Chapter 1

Introduction and Review of Basic Plasma Properties

1.1 Preliminaries

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- Important: participation, criticism, and suggestions
- Scope of the course and contents (see handout or web page)
  - Plasma Kinetic Equations and Collisions
  - Kinetic (Warm Plasma) Dispersion Relation
    * Electrostatic Plasma Waves
    * Electromagnetic Plasma Waves
  - Fluid and Kinetic Plasma Equilibria and Steady States
  - Fluid Plasma Properties
  - Micro- (Kinetic) Instabilities
    * Concept of Instability (Bunemann and Beam instabilities)
    * Electrostatic Instabilities
    * Electromagnetic Instability
    * Drift waves

- Macro-Instabilities
  - Rayleigh-Taylor, Kelvin-Helmholtz, Firehose, Mirror
  - Tearing mode
- Magnetic Reconnection
  - Simple Models of Reconnection
  - Effects of Hall Physics and Dissipation
  - Three-Dimensional Reconnection and Magnetic Topology
- Nonlinear Waves
- Turbulence and Collective Effects (Anomalous resistivity, Diffusion, Particle Acceleration)

- Conduct (see handout or web page)
  - Textbooks, some lecture notes on the web, but emphasis on lecture notes
  - Homework: analytical
  - Grading
  - Midterm test and final exam
- Questions

1.2 Review of Basic Plasma Properties

1.2.1 Debye Shielding and Plasma Parameter

Plasma definition: A plasma is a gas of charged particles, which consists of 'free' positive and/or negative charge carriers.

- A plasma is a (partially) ionized gas in which the potential energy of a particle due to its nearest neighbor force is much smaller than its kinetic energy.

Debye Length

Coulomb potential of a test charge q_t at $\mathbf{r} = 0$:

$$\phi_D = q_t/(4\pi\varepsilon_0 r). \tag{1.1}$$

In a plasma the test charge (of infinite mass) modifies the distribution of the particles in its vicinity.

Distribution function from equilibrium statistical mechanics

$$f_s(\mathbf{r}, \mathbf{v}) = const \exp(-H_s/k_B T_s) \tag{1.2}$$

with Hamiltonian $H_s = m_s v^2/2 + q_s \Phi$ yields the density distributions

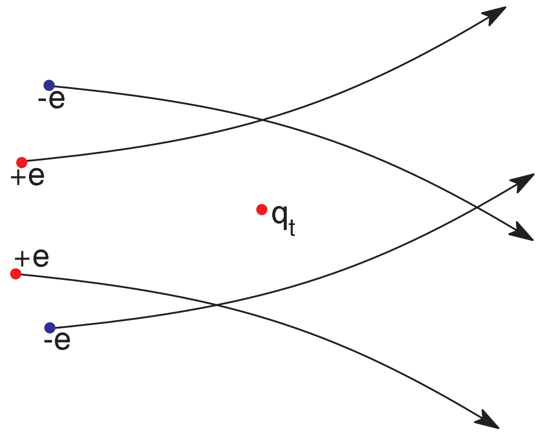

Figure 1.1: Deflection of Charged particles around the test charge q_t.

$$
\begin{aligned}
n_i &= n_0 \exp(-e\Phi/k_B T_e) \\
n_e &= n_0 \exp(e\Phi/k_B T_i)
\end{aligned}
$$

Poisson's equation:

$$
-\nabla^2 \Phi = \frac{1}{\varepsilon_0} \rho_c(\mathbf{r}, t) = \frac{1}{\varepsilon_0} \left[q_t \delta(\mathbf{r}) + e(n_i - n_e) \right] \tag{1.3}
$$

Assume $e\Phi \ll \{k_B T_e, k_B T_i\}$ such that $\exp(e\Phi/k_B T_e) \approx 1 + e\Phi/k_B T_e$

=>

$$
\nabla^2 \Phi = -\frac{1}{\varepsilon_0} q_t \delta(\mathbf{r}) + \frac{1}{\lambda_D^2} \Phi(\mathbf{r})
$$

with the **Debye length** as

$$
\lambda_D^{-2} = \frac{n_0 e^2}{\varepsilon_0 k_B} \left(\frac{1}{T_e} + \frac{1}{T_i} \right) \tag{1.4}
$$

At large distances $\Phi \propto \exp(-r/\lambda_D)$

Full solution:

$$
\phi_D = \frac{q_t}{4\pi\varepsilon_0 r} \exp\left(-\frac{r}{\lambda_D} \right) \tag{1.5}
$$

This potential is sometimes called the Yukawa potential. In comparison to the coulomb potential the Yukawa potential converges much faster to 0 for length scales larger than the Debye length due to the exponential factor. Thus the electric field tends to 0 much faster or in other words the electric field from the test charge is effectively shielded at distances larger than the Debye length.

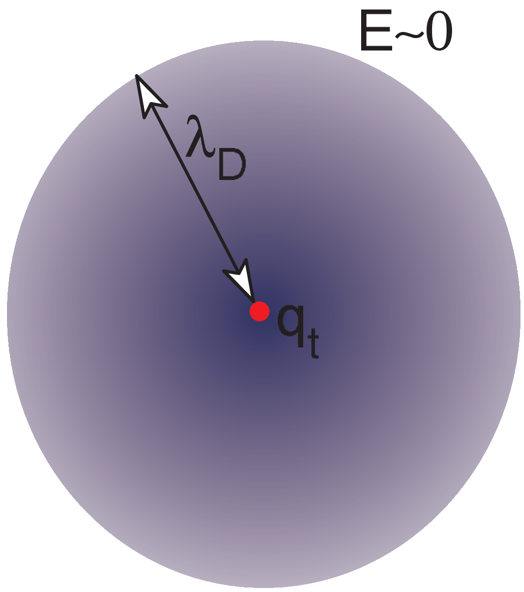

Figure 1.2: Illustration of the Debye sphere which shields the individual particle charge from its vicinity.

Defining the ion/electron Debye length $\lambda_{Di,e} = \left(\frac{\varepsilon_0 k_B T_{i,e}}{n_0 e^2}\right)^{1/2}$ we have $\lambda_D^{-2} = \lambda_{Di}^{-2} + \lambda_{De}^{-2}$.

=> Quasi-neutrality for any physical length $L \gg \lambda_D$ otherwise binary interaction should be considered.

Exercise: Compute the net charge of the shielding cloud.

The Plasma Parameter:

Using the average potential

$$\langle\Phi\rangle \sim \frac{e^2}{4\pi\varepsilon_0 \langle r\rangle} \sim \frac{n_0^{1/3} e^2}{4\pi\varepsilon_0}$$

and average kinetic energy with the thermal speed $v_s = (k_B T_s/m_s)^{1/2}$:

$$\langle E_{kin,s}\rangle = \frac{1}{2} m_s \langle v^2\rangle = \frac{3}{2} k_B T_s \equiv \frac{3}{2} m_s v_s^2$$

it is easy to demonstrate using $\langle E_{kin,s}\rangle \gg \langle\Phi\rangle$ that the so-called plasma parameter satisfies

$$\Lambda_s = n_0 \lambda_{Ds}^3 \gg 1 \tag{1.6}$$

which is implies that the number of particles in a Debye sphere $N = \frac{4\pi}{3} n_0 \lambda_D^3$ is much larger than unity. This is consistent with the shielding. A considerable shielding of individual charges can occur only on the Debye length if there are sufficient charges in the Debye sphere of each individual particle.

Exercise: Calculate the electron thermal speed, Debye length, and the plasma parameter for
(a) a tokamak plasma with $T_e = 10^8$ K, $n_0 = 10^{19}$ m^{-3}
(b) the tail magnetosphere with $T_e = 10^7$ K, $n_0 = 10^6$ m^{-3}
(b) the ionosphere with $T_e = 10^3$ K, $n_0 = 10^{12}$ m^{-3}
(b) the solar atmosphere with $T_e = 10^4$ K, $n_0 = 10^{20}$ m^{-3}
(b) a laser fusion plasma with $T_e = 10^7$ K, $n_0 = 10^{29}$ m^{-3}

Exercise: Can a plasma be maintained at temperatures of $T_e = 100$ K. Hint: Calculate the density limit using the plasma parameter and explain.

Exercise: $\Lambda \propto n_0^{-1/2} T^{3/2}$ While the dependence on temperature seems intuitively clear the density dependence appears odd because lower densities mean less particles and less shielding. Why does the plasma parameter improve (increase) with decreasing density?

1.2.2 Other Typical Plasma Frequencies and Length Scales

Plasma and Gyro Frequency

Consider an infinite slab of electrons and ion with a width of L (in x) and particle density of n_0. Assume that the electrons are displaced by a small distance $\xi \ll L$ in the x direction.

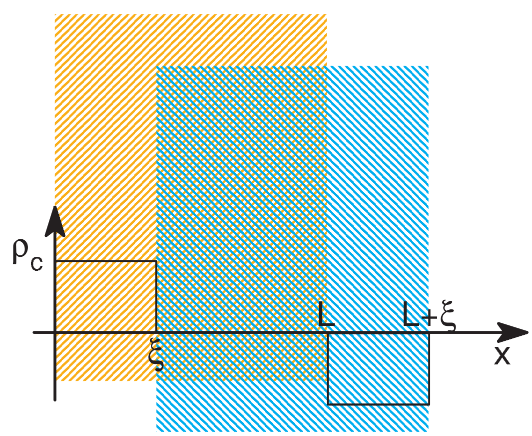

Figure 1.3: Illustration of the displacement between electrons and ions to illustrate plasma oscillations.

This creates two regions of nonzero charge density. Evaluating the resulting force for the electrons yields a harmonic oscillation with the electron **plasma frequency**:

$$\omega_{pe} = \left(\frac{n_0 e^2}{m_e \varepsilon_0} \right)^{1/2} \tag{1.7}$$

Similarly one can define an ion plasma frequency, however, because of the much larger inertia the basic plasma frequency is given by the electron plasma frequency.

Another important set of frequencies in many (electromagnetic) kinetic applications are the electron and ion gyro frequencies:

$$\omega_{gs} = \frac{q_s B}{m_s}$$

Gyro and Inertial Length Scales

In addition to the Debye length there are two other intrinsic plasma length scale which are associated with the frequency of the physical process. The ion and electron plasma frequencies define the ion inertial and electron inertial length scales (or plasma skin depth)

$$\lambda_s = \frac{c}{\omega_{ps}}$$

The second set of length scales in a typical plasma is the gyro-radius

$$r_{gs} = \frac{v_{ths}}{\omega_{gs}}$$

Both, the inertial and the gyro-scales are much larger for ions. The ordering of plasma processes according to temporal and length scales helps to introduce a hierarchy which can identify the important physics on the respective scales.

Coulomb Collision Frequency and Mean Free Path

The name is actually a bit misleading because the total scattering cross section for coulomb collisions diverges for the large numbers of particles which have large collision impact parameters. However, keeping in mind that electric charges are strongly shielded on the Debye scale impact parameters larger the the Debye length do not contribute to the collision process. The resulting collision frequency is

$$\nu_c = \sqrt{\frac{\pi}{2}} \frac{n_0 e^4}{32\pi\varepsilon_0^2 m^{1/2} (k_B T)^{3/2}} \ln[12\pi\Lambda] \tag{1.8}$$

Exercise: Compute $\langle v^3 \rangle$

It is interesting to note that using the plasma frequency and the plasma parameter

$$\frac{\nu_c}{\omega_{pe}} = \sqrt{\frac{\pi}{2}} \frac{1}{32\pi} \frac{1}{\Lambda} \ln[12\pi\Lambda]$$

For $\Lambda \gg 1$ => $\nu_c/\omega_p \ll 1$

Binary collisions are less important than collective plasma effects! Here the term $\ln[12\pi\Lambda]$ is called the Coulomb logarithm. No coincidence: The plasma parameter (and powers thereof) is the only possibility to create a dimensionless parameter that is a function of m, e, n, and T.

Exercise: Compute the mean free path and the ratio of the mean free path to the Debye length.

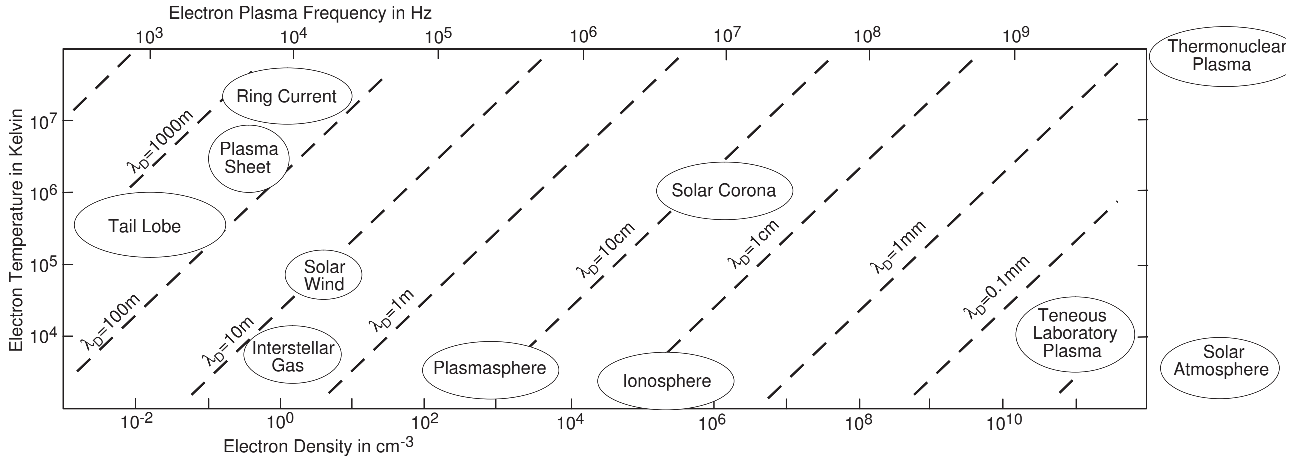

Figure 1.4: Plasma parameters.

1.2.3 Plasma in a Fluid Limit

Consider limit of $m, e \sim \varepsilon \to 0$ under the constraint of the following conserved properties:

- mass density: mn_0
- charge density: en_0
- kinetic energy density: $n_0 k_B T$

=> $n_0 \sim 1/m$, $e \sim m$, and $T \sim 1/n_0 \sim m$

In this limit discreteness vanishes and fluid-like properties survive and

$$\lambda_D = \left(\frac{\varepsilon_0 k_B n_0 T}{n_0^2 e^2}\right)^{1/2}$$

$$\omega_p = \left(\frac{n_0^2 e^2}{m n_0 \varepsilon_0}\right)^{1/2}$$

Debye length and plasma frequency remain unchanged.

Plasma parameter: $\Lambda = n_0 \lambda_D^3 \to \infty$

Collision frequency: $\nu_c \to 0$

Exercise: Determine the fluid limit for the plasma parameter, collision frequency, thermal speed, and gyro frequency eB/m. Discuss the results

1.3 Basic plasma equations

1.3.1 Maxwell's Equations

In general magnetic and electric fields are determined by Maxwell's equations, corresponding boundary conditions and the source (charges and currents) distributions. In a vacuum these equations are

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho_c \tag{1.9}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \tag{1.10}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.11}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.12}$$

where **E** and **B** are electric and magnetic fields.

$c = 3 \cdot 10^8 ms^{-1}$, $\varepsilon_0 = 8.85 \cdot 10^{-12} Fm^{-1}$, and $\mu_0 = 4\pi \cdot 10^{-7} Hm^{-1}$. Sometimes it is convenient to express the electromagnetic fields in terms of an electric potential Φ and a vector potential **A** such that

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t)$$

which requires to solve the electromagnetic field equations for the potentials for instance in the form

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{1}{\varepsilon_0}\rho_c(\mathbf{r},t) \tag{1.13}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}(\mathbf{r},t) \tag{1.14}$$

where Φ and **A** satisfy the Lorentz gauge $\partial\Phi/\partial t + c^2 \nabla \cdot \mathbf{A} = 0$.

Exercise: Derive the equations for the scalar and the vector potentials using the Lorentz gauge.

Exercise: Derive the equations for the scalar and the vector potentials using the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Are the equations for the potential still valid for the Coulomb gauge?

Electromagnetic energy density w and Poynting flux P are

$$w = \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\mathbf{B}^2}{2\mu_0}$$

$$P = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

For completeness, electromagnetic properties in media are often described by introducing the electric displacement and $\mathbf{D}$ and the magnetic field strength $\mathbf{H}$

$$
\begin{aligned}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}
\end{aligned}
$$

with the electric polarization $\mathbf{P}$ and the magnetization $\mathbf{M}$. Maxwell's equations are

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_c \\
\nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0.
\end{aligned}
$$

1.3.2 Lorentz Equations of Motion

In electromagnetic fields the motion of charged particles is determined by the fields through the equations of motion

$$
\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i \tag{1.15}
$$

$$
\frac{d\mathbf{v}_i}{dt} = \frac{q_i}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{r}_i, t}. \tag{1.16}
$$

The electromagnetic forces in a plasma depends on the current and charge densities which are determined by the **collective particle interaction.** In a plasma the number of particles in a physical system is usually rather large. In addition the overwhelming majority of problems deal with the collective particle behavior rather than the individual one. Discrete particle dynamics is important in some areas of plasma physics for sufficiently small ('microscopic') length or time scales.

1.3.3 Basic Kinetic Equations

Boltzmann equation: The most important kinetic description of the collective plasma dynamics is based on the so-called single particle distribution function and the Boltzmann equation: To describe a plasma one can solve the coupled system of Maxwell's equations and the particle equations of motion.

$$
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \left. \frac{\partial f}{\partial t} \right|_{collisions} \tag{1.17}
$$

Here the rhs of the equation considers collisional effects. To solve the Boltzmann equation one needs to evaluate charge and current density from

$$
\begin{aligned}
q_s n_s &= q_s \int_{-\infty}^{\infty} d^3 v f_s(\mathbf{r},\mathbf{v},t) \\
q_s n_s \mathbf{u}_s &= q_s \int_{-\infty}^{\infty} d^3 v \mathbf{v} f_s(\mathbf{r},\mathbf{v},t)
\end{aligned}
$$

and solve Maxwell's equations.

In the case of thermal equilibrium f assumes locally a Maxwell distribution in velocity space and the collision term on the rhs of 1.17 vanishes. Equation 1.17is a fluid (advection) equation though in a 6 dimensional space. The lhs can be interpreted as the total derivative of $f(\mathbf{r},\mathbf{u},t)$ along a trajectory given by the 6 dimensional velocity $\mathbf{v}^{(6)} = (\mathbf{v}, \frac{\mathbf{F}}{m})$ where $\mathbf{F}$ is the Lorentz force.

The collision term on the rhs can consider many different physical or chemical processes. Chemical reactions, ionization or recombination, friction, diffusion, and energy exchange collisions are contained in the collision term. Details depend on the corresponding processes.

Vlasov Equations: Often collisions can be neglected in a high temperature plasma. This is specifically the case if the mean free path is much larger than the size of the system under consideration or if the collision time is much larger than the typical time scale of a plasma process. almost everywhere except for small regions in space. In this case the system of equations is called the Vlasov equations and consist of the collisionless Boltzmann equation

$$
\begin{aligned}
\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} f + \frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}}\cdot f &= 0 \qquad (1.18)\\
+ \textit{Maxwell equations} & \qquad (1.19)
\end{aligned}
$$

Defining the total derivative along the 6-dimensional path $[\mathbf{r},\mathbf{v}]$ by $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \dot{\mathbf{v}}\cdot\nabla_{\mathbf{v}}$ with $\dot{\mathbf{v}} = d\mathbf{v}/dt$ the collisionless Boltzmann equation reduces to $df/dt = 0$. If the forces are derived from a Hamiltonian the collisionless Boltzmann equation is equivalent to

$$
\frac{\partial f}{\partial t} + \nabla_{\mathbf{x}}\cdot(\mathbf{v}f) + \nabla_{\mathbf{v}}\cdot\left(\frac{\mathbf{F}}{m}f\right) = 0 \qquad (1.20)
$$

Note that this is actually the more basic equation because it implies that the number of particles is conserved in any volume unless there is a 6-dimensional particle flux through the boundary of the volume. These equations also imply that the phase space volume for a fixed range of f remains constant for collisionless plasma processes and because Hamiltonian forces imply incompressible dynamics in phase space, i.e. $\nabla_{\mathbf{r}}\cdot\mathbf{v} + \nabla_{\mathbf{v}}\cdot(\mathbf{F}/m) = 0$ as illustrated in Figure 1.5.

Exercise: Consider an ordinary continuity equation $\partial n/\partial t + \nabla\cdot(\mathbf{v}n) = 0$. The number of particles in an arbitrary volume is $N = \int_V n d^3 r$. Show that the number of particles changes only due to particle flux trough the surface of the volume V.

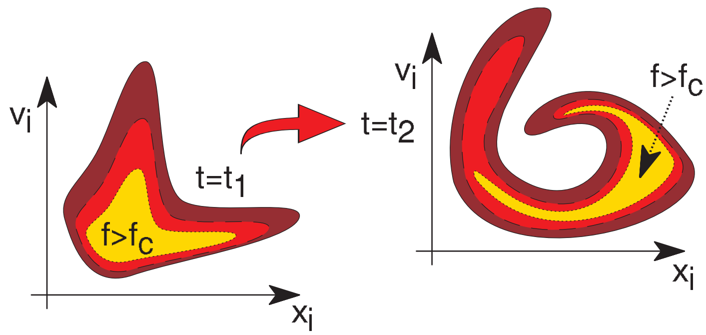

Figure 1.5: Illustration of the evolution of a collisionless distribution conserving the area inside contours of constant f.

1.4 Single Particle Dynamics

1.4.1 Electric and Magnetic Field Drifts

The motion of charged particles in assumed electric and magnetic field can provide insight into many important physical properties of plasmas. While it does not provide the full plasma dynamics it can provide insight into the collective behavior. In many cases the gyro motion of charged particles is much faster than superposed particle drifts or the evolution of the electric and magnetic fields. In these cases the typical particle drifts and respective currents are summarized below.

$$\text{Electric Field}: \quad \mathbf{v}_E = \frac{1}{B^2}\mathbf{E}\times\mathbf{B} \tag{1.21}$$

$$\text{General Force}: \quad \mathbf{v}_F = \frac{1}{qB^2}\mathbf{F}\times\mathbf{B} \tag{1.22}$$

$$\text{Polarisation}: \quad \mathbf{v}_P = \frac{m}{qB^2}\frac{d\mathbf{E}_\perp}{dt} \tag{1.23}$$

$$\text{Curvature}: \quad \mathbf{v}_C = \frac{mv_\parallel^2}{qB^4}\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}] \tag{1.24}$$

$$\text{Gradient}: \quad \mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}(\mathbf{B}\times\nabla B) \tag{1.25}$$

Most drifts are associated with an electric current which is given by $\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e)$.

$$\textit{Polarisation}: \quad \mathbf{j}_P = \frac{n(m_i+m_e)}{B^2}\frac{d\mathbf{E}_\perp}{dt} \tag{1.26}$$

$$\textit{Curvature}: \quad \mathbf{j}_C = \frac{n\left(m_i v_{i\parallel}^2 + m_e v_{e\parallel}^2\right)}{B^4}\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}] \tag{1.27}$$

$$\textit{Gradient}: \quad \mathbf{j}_\nabla = \frac{n\left(m_i v_{i\perp}^2 + m_e v_{e\perp}^2\right)}{2B^3}(\mathbf{B}\times\nabla B) \tag{1.28}$$

These drifts have been determined by model electric and magnetic fields. Thus they describe test particle motion if the electric and magnetic fields were in fact as assumed. However, it should be reminded that the currents due to the drifts alter the fields. If these changes are small compared to the background field it is justified to apply the drift model. The derived particle drifts do not contain any collective behavior. For this reason it is a nontrivial aspect to compare particle and fluid plasma drifts. We will return to this issue in a later chapter.

1.4.2 Magnetic Moment and Adiabatic Invariants

First Adiabatic Invariant

The magnetic moment of a closed current loop is

$$\mu = \frac{1}{2} I \oint_C \mathbf{r} \times d\mathbf{l} = \frac{m v_\perp^2}{2B} \tag{1.29}$$

or periodic motion with a period smaller than changes of the overall system (slowly varying electric and magnetic fields) the action integral

$$J_i = \oint P_i dq_i \tag{1.30}$$

is a constant of motion and an adiabatic invariant. The first adiabatic invariant is associated with the gyro motion with the generalized coordinate $\mathbf{l}$ along the circular particle path and the associated generalized momentum $\mathbf{P}_g = m\mathbf{v}_\perp + q\mathbf{A}$:

$$J_1 = \oint (m\mathbf{v}_\perp + q\mathbf{A}) \cdot d\mathbf{l}$$

Properties

The force associated with the conservation of the magnetic moment is

$$\mathbf{F}_\mu = -\frac{m v_\perp^2}{2B} \nabla B \tag{1.31}$$

Adiabatic (Betatron) heating can occur due to a compression of the magnetic field:

$$W_{\perp 2} = \frac{B_2}{B_1} W_{\perp 1} \tag{1.32}$$

Magnetic mirror motion occurs if a particle moves in an increasing magnetic field. Defining the pitch angle α as the angle between the magnetic field and the particle velocity, the perpendicular component of the particle velocity is $v_\perp = v \sin\alpha$. If the pitch angle at a location 1 is α_1 the pitchangle changes during the motion along the magnetic field as

$$\sin^2\alpha = \frac{B}{B_1} \sin^2\alpha_1$$

At the mirror point the pitch angle becomes 90° and the parallel velocity of the particle is 0 with the condition

$$\sin\alpha_1 = \sqrt{B_1/B_{mirror}} \tag{1.33}$$

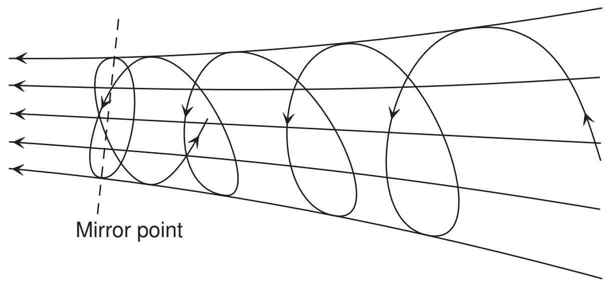

Figure 1.6: Illustration of magnetic mirror motion.

The force responsible for decelerating the particle is the force in (1.31).

Second (Longitudinal) Adiabatic Invariant

The mirror motion implies a second quasi-periodic motion for a particle in a mirror magnetic field , i.e., the motion from one mirror point to the opposite and back with a bounce frequency of ω_b. For configurational changes on a time scale $\tau \gg 1/\omega_b$ the corresponding action integral

$$J = \oint m v_{\|} ds \tag{1.34}$$

is a longitudinal invariant of the particle motion.

In terms of an average parallel velocity $\langle v_{\|} \rangle$ the invariant is $J = 2ml \langle v_{\|} \rangle$ with l being the length of the entire field line between the mirror points. The square of this invariant implies for the parallel energy

$$\frac{\langle W_{\|} \rangle_2}{\langle W_{\|} \rangle_1} = \frac{l_1^2}{l_2^2} \tag{1.35}$$

Therefore as the length of the field line between mirror points changes, so does the parallel energy which is basic for so-called Fermi acceleration.

Exercise: Demonstrate that the momentum of an ideally reflecting ball which bounces between two walls which approach each other with a velocity u, satisfies $p_{\|} d = const$ where d is the distance between the walls and $p_{\|}$ is the momentum normal to the wall surface.

Third (Drift) Adiabatic Invariant

The third adiabatic invariant is the magnetic flux encircled by the (periodic) drift path of a particle.

$$\Phi = \oint v_d r d\psi \tag{1.36}$$

Similar to the other invariants it requires slow configurational changes $\tau \gg 1/\omega_d$ where ω_d is the frequency of the drift motion.

1.4.3 Drift Kinetic Equations

Considering slow changes (compared to the gyro period) of the electric and magnetic fields and gradients of these which length scales much larger than the gyro radius of charged particles it is convenient to split the particles location vector into the slowly changing location of the gyro center $\mathbf{R}(t)$ and a fast changing actual particle location relative to the gyro center $\mathbf{r}_g(t)$:

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}_g(t) \tag{1.37}$$

In this representation on can consider the parameters of the gyro motion as functions of the guiding center location: $\mathbf{B}(\mathbf{R}(t),t)$ => $\omega_g(\mathbf{R}(t))$ and $\mu(\mathbf{R}(t)) = mv_\perp^2(t)/(2B(\mathbf{R}(t),t))$. However, for slowly changing fields the magnetic moment is approximately constant and changes in the total kinetic energy $\varepsilon = m\left(v_\parallel^2 + v_\perp^2\right)/2$ are due to drifts along a component of the electric field and changes of the magnetic field. In combination the basic equations for the drift kinetic equations are

$$\mu = \frac{mv_\perp^2}{2B} = const \tag{1.38}$$

$$\frac{d\varepsilon}{dt} = e\mathbf{E}\cdot\frac{d\mathbf{R}}{dt} + \frac{mv_\perp^2}{2B}\frac{\partial B}{\partial t} \tag{1.39}$$

$$\frac{d\mathbf{R}}{dt} = v_\parallel\frac{\mathbf{B}}{B} + \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{mv_\perp^2}{2qB^3}(\mathbf{B}\times\nabla B) + \frac{mv_\parallel^2}{qB^4}\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}] \tag{1.40}$$

for given electric and magnetic fields which satisfy the slowness (adiabatic) conditions. The last equation in this system is the equation describing the actual motion of the guiding center. The first term in this equation is the motion parallel to the magnet field. The next three terms are the $\mathbf{E}\times\mathbf{B}$, magnetic gradient, and magnetic curvature drift.

Note that the polarization drift is not contained here because it is higher order than the other drifts because it involves the time derivative of the electric field which needs to be small in the drift kinetic approximation.

These equations can also be used for a self-consistent solution of a plasma problem if the system is coupled to Maxwell's equations and the adiabatic (slow) evolution is satisfied for the actual solution.

1.5 Fluid Plasma and Magnetohydrodynamic Equations

1.5.1 Definitions

The equations of ordinary fluids and gases as well as those for magnetofluids (plasmas) can be obtained from the Boltzmann equation 1.17 in a systematic manner. Defining the 0th, 1st, and 2nd moment of the integral over the distribution function f_s as mass density ρ_S, fluid bulk velocity $\mathbf{u}_s$, and pressure tensor $\underline{\Pi}_s$

$$\rho_s(\mathbf{r},t) = m_s \int_{-\infty}^{\infty} d^3v f_s(\mathbf{r},\mathbf{v},t) \tag{1.41}$$

$$\mathbf{u}_s(\mathbf{r},t) = \frac{1}{n_s} \int_{-\infty}^{\infty} d^3v \mathbf{v} f_s(\mathbf{r},\mathbf{v},t) \tag{1.42}$$

$$\underline{\Pi}_s(\mathbf{r},t) = m_s \int_{-\infty}^{\infty} d^3v (\mathbf{v}-\mathbf{u}_s)(\mathbf{v}-\mathbf{u}_s) f_s(\mathbf{r},\mathbf{v},t). \tag{1.43}$$

where the index s indicates the particle species (electrons and different ion species if present). With these definitions one also obtains number density $n_s(\mathbf{r},t) = \rho_s/m_s$, charge density $\rho_{c,s}(\mathbf{r},t) = q_s n_s$, momentum density $\mathbf{p}_s(\mathbf{r},t) = \rho_s \mathbf{u}_s$, current density $\mathbf{j}_s(\mathbf{r},t) = q_s \mathbf{u}_s$, and scalar pressure (the isotropic portion of the pressure) $p_s(\mathbf{r},t) = \frac{1}{3} Tr(\underline{\Pi}_s)$ where the individual particle mass m_s and charge q_s are used. In the following section we will drop the index s for a more compact representation but remind the reader that there is a separate set of fluid equations for each particle species.

1.5.2 Fluid Moments

The fluid equations are determined by the moments of the Boltzmann equation, i.e.,

$$\int_{-\infty}^{\infty} d^3v \, \mathbf{v}^{\mathbf{i}} (Boltzmann\, Equ.)_s$$

To account for the collision term in (1.17) we define

$$Q_s^{\rho}(\mathbf{r},t) = m_s \int_{-\infty}^{\infty} d^3v \left. \frac{\partial f_s}{\partial t} \right|_c \tag{1.44}$$

$$\mathbf{Q}_s^{p}(\mathbf{r},t) = m_s \int_{-\infty}^{\infty} d^3v (\mathbf{v}-\mathbf{u}_s) \left. \frac{\partial f_s}{\partial t} \right|_c \tag{1.45}$$

$$Q_s^{E}(\mathbf{r},t) = \frac{1}{2} m_S \int_{-\infty}^{\infty} d^3v (\mathbf{v}-\mathbf{u}_s)^2 \left. \frac{\partial f_s}{\partial t} \right|_c \tag{1.46}$$

The precise form of these terms depends on the particular collisional properties of the systems and will not be specified at this point. The respective integrals over the velocity yield

$$\frac{\partial \rho_s}{\partial t} = -\nabla \cdot (\rho_s \mathbf{u}_s) + Q_s^\rho \quad (1.47)$$

$$\frac{\partial \rho_s \mathbf{u}_s}{\partial t} = -\nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s) - \nabla \cdot \underline{\Pi}_s + q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \mathbf{u}_s Q_s^\rho + \mathbf{Q}_s^p \quad (1.48)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\gamma - 1} p_s + \frac{1}{2}\rho_s u_s^2\right) = -\nabla \cdot \left(\frac{1}{2}\rho_s u_S^2 \mathbf{u}_s + \frac{1}{\gamma_s - 1} p_s \mathbf{u}_s + \mathbf{u}_s \cdot \underline{\Pi}_s + \mathbf{L}_s\right)$$
$$+ q_s n_s \mathbf{u}_s \cdot \mathbf{E} + \frac{1}{2} u_s^2 Q_s^\rho + \mathbf{u}_s \cdot \mathbf{Q}_s^p + Q_s^E \quad (1.49)$$

with the heat flux $\mathbf{L}_s(\mathbf{r},t) = \frac{1}{2} m_s \int_\infty^\infty d^3 v (\mathbf{v} - \mathbf{u}_s)(\mathbf{v} - \mathbf{u}_s)^2 f(\mathbf{r},\mathbf{v},t)$ and γ_s is the ratio of specific heats, i.e., $\gamma_s = 5/3$ if a gas has 3 degrees of freedom for motion.

Elimination of $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_s u_s^2\right)$ in the energy equation (with the aid of (1.47) and (1.47)) yields

$$\frac{1}{\gamma_s - 1}\left(\frac{\partial}{\partial t} p_s + \nabla \cdot p_s \mathbf{u}_s\right) = -(\underline{\Pi}_s \cdot \nabla) \cdot \mathbf{u}_s - \nabla \cdot \mathbf{L}_s + Q_s^E \quad (1.50)$$

As indicated by the index s a set consisting of continuity, momentum, and energy equation is present for each particle species, specifically in a fully ionized proton and electron plasma. There are many applications and further approximations of the above sets of equations. A specifically important approximation valid on scales much larger than the Debye length is neutrality. The set of two-fluid equations is also often used to examine waves in a plasma. Finally it is important to note that the above set of equations combined with Maxwell's equations must conserve total mass, momentum, and energy. The only exception to this are sources that are not contained in the basic approximation, for instance strong radiation that interacts with the plasma or large energy fluxes into a plasma through separate energetic particle fluxes (such as electron precipitation leading to aurora) which are not accounted for in these equations.

Exercise: Determine the integral of the 1st order moment for the first two terms in the Boltzmann equation.

Exercise: Derive the 1st order moment force term for a gravitational force and the Lorentz force (velocity dependent).

Exercise: Do the same for the energy equation (i.e., multiply the Boltzmann equation (1.17) with $\frac{1}{2}mv^2$ and integrate).

Exercise: Derive the equation for heat conduction with the stated assumptions.

Exercise: Derive the heat conduction equation for nonzero velocity **u**.

Exercise: Derive the continuity equation and momentum equation for irrotational flow.

Exercise: Assume a scalar pressure, $\mathbf{L} = 0$, and $Q^E = 0$ in the pressure equation (1.50). Consider a function $g = p^a \rho^b$ and determine a and b such that the resulting equation for g assumes a conservative form, i.e., $\partial g/\partial t + \nabla \cdot g\mathbf{u} = 0$.

Exercise: Assume a scalar pressure, $\mathbf{L} = 0$, and $Q^E = 0$ in the pressure equation (1.50). Consider a function $h = p^a \rho^b$ and determine a and b such that the resulting equation for h assumes a total derivative, i.e., $\partial h/\partial t + \mathbf{u} \cdot \nabla h = 0$. For $\gamma = 5/3$ this equation becomes a measure for entropy because entropy is conserved for adiabatic changes.

1.5.3 General Ohm's law and MHD Equations

Assuming electrons and single charged ions with $q_i = -q_e = e$, a charge neutral plasma $n_e = n_i$, and the following definitions for total mass density ρ, effective mass M, and bulk velocity or total mass density flux $\rho\mathbf{u}$, and total current density $\mathbf{j}$

$$
\begin{aligned}
\rho &= n(m_i + m_e) \\
M &= m_i + m_e \\
\rho\mathbf{u} &= n(m_i\mathbf{u}_i + m_e\mathbf{u}_e) \\
\mathbf{j} &= en(\mathbf{u}_i - \mathbf{u}_e)
\end{aligned}
$$

we can express ion and electron velocities through

$$
\begin{aligned}
\mathbf{u}_i &= \mathbf{u} + \frac{m_e}{m_i}\frac{\mathbf{j}}{ne} \simeq \mathbf{u} \\
\mathbf{u}_e &= \mathbf{u} - \frac{\mathbf{j}}{ne}
\end{aligned}
$$

Further we can combine the two fluid equation into a single set of equations complemented by Ohm's law. This latter is obtained by multiplying the ion equation with q_i/m_i and the electron equation with q_e/m_e and the sum of the modified equations:

$$
\mathbf{E} + \mathbf{u}\times\mathbf{B} = \frac{m_e m_i}{e^2\rho}\left[\frac{\partial\mathbf{j}}{\partial t} + \nabla\cdot(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u})\right] - \frac{M}{e\rho}\nabla p_e + \frac{m_i}{e\rho}\mathbf{j}\times\mathbf{B} + \eta\mathbf{j} \tag{1.51}
$$

with the resistivity $\eta = m_e\nu_c/ne^2$ where ν_c is the collision frequency between electrons and ions (or neutrals). This equation is usually termed **generalized Ohm's law**. In the above equation the first term on the rhs is often called the inertia term because it represents the electron inertia in this equation. Using a scaling analysis this term scales with the so-called electron inertia scale (or plasma skin depth)) $c/\omega_{pe} = \left(\varepsilon_0 m_e c^2/ne^2\right)^{1/2}$. The same analysis demonstrates that the 2nd and 3rd terms on the rhs scale with the ion inertia scale $c/\omega_{pi} = (m_i/m_e)^{1/2}c/\omega_{pe} \gg c/\omega_{pe}$ this scaling provides a hierarchy of length scales for which plasma processes involve the physics associated with these terms. Specifically a scaling which retains the ion inertia terms but neglects the electron inertia term is often addressed as Hall Magnetohydrodynamics.

Taking sum of the two fluid continuity, momentum and energy equations, neglecting electron and ion inertial effects, using total pressure as $p = p_e + p_i$, assuming isotropic pressure, and including Maxwell's equations leads to the so-called resistive magnetohydrodynamic (or MHD) equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \tag{1.52}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{1.53}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \tag{1.54}$$

$$\frac{1}{\gamma - 1}\left(\frac{\partial}{\partial t} p + \nabla \cdot p\mathbf{u}\right) = -p\nabla \cdot \mathbf{u} + \eta \mathbf{j}^2 \tag{1.55}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{1.56}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.57}$$

Here charge continuity results in $\nabla \cdot \mathbf{j} = 0$. The above equations do not contain $\nabla \cdot \mathbf{B} = 0$. This equation enters actually as an initial condition. If $\nabla \cdot \mathbf{B} = 0$ is satisfied initially then the induction equation implies $\nabla \cdot \mathbf{B} = 0$ at all times.

Exercise: Derive the above equations.

Exercise: Derive Ohm's law from the two fluid approximation.

Exercise: Use Ampere's law and $\nabla \cdot \mathbf{B} = 0$ to show that the momentum equation can also be written as

$$\frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu_0}\right)\underline{\underline{1}} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}\right]$$

In the above equation the term $B^2/(2\mu_0)$ is referred to as a magnetic pressure. This terminology makes sense as will be shown in simple magnetic equilibrium situations.

Exercise: Assume a plasma density of 1 cm^{-3}, temperature equivalent to 1 keV, and a magnetic field of 20 nT which are typical for the near Earth magnetotail. Determine electron and ion inertia scales. Assume that quasi-neutrality is violated in a sphere with the radius of the electron inertia length by 1 % (e.g. 1% of the ion charge is not compensated by electrons. If outside were a vacuum what is the electric field outside the sphere? What velocity perpendicular to the magnetic field is required by Ohm's law to generate an electric field magnitude equal to that on the surface of the sphere?

Exercise: For the plasma in the prior exercise, determine the temperature in degrees Kelvin. Determine the energy density in kW hours/m^3 and kW hours $/R_e^3$ (1 $R_E = 6370$ km). For the sake of simplicity assume that the plasma sheet is represented by a cylinder with 10 R_E radius and 100 R_E length. How long could a power plant with an output of 1000 MW operate on the energy stored in the plasma?

1.5.4 Properties of the MHD equations

Frozen-in Condition

Considering a closed contour C in an ideal ($\eta = 0$) MHD plasma where the contour moves with the plasma bulk velocity $\mathbf{u}$ such that Ohm's law reduces to

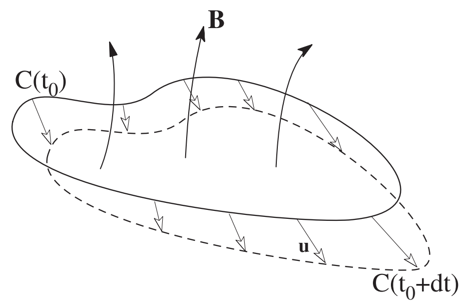

Figure 1.7: Illustration of the frozen-in condition.

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

This implies that the magnetic flux is frozen into the plasma motion in the following way. Assuming the magnetic flux through the surface C is the surface integral

$$\Phi_C = \int_C \mathbf{B} \cdot d\mathbf{s}$$

with $d\mathbf{s}_C$ being the surface element of the contour C. The contour elements move with the fluid velocity $\mathbf{u}$. It is straightforward to demonstrate that the change of magnetic flux through the contour C is 0

$$\frac{d\Phi_C}{dt} = 0$$

A more complete form of Ohm's law should be considered if gradients on smaller scales exist in a plasma. Including the ion inertia terms in Ohm's law yields

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = -\frac{M}{e\rho} \nabla p_e \tag{1.58}$$

such that the electron velocity substitutes the bulk velocity of ideal Ohm's law. Here again the frozen-in condition $d\Phi/dt = 0$ is satisfied if the contour C moves with the electron velocity

$$\mathbf{u}_e = \mathbf{u} - \frac{m_i}{e\rho} \mathbf{j} \tag{1.59}$$

Comparing this with generalized Ohm's law we can re-write this neglecting the electron inertia scale as

There are various applications using the fluid and the kinetic equations. Typical applications consider waves, discontinuities and shocks, instabilities, steady state solutions, and equilibrium solutions. Particularly for the last topic it is important to note the following terminology.

Steady state assumes time stationary solutions with nonzero velocity, $\partial/\partial t = 0$ and $\mathbf{u} \neq 0$.

Equilibrium solutions assume $\partial/\partial t = 0$ and $\mathbf{u} = 0$. Note that for kinetic systems the velocity in phase space is always nonzero for physical systems. Also the electron velocity is nonzero in current regions.

Electrostatic solutions assume $\partial \mathbf{B}/\partial t = 0$. This implies $\nabla \times \mathbf{E} = 0$ or $\mathbf{E} = -\nabla\phi$. In this case Ohm's law must be replaced by the Coulomb equation.

Entropy and Adiabatic Convection

Assuming an ideal MHD plasma the combination of the continuity equation and the internal energy equation

$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p+\nabla\cdot p\mathbf{u}\right)=-p\nabla\cdot\mathbf{u}$$

demonstrates that the quantity $s=p\rho^{-\gamma}$ is conserved along the path of any fluid element or

$$\frac{dp\rho^{-\gamma}}{dt}=\frac{\partial p\rho^{-\gamma}}{\partial t}+\mathbf{u}\cdot\nabla\left(p\rho^{-\gamma}\right)=0 \tag{1.60}$$

It is possible to identify s with the local entropy in the fluid. In other words for sufficiently slow changes local entropy is conserved along the path of any fluid element in the absence of resistive (or viscous) heating.

It is also instructive to note that the quantity $h=p^{1/\gamma}$ satisfies a continuity equation,

$$\frac{\partial p^{1/\gamma}}{\partial t}+\nabla\cdot\mathbf{u}p^{1/\gamma}=0$$

i.e., the integral of h in any finite volume of space changes only due to in- and outflow into the volume for ideal MHD dynamics. Let us consider specifically the integral over the volume of a magnetic flux tube (without loss at the ends of the flux tube). It can be shown that any conserved quantity (i.e., satisfying a continuity equation) is also conserved over this flux tube volume. Using

$$N_C=\int_{l_1}^{l_2}\left(\int\int_{A_c(l)}nds\right)dl \quad\hookrightarrow\quad \frac{dN_C}{dt}=0$$

In the limit where the cross section of the flux tube approaches 0 we can express this for instance for the number of particles in a flux tube by

$$N=\int_{l_1}^{l_2}\frac{ndl}{B} \quad\hookrightarrow\quad \frac{dN}{dt}=0$$

(Note that the with cross section of a flux tube center on a specific field line is proportional to $1/B$). Since $h$satisfies a continuity it is clear that for

$$H=\int_{l_1}^{l_2}\frac{p^{1/\gamma}dl}{B} \quad\hookrightarrow\quad \frac{dH}{dt}=0$$

These relations do not require that the system is in an equilibrium. However, if an equilibrium is obtained the pressure is constant along a field line and

$$H=p^{1/\gamma}\int_{l_1}^{l_2}\frac{dl}{B}=p^{1/\gamma}V=const$$

where $V = \int dl/B$ is called the specific flux tube volume. Taking $S = H^\gamma = pV^\gamma$ provides the flux tube equivalent of the local entropy function, in other words H or S are flux tube entropy functions. This is a generalisation of the local entropy concept and similarly to local entropy these are constant in time unless (a) there is some nonadiabatic heating or (b) the integrity of magnetic flux tubes is destroyed through resistivity.

Exercise: Derive equation (1.60) from the pressure equation.

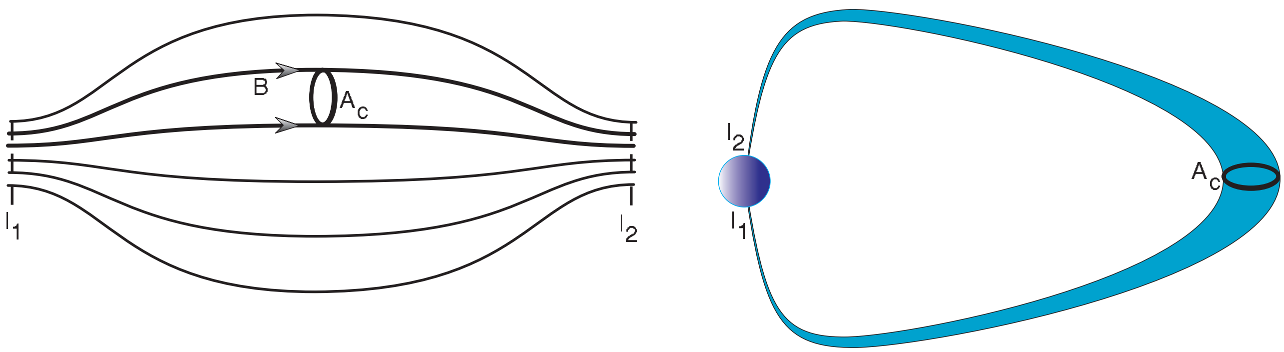

Figure 1.8: Examples of magnetic flux tubes (magnetic bottle and magnetosphere)

MHD Conservation Laws:

The MHD equation satisfy mass, momentum, and energy conservation.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right) \underline{\underline{1}} - \frac{1}{\mu_0} \mathbf{B}\mathbf{B} \right] \\
\frac{\partial w_{tot}}{\partial t} &= -\nabla \cdot \left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1} + \frac{1}{\mu_0} B^2 \right) \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{B}}{\mu_0} \mathbf{B} + \frac{\eta}{\mu_0} \mathbf{j} \times \mathbf{B} \right]
\end{aligned}$$

with the total energy density

$$w_{tot} = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} + \frac{1}{2\mu_0} B^2 \qquad (1.61)$$

Exercise: Demonstrate the validity of the energy conservation equation.

The various terms in (1.61) are the energy densities of the bulk flow $\frac{1}{2}\rho u^2$, thermal energy $\frac{p}{\gamma-1}$, and magnetic field energy density $\frac{1}{2\mu_0} B^2$.

Other Properties of the MHD approximation

Important insight into the large scale plasma behaviour is obtained in the framework of

- MHD equilibrium theory
- Stability and energy principles
- MHD waves
- Macroscopic instabilities