(r) &= -\frac{\mu_0 j_0^2}{4}r^2 + p_0 \\
B &= \frac{\mu_0 j_0}{2}r \\
\beta &= \frac{2\mu_0 p}{B^2} = \frac{-\mu_0 j_0^2 r^2 + 4p_0}{4}\frac{8\mu_0}{\mu_0^2 j_0^2 r^2} = \frac{2\mu_0 p_0}{B^2} - 2 \\
-\frac{d\ln p}{d\ln r} &= \frac{r}{p}\frac{\mu_0 j_0^2}{2}r = \frac{2B^2}{\mu_0 p} = \frac{4}{\beta}
\end{aligned}$$

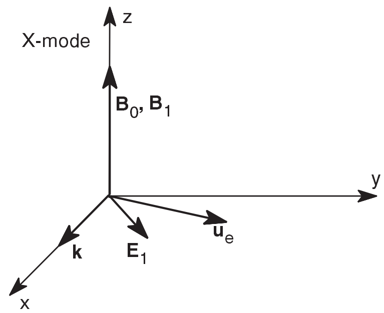
or

$$\begin{aligned}
\frac{\gamma\beta + 2}{\beta} &< \gamma \\
&\mathit{or} \\
\gamma\beta + 2 &< \gamma\beta
\end{aligned}$$

This relation cannot be satisfied and demonstrates that the configuration is always unstable for constant current density.

B.2 X-mode dispersion relation

X(extraordinary)-mode: If the electric field is in the the x, y plane the electric field generates a velocity $\mathbf{u}$ in the x, y plane and the $\mathbf{u} \times \mathbf{B}_0$ term generates a component perpendicular to the original velocity and perpendicular to $\mathbf{B}_0$ such that this velocity is also entirely in the x, y plane. Thus we do not need to consider the z component of the momentum equation. This illustrates that the ordinary and extraordinary modes separate. In this case with $\mathbf{E}_1 = E_x \mathbf{e}_x + E_y \mathbf{e}_y$ the components of the linear equations are



$$
\begin{aligned}
-i\omega m_e u_x &= -eE_x - eu_y B_0 \\
-i\omega m_e u_y &= -eE_y + eu_x B_0 \\
ikE_y &= i\omega B_1 \\
0 &= -\mu_0 e n_0 u_x - \frac{i\omega}{c^2} E_x \\
-ikB_1 &= -\mu_0 e n_0 u_y - \frac{i\omega}{c^2} E_y
\end{aligned}
$$

These are 5 equations for 5 unknowns. Using the last three equations we express the velocities in terms of the electric field

$$
\begin{aligned}
u_x &= -\frac{i\omega}{\mu_0 e n_0 c^2} E_x \\
u_y &= -\frac{i\omega}{\mu_0 e n_0 c^2} E_y + \frac{ik}{\mu_0 e n_0} B_1 = -\frac{i\omega}{\mu_0 e n_0 c^2} \left(1 - \frac{c^2 k^2}{\omega^2}\right) E_y
\end{aligned}
$$

Substitution in the first two equations

$$
\begin{aligned}
-i\omega m_e u_x + eE_x + eB_0 u_y &= 0 \\
-eB_0 u_x - i\omega m_e u_y + eE_y &= 0
\end{aligned}
$$

yields

$$
\begin{aligned}
\left(-\frac{\omega^2 m_e}{\mu_0 e n_0 c^2} + e\right) E_x - i\frac{\omega B_0}{\mu_0 c^2 n_0} \left(1 - \frac{c^2 k^2}{\omega^2}\right) E_y &= 0 \\
i\frac{\omega B_0}{\mu_0 c^2 n_0} E_x - \frac{\omega^2 m_e}{\mu_0 e n_0 c^2} \left(1 - \frac{c^2 k^2}{\omega^2}\right) E_y + eE_y &= 0
\end{aligned}
$$

which is easily modified to

$$\begin{aligned} \left(-\frac{\omega^2}{\omega_{pe}^2}+1\right)eE_x - i\frac{\omega\omega_{ge}}{\omega_{pe}^2}\left(1-\frac{c^2k^2}{\omega^2}\right)eE_y &= 0 \\ i\frac{\omega\omega_{ge}}{\omega_{pe}^2}eE_x - \left[\frac{\omega^2}{\omega_{pe}^2}\left(1-\frac{c^2k^2}{\omega^2}\right)-1\right]eE_y &= 0 \end{aligned}$$

For nontrivial solutions the determinant must vanish such that the dispersion relation is

$$\left(1-\frac{\omega^2}{\omega_{pe}^2}\right)\left[1-\frac{\omega^2}{\omega_{pe}^2}+\frac{c^2k^2}{\omega_{pe}^2}\right]-\frac{\omega^2\omega_{ge}^2}{\omega_{pe}^4}+\frac{\omega_{ge}^2c^2k^2}{\omega_{pe}^4}=0$$

With some manipulation we can rewrite the dispersion relation as

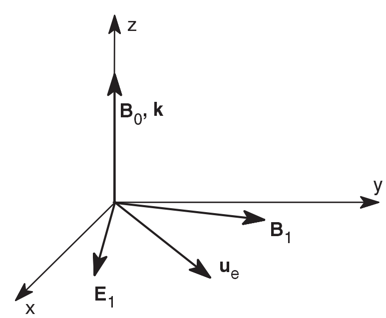
$$n^2=\frac{k^2c^2}{\omega^2}=1-\frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2-\omega_{pe}^2}{\omega^2-\omega_{uh}^2}$$

with $\omega_{uh}^2=\omega_{pe}^2+\omega_{ge}^2$ for the extraordinary or X-mode. Here n is the index of refraction

B.3 Dispersion relation for electromagnetic wave along B_0

To discuss electromagnetic waves along $\mathbf{B}_0$ we use the same set of linear equations as in the case of waves perpendicular to B_0 which for the plane wave solutions assume the form

$$\begin{aligned} i\omega m_e n_0\mathbf{u} &= -en_0\mathbf{E}-en_0\mathbf{u}\times\mathbf{B}_0 \\ i\mathbf{k}\times\mathbf{E} &= i\omega\mathbf{B} \\ i\mathbf{k}\times\mathbf{B} &= -\mu_0 e n_0\mathbf{u}-\frac{i\omega}{c^2}\mathbf{E} \end{aligned}$$



and $\mathbf{E}=\mathbf{E}_1$, $\mathbf{B}=\mathbf{B}_1$, $\mathbf{u}=\mathbf{u}_{e1}$, and $\mathbf{B}_0=B\,\mathbf{e}_z$. For $\mathbf{k}=k\,\mathbf{e}_z$ a consistent solution can be found by assuming that all perturbations $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{u}$ are in the x y plane. Writing out the components of the linear equations

$$\begin{aligned} -i\omega m_e u_x &= -eE_x-eu_yB_0 \\ -i\omega m_e u_y &= -eE_y+eu_xB_0 \\ -ikE_y &= i\omega B_x \\ ikE_x &= i\omega B_y \\ -ikB_y &= -\mu_0 en_0u_x-\frac{i\omega}{c^2}E_x \\ ikB_x &= -\mu_0 en_0u_y-\frac{i\omega}{c^2}E_y \end{aligned}$$

Using the induction equation we can substitute B_x and B_y in the last two equations.

$$\begin{aligned} -i\frac{k^2}{\omega}E_x &= -\mu_0 e n_0 u_x - \frac{i\omega}{c^2}E_x \\ -i\frac{k^2}{\omega}E_y &= -\mu_0 e n_0 u_y - \frac{i\omega}{c^2}E_y \end{aligned}$$

with the result

$$\begin{aligned} u_x &= -\frac{i\omega}{c^2\mu_0 e n_0}\left(1-\frac{c^2k^2}{\omega^2}\right)E_x \\ u_y &= -\frac{i\omega}{c^2\mu_0 e n_0}\left(1-\frac{c^2k^2}{\omega^2}\right)E_y \end{aligned}$$

Substitution into the momentum (first two) equations

$$\begin{aligned} \left[e-\frac{m_e\omega^2}{c^2\mu_0 e n_0}\left(1-\frac{c^2k^2}{\omega^2}\right)\right]E_x - i\frac{i\omega B_0}{c^2\mu_0 n_0}\left(1-\frac{c^2k^2}{\omega^2}\right)E_y &= 0 \\ -i\frac{\omega B_0}{c^2\mu_0 n_0}\left(1-\frac{c^2k^2}{\omega^2}\right)E_x + \left[e-\frac{m_e\omega^2}{c^2\mu_0 e n_0}\left(1-\frac{c^2k^2}{\omega^2}\right)\right]E_y &= 0 \end{aligned}$$

or in matrix form

$$\left[\begin{array}{cc} 1-\frac{\omega^2}{\omega_{pe}^2}+\frac{c^2k^2}{\omega_{pe}^2} & -i\frac{\omega\omega_{ge}}{\omega_{pe}^2}\left(1-\frac{c^2k^2}{\omega^2}\right) \\ -i\frac{\omega\omega_{ge}}{\omega_{pe}^2}\left(1-\frac{c^2k^2}{\omega^2}\right) & 1-\frac{\omega^2}{\omega_{pe}^2}+\frac{c^2k^2}{\omega_{pe}^2} \end{array}\right]\left(\begin{array}{c} E_x \\ E_y \end{array}\right) = 0$$

Setting the determinant to zero yields

$$\left(1-\frac{\omega^2}{\omega_{pe}^2}+\frac{c^2k^2}{\omega_{pe}^2}\right)^2 = \frac{\omega_{ge}^2}{\omega_{pe}^4}\left(1-\frac{c^2k^2}{\omega^2}\right)^2$$

or by taking the square root

$$1-\frac{\omega}{\omega_{pe}^2}\left(\omega-\frac{c^2k^2}{\omega}\right) = \pm\frac{\omega_{ge}}{\omega_{pe}^2}\left(\omega-\frac{c^2k^2}{\omega}\right)$$

or

$$1 = \frac{1}{\omega_{pe}^2}\left(\omega \pm \omega_{ge}\right)\left(\omega-\frac{c^2k^2}{\omega}\right)$$

B.4 Magnetohydrodynamic Waves

In the following we want to examine typical waves in the single fluid plasma. The treatment is similar to for instance sound waves but clearly one expects that the magnetofluid has more degrees of freedom and wave propagation should depend on the orientation of the magnetic field, i.e. it is not isotropic as in the case of sound waves. Note that in the cause of waves in gravitationally stratified atmosphere wave propagation is also anisotropic and gravity allows for a new wave mode the so-called gravity wave.

In the case of MHD we start from the full set of linearized ideal MHD equations.

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) \\
\frac{\partial p}{\partial t} &= -\mathbf{u} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{u}
\end{aligned}
$$

where we have already combined the induction equation and Ohm's law. Now we linearize the equation as in the previous section, however, we do not introduce a small displace but rather keep $\mathbf{u}_1$ as a variable. Specifically the MHD variables are expressed as

$$
\begin{aligned}
\rho &= \rho_0 + \rho_1 \\
\mathbf{u} &= \mathbf{u}_1 \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1 \\
p &= p_0 + p_1
\end{aligned}
$$

For simplicity we assume all equilibrium quantities to be constant, i.e., we do not consider and inhomogeneous plasma. Substitution of the perturbations into the MHD equations yields similar to the small displacement treatment

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\rho_0 \nabla \cdot \mathbf{u}_1 \\
\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} &= -\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \\
\frac{\partial p_1}{\partial t} &= -\gamma p_0 \nabla \cdot \mathbf{u}_1
\end{aligned}
$$

Taking the second derivative in time of the momentum equation yields

$$\rho_0 \frac{\partial^2 \mathbf{u}_1}{\partial t^2} = -\nabla \frac{\partial p_1}{\partial t} + \frac{1}{\mu_0}\left(\nabla \times \frac{\partial \mathbf{B}_1}{\partial t}\right) \times \mathbf{B}_0$$
$$= \gamma p_0 \nabla \left(\nabla \cdot \mathbf{u}_1\right) + \frac{B_0^2}{\mu_0}\left(\nabla \times \left(\nabla \times \left(\mathbf{u}_1 \times \mathbf{e}_B\right)\right)\right) \times \mathbf{e}_B$$

where we used $\mathbf{B}_0 = B_0 \mathbf{e}_B$. Now dividing by ρ_0 and with the definitions for the speed of sound c_s and the Alfven speed v_A

$$c_s^2 = \frac{\gamma p_0}{\rho}$$
$$v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$$

we obtain

$$\frac{\partial^2 \mathbf{u}_1}{\partial t^2} = c_s^2 \nabla \left(\nabla \cdot \mathbf{u}_1\right) + v_A^2 \left(\nabla \times \left(\nabla \times \left(\mathbf{u}_1 \times \mathbf{e}_B\right)\right)\right) \times \mathbf{e}_B$$

We will now try to find the solution as plane waves by assuming

$$\mathbf{u}_1 = \widehat{\mathbf{u}}_1 \exp\left(i\mathbf{k} \cdot \mathbf{x} - i\omega t\right)$$

which leads us to

$$\omega^2 \mathbf{u}_1 = c_s^2 \mathbf{k}\left(\mathbf{k} \cdot \mathbf{u}_1\right) - v_A^2 \mathbf{e}_B \times \left(\mathbf{k} \times \left(\mathbf{k} \times \left(\mathbf{u}_1 \times \mathbf{e}_B\right)\right)\right)$$

We now choose a coordinate system where the z axis points along the equilibrium (constant) magnetic field and the $\mathbf{k}$ vector is in the x, z plane and given by

$$\mathbf{k} = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z$$

This leads to the following three components for the equation:

$$\left(\omega^2 - v_A^2 k_\parallel^2 - \left(c_s^2 + v_A^2\right) k_\perp^2\right) u_{1x} - c_s^2 k_\parallel k_\perp u_{1z} = 0$$
$$\left(\omega^2 - v_A^2 k_\parallel^2\right) u_{1y} = 0$$
$$\left(\omega^2 - c_s^2 k_\parallel^2\right) u_{1z} - c_s^2 k_\parallel k_\perp u_{1x} = 0$$

or in matrix form $\underline{\underline{\mathbf{W}}} \cdot \mathbf{u}_1 = 0$

$$\left[\begin{array}{ccc} \omega^2 - v_A^2 k_\parallel^2 - \left(c_s^2 + v_A^2\right) k_\perp^2 & 0 & -c_s^2 k_\parallel k_\perp \\ 0 & \omega^2 - v_A^2 k_\parallel^2 & 0 \\ -c_s^2 k_\parallel k_\perp & 0 & \omega^2 - c_s^2 k_\parallel^2 \end{array}\right] \left(\begin{array}{c} u_{1x} \\ u_{1y} \\ u_{1z} \end{array}\right) = 0$$

Exercise: Derive the above matrix.

These equations are linear independent and provide nontrivial solutions if the determinant is equal to zero.

a) Alfvén wave:

The first nontrivial solution is determined by the y component of the above equations. This component separates from the other two because u_{1y} does not appear in the first and third equation. In the determinant $\omega^2 - v_A^2 k_\parallel^2$ appears a a common factor. The solution is

$$\omega^2 = v_A^2 k_\parallel^2 = \frac{k_\parallel^2 B_0^2}{\mu_0 \rho_0}$$

or

$$\omega = \pm \frac{\mathbf{k} \cdot \mathbf{B}_0}{(\mu_0 \rho_0)^{1/2}} = \pm k v_A \cos\theta$$

The group velocity $\mathbf{v}_g = d\omega / d\mathbf{k}$ is

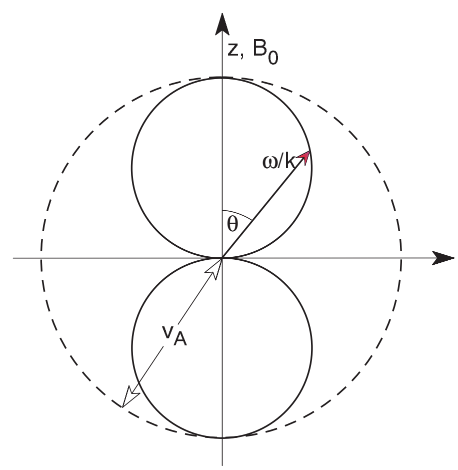
$$\mathbf{v}_g = \pm \frac{\mathbf{B}_0}{(\mu_0 \rho_0)^{1/2}} = \pm v_A \mathbf{e}_B$$

which is independent of $\mathbf{k}$ and always along the magnetic field. The phase velocity is

$$v_{ph} = \frac{\omega}{k} = \pm v_A \cos\theta$$

where theta is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. This wave is the Alf\ven wave.

A common representation of phase and group velocities is the Clemmov-Mullaly-Allis diagram which represents phase and group velocities in a polar plot where the radius vector represents the magnitude of the velocity and the angle is the propagation angle theta. Note that the direction of the group velocity for Alfv\'en waves (which is the direction in which energy and momentum is carried) is always along the magnetic field! Note also that the case for $\theta = 90°$ is singular in that wave does not propagate and the points with $\theta = 90°$ should be excluded from the group velocity plot.

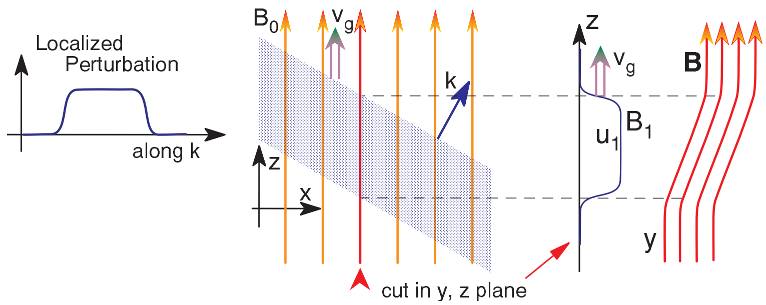


Further properties of Alfvén waves:

- Velocity perturbation $\mathbf{u}_1 \perp \mathbf{B}_0$
- Wave is incompressible: $\nabla \cdot \mathbf{u}_1 = i\mathbf{k} \cdot \mathbf{u}_1 = 0$ or $\mathbf{u}_1 \perp \mathbf{k}$
- Magnetic field perturbation: $\omega \mathbf{B}_1 = -\mathbf{k} \times (\mathbf{u}_1 \times \mathbf{B}_0) = -k_\parallel \mathbf{u}_1$, i.e., the magnetic field perturbation is aligned with the velocity perturbation (and both are perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$.

- Current density $\mathbf{j}_1 \sim \mathbf{k} \times \mathbf{B}_0$, i.e., the current perturbation is perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$.
- Alfvén wave also exist without steepening with nonlinear amplitudes.

Considering a localized perturbation for $\mathbf{B}_1$ along the $\mathbf{k}$ vector propagation of an Alfven wave and the deformation of the magnetic field is presented in the following figure.



The Alfvén wave is of large importance in many plasma systems. It is very effective in carrying energy and momentum along the magnetic field. Among many space plasma applications the Alfvén wave is particularly important for the coupling between the magnetosphere and the ionosphere. Here perturbations (convection) from the magnetosphere are carried into the ionosphere and depending on ionospheric conditions cause convection in the high latitude ionosphere.

A coomon diagnostic to identify Alfvén waves in space physics is the use of the so-called the Walen relation.

$$\Delta \mathbf{u} = \pm \Delta \mathbf{v}_A = \pm \Delta \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}}$$

This relation is valid for linear and nonlinear Alfvén waves, and can also be used to identify the corresponding (nonlinear) discontinuities.

b) **Fast and slow mode:**

The other nontrivial solution is given by $\det \underline{\underline{\mathbf{W}}} = 0$ or

$$\left(\omega^2 - v_A^2 k_\parallel^2 - \left(c_s^2 + v_A^2\right) k_\perp^2\right)\left(\omega^2 - c_s^2 k_\parallel^2\right) - c_s^4 k_\parallel^2 k_\perp^2 = 0$$

This is the dispersion relation for 4 additional solutions which are given by the roots of the equation. Since the equation is double quadratic, i.e., depends only on ω^2 there are two different solutions for ω^2. Abbreviating $c_f^2 = c_s^2 + v_A^2$ (where the f stands for fast mode - the reason for this will be clear in a moment) one can re-write the equation as

$$\omega^4 - c_f^2 k^2 \omega^2 + v_A^2 c_s^2 k_\parallel^2 k^2 = 0 \tag{B.1}$$

with the solutions

$$\omega^2 = \frac{k^2}{2}\left\{c_f^2 \pm \left[c_f^4 - 4v_A^2 c_s^2 \frac{k_\parallel^2}{k^2}\right]^{1/2}\right\}$$
$$= \frac{k^2}{2}\left\{v_A^2 + c_s^2 \pm \left[\left(v_A^2 + c_s^2\right)^2 - 4v_A^2 c_s^2 \cos^2\theta\right]^{1/2}\right\}$$

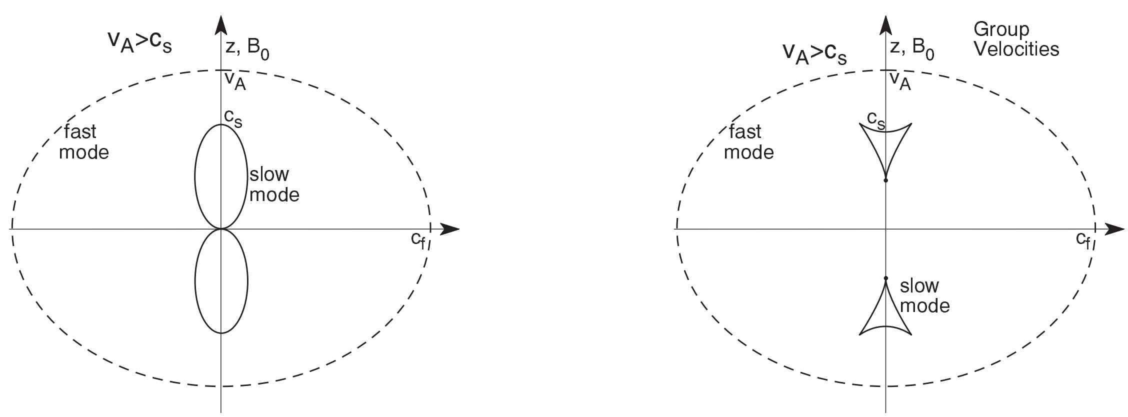
Here the solution with the $+$ sign is called the fast mode and the solution with the $-$ sign the slow mode.

Fast wave:

This is the MHD wave with the fastest propagation speed, i.e., information in an MHD plasma cannot travel faster thant this velocity.

- Phase speed along the magnetic field: $\omega^2/k^2 = \max\left[v_A^2,\, c_s^2\right]$
- Phase speed perpendicular to the magnetic field $\omega^2/k^2 = c_f^2 \equiv v_A^2 + c_s^2$
- Limit for $c_s^2 \ll v_A^2$: $\omega^2 - v_A^2 k^2 = 0$ => the fast mode becomes the so-called compressional Afvén wave. Note that the compressional Alfven waves has a group velocity equal to its phase velocity (different from the regular or shear Alfven wave).
- Limit for $v_A^2 \ll c_s^2$: $\omega^2 - c_s^2 k^2 = 0$ => the fast mode becomes a sound wave.

The most prominant example of a fast wave in space physics is the Earth's bow shock. This shock forms as a result of the solar wind velocity which is faster than the fast mode speed. The fast wave is also important in terms of transporting energy perpendicular to the magnetic field. The slow mode group velocity is as for the Alfven wave along the magnetic feild and it is comparably small. Thus the fast wave is the only MHD wave able to carry energy perpendicular to the magnetic field.



Slow wave:

The slow wave is the third of the basic MHD waves. The phase velocity is always smaller than or equal to $\min\left[v_A^2,\, c_s^2\right]$. Basic properties:

Phase speed along the magnetic field: $\omega^2/k^2 = \min\left[v_A^2,\, c_s^2\right]$.

Phase speed perpendicular to the magnetic field $\omega^2/k^2 = 0$.

In the limits of $c_s^2 \ll v_A^2$ and $v_A^2 \ll c_s^2$ the slow wave disappears.

Applications of the slow wave are not as eye catching than those for the other MHD wave but the wave is important to explain plasma structure in the region between the bow shock and the magnetosphere, and well know for applications such as magnetic reconnection.

Concluding remarks:

There is an additional wave which in MHD which is the sound wave. This is obtained by using $\mathbf{k}_\perp = 0$ and $\mathbf{u}_\perp = 0$ such that the velocity perturbation is along the magnetic field and thus the magnetic field perturbation is 0. However, it is singular in that it exists only for propagation exactly along the magnetic field, i.e. the velocity is exactly parallel to $\mathbf{B}_0$ and the $\mathbf{k}$ vector is also exactly parallel to $\mathbf{B}_0$. In fact the sound wave can be either part of the slow wave or part of the fast wave solution depending on whether $c_s^2 < v_A^2$ or $c_s^2 > v_A^2$.

Exercise: Derive the dispersion relations for $\mathbf{k}$ parallel to $\mathbf{B}_0$. What are the three wave solutions.

It should also be remarked that similar to a sonic shock wave the MHD waves are associated with respective shocks or discontinuities, I.e., there is a fast shock for the transition from plasma flow that is super-fast (faster than the fast mode speed) to sub-fast (slower than the fast mode speed), a slow shock, and an intermediate shock corresponding the Alfvén wave.