Chapter 2

Plasma Kinetic Theory

2.1 Klimontovich Equation

2.1.1 Introduction

Start from first principles => exact plasma description

Single particle:

- location $\mathbf{R}_1(t)$, velocity $\mathbf{V}_1(t)$
  => 6 degree of freedom
  => six-dimensional space
- Density of the particle in this space: $N(\mathbf{r},\mathbf{v},t) = \delta(\mathbf{r}-\mathbf{R}_1(t))\,\delta(\mathbf{v}-\mathbf{V}_1(t))$

with: $\delta(\mathbf{r}-\mathbf{R}_1(t)) = \delta(x-X_1(t))\,\delta(y-Y_1(t))\,\delta(z-Z_1(t))$

$\delta -$ Dirac delta function

Consider N_{0s} **particles** of species s:

Density of this distribution in phase space:

$$N_s(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N_{0s}} \delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))$$

and for all species:

$$N = \sum_s N_s(\mathbf{r},\mathbf{v},t)$$

Particle motion:

$$\begin{aligned} \dot{\mathbf{R}}_i &= \mathbf{V}_i(t) \\ \dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\mathbf{E}^m(\mathbf{R}_i(t),t) + \frac{q_s}{m_s}\mathbf{V}_i(t)\times\mathbf{B}^m(\mathbf{R}_i(t),t) \end{aligned}$$

To solve the equations of motion we need Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \mathbf{E}^m(\mathbf{r},t) &= \frac{1}{\varepsilon_0}\rho_c(\mathbf{r},t) \\
\nabla \cdot \mathbf{B}^m(\mathbf{r},t) &= 0 \\
\nabla \times \mathbf{E}^m(\mathbf{r},t) + \frac{\partial \mathbf{B}^m(\mathbf{r},t)}{\partial t} &= 0 \\
\nabla \times \mathbf{B}^m(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial \mathbf{E}^m(\mathbf{r},t)}{\partial t} &= \mu_0 \mathbf{j}^m(\mathbf{r},t)
\end{aligned}
$$

(m stands for microscopic fields) with the charge and current densities (sources)

$$
\begin{aligned}
\rho_c^m(\mathbf{r},t) &= \sum_s q_s \int_{-\infty}^{\infty} d^3v N_s(\mathbf{r},\mathbf{v},t) \\
\mathbf{j}^m(\mathbf{r},t) &= \sum q_s \int_{-\infty}^{\infty} d^3v \mathbf{v} N_s(\mathbf{r},\mathbf{v},t)
\end{aligned}
$$

The above equations fully determine the entire system of particles.

Initial value problem:

$$
N_s(\mathbf{r},\mathbf{v},t=0) \quad => \quad \mathbf{E}^m(\mathbf{r},t=0), \mathbf{B}^m(\mathbf{r},t=0)
$$

=> Integrate equations in time.

2.1.2 Klimontovich Equation

Time evolution of the distribution function $N_s(\mathbf{r},\mathbf{v},t)$:

$$
\begin{aligned}
\frac{\partial N_s(\mathbf{r},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \dot{\mathbf{R}}_i \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r}-\mathbf{R}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t)) \\
&\quad -\sum_{i=1}^{N_{0s}} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r}-\mathbf{R}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))
\end{aligned}
$$

Note:

$$
\begin{aligned}
\frac{\partial f(a-b)}{\partial a} &= -\frac{\partial f(a-b)}{\partial b} \\
\frac{df(g(t))}{dt} &= \frac{df}{dg}\frac{dg}{dt}
\end{aligned}
$$

Substitute: $\dot{\mathbf{R}}_i$ and $\dot{\mathbf{V}}_i$

$$
\begin{aligned}
\frac{\partial N_s(\mathbf{r},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \mathbf{v}\cdot\nabla_{\mathbf{r}}\delta(\mathbf{r}-\mathbf{R}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t)) \\
&\quad -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{r},t)+\frac{q_s}{m_s}\mathbf{v}\times\mathbf{B}^m(\mathbf{r},t)\right\}\cdot\nabla_{\mathbf{v}}\delta(\mathbf{r}-\mathbf{R}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))
\end{aligned}
$$

where we used $f(a)\delta(a-b) = f(b)\delta(a-b)$.

Exercise: Prove that the last equation for $N_s(\mathbf{r},\mathbf{v},t)$ is correct and in particular that one can replace $\mathbf{V}_i(t)\times\mathbf{B}^m(\mathbf{R}_i(t),t)$ with $\mathbf{v}\times\mathbf{B}^m(\mathbf{r},t)$ in this equation.

As the final step we can now take the $\mathbf{v}\cdot\nabla_{\mathbf{r}}$ and $\left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{r},t)+\frac{q_s}{m_s}\mathbf{v}\times\mathbf{B}^m(\mathbf{r},t)\right\}$ in front of the summation which yields the **Klimontovich equation**

$$
\frac{\partial N_s}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}N_s+\frac{q_s}{m_s}(\mathbf{E}^m+\mathbf{v}\times\mathbf{B}^m)\cdot\nabla_{\mathbf{v}}N_s = 0 \tag{2.1}
$$

Together with the Maxwell's equation and the definitions for charge and current densities this provides a full description of the plasma dynamics!

However, since the distribution is a distribution of delta functions it still requires basically to follow all individual particles which in typical application is not feasible even on modern supercomputers. Similarly the electric and magnetic field are microscopic and contain all the fluctuations present in the description of all individual particles. In order to derive an equation for an averaged smooth distribution function we need to average the Klimontovich equation and split the field and the distribution function into fluctuating and averaged contribution, i.e.,

$$
\begin{aligned}
N_s(\mathbf{r},\mathbf{v},t) &= f_s(\mathbf{r},\mathbf{v},t)+\delta f_s(\mathbf{r},\mathbf{v},t) \\
\mathbf{E}^m(\mathbf{r},\mathbf{v},t) &= \mathbf{E}(\mathbf{r},\mathbf{v},t)+\delta\mathbf{E}(\mathbf{r},\mathbf{v},t) \\
\mathbf{B}^m(\mathbf{r},\mathbf{v},t) &= \mathbf{B}(\mathbf{r},\mathbf{v},t)+\delta\mathbf{B}(\mathbf{r},\mathbf{v},t)
\end{aligned}
$$

with $f_s(\mathbf{r},\mathbf{v},t) = \langle N_s(\mathbf{r},\mathbf{v},t)\rangle$, $\mathbf{E}(\mathbf{r},\mathbf{v},t) = \langle\mathbf{E}^m(\mathbf{r},\mathbf{v},t)\rangle$, and $\mathbf{B}(\mathbf{r},\mathbf{v},t) = \langle\mathbf{B}^m(\mathbf{r},\mathbf{v},t)\rangle$. Now we can use the averaging procedure for the Klimontovich equation itself to obtain

$$
\frac{\partial f_s}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}f_s+\frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}}f_s = \frac{q_s}{m_s}\langle(\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B})\cdot\nabla_{\mathbf{v}}\delta f_s\rangle
$$

Here the rhs contains the contribution from the fluctuations for the averaged distribution function.

Properties of the Klimontovich equation

- Incompressibility in phase space: Hypothetical point particle at $\mathbf{r}$, $\mathbf{v}$ total time derivative

$$
\frac{D}{Dt} = \frac{\partial}{\partial t}+\frac{d\mathbf{r}}{dt}\cdot\nabla_{\mathbf{r}}+\frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}}
$$

=> Klimontovich equation

$$\frac{DN_s(\mathbf{r},\mathbf{v},t)}{Dt} = 0$$

=> along each (hypothetical) path N_s is constant!

- Conservation of particles (continuity): $\partial f/\partial t + \nabla_{\mathbf{r}} \cdot (\mathbf{v}f) = 0$
  In 6-dimensional phase space we can define $\nabla_{\mathbf{R^6}} = (\nabla_{\mathbf{r}}, \nabla_{\mathbf{v}})$ and $\mathbf{V}^6 = (d\mathbf{r}/dt, d\mathbf{v}/dt)$ =>

$$\frac{\partial N_s}{\partial t} + \nabla_{\mathbf{R}^6} \cdot \left(\mathbf{V}^6 N_s\right) = 0$$

  Klimontovich equation must satisfy continuity!

2.1.3 Plasma Kinetic Equation

The Klimontovich distribution is a distribution of δ functions => need to reduce amount of information (we know that a plasma behave collectively so it is not necessary to follow each individual particle.).

=> generate smooth distribution using an appropriate average

Rigorous way:

- Ensemble average over infinite number of realizations, e.g., with a temperature contact => statistical mechanics

Alternatively:

- Define boxes size Δx, Δv with $\Delta x \ll \lambda_{de}$ and count particles in range $[\mathbf{r},\mathbf{v}]$ to $[\mathbf{r}+\Delta\mathbf{r},\mathbf{v}+\Delta\mathbf{v}]$
  => $f_s = \frac{n_s}{\Delta r^3 \Delta v^3}$

Define fluctuations

$$\begin{aligned} N_s(\mathbf{r},\mathbf{v},t) &= f_s(\mathbf{r},\mathbf{v},t) + \delta N_s(\mathbf{r},\mathbf{v},t) \\ \mathbf{E}^m(\mathbf{r},t) &= \mathbf{E}(\mathbf{r},t) + \delta\mathbf{E}(\mathbf{r},t) \\ \mathbf{B}^m(\mathbf{r},t) &= \mathbf{B}(\mathbf{r},t) + \delta\mathbf{B}(\mathbf{r},t) \end{aligned}$$

such that: $\langle\delta N_s\rangle$, $\langle\delta\mathbf{E}\rangle$, $\langle\delta\mathbf{B}\rangle = 0$

=>

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = \frac{q_s}{m_s}\langle(\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B})\cdot\nabla_{\mathbf{v}}\delta N_s\rangle \tag{2.2}$$

- Left side - collective effects
- right side -collisional effects

Continuum limit: $N_0 \rightarrow \infty,\ f_s \rightarrow \infty$ with $eN_0 = const$ and $mN_0 = const$

- right side: fluctuations $\delta N_s \sim N_0^{1/2}$ (statistical mechanics)
  $\delta \mathbf{E} \sim e\delta N_s \sim \frac{1}{N_0} N_0^{1/2} \sim N_0^{-1/2}$

=> right side $\rightarrow$ const

=> left side $\sim N_0 \rightarrow \infty$

Which yields the collisionless **Boltzmann** equations:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = 0$$

Complemented with Maxwell's equations and with the definitions for charge and current density

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r},t) &= \frac{1}{\varepsilon_0} \rho_c(\mathbf{r},t) \\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \\
\nabla \times \mathbf{B}(\mathbf{r},t) &= \mu_0 \mathbf{j}(\mathbf{r},t) + \frac{1}{c^2} \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \\
\rho_c(\mathbf{r},t) &= \sum_s q_s \int_{-\infty}^{\infty} d^3v f_s(\mathbf{r},\mathbf{v},t) \\
\mathbf{j}(\mathbf{r},t) &= \sum q_s \int_{-\infty}^{\infty} d^3v \mathbf{v} f_s(\mathbf{r},\mathbf{v},t)
\end{aligned}$$

yield the **Vlasov** equations.

2.2 Liouville Equation

2.2.1 Concept (or state) of a system

Motivation:

Use Liouville equation => derivation of a kinetic equation (right hand side of the Boltzmann equation)

Note: Klimontovich equation - Behaviour of individual particles

One particle:

- spatial coordinate of the system $\mathbf{r}_1 = (x_1, y_1, z_1)$
- velocity coordinate of the system $\mathbf{v}_1 = (v_{x1}, v_{y1}, v_{z1})$
- Particle orbit (as before) by $\mathbf{R}_1(t)$ and $\mathbf{V}_1(t)$
- System coordinates: $(\mathbf{r}_1, \mathbf{v}_1) = (x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1})$ (6 coord)

- Density of systems:
  $N(\mathbf{r},\mathbf{v},t) = \delta(\mathbf{r}_1 - \mathbf{R}_1(t))\,\delta(\mathbf{v}_1 - \mathbf{V}_1(t))$
  One system consisting of one particle

2 particles:

- 12 coordinates for our system
- Phase space: $(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2) = \left(x_1,y_1,z_1,v_{x1},v_{y1},v_{z1},x_2,y_2,z_2,v_{x2},v_{y2},v_{z2}\right)$
- Density:
  $N(\mathbf{r},\mathbf{v},t) = \delta(\mathbf{r}_1 - \mathbf{R}_1(t))\,\delta(\mathbf{v}_1 - \mathbf{V}_1(t))\,\delta(\mathbf{r}_2 - \mathbf{R}_2(t))\,\delta(\mathbf{v}_2 - \mathbf{V}_2(t))$
  1 system consisting of 2 particles, or equivalent $N(\mathbf{r},\mathbf{v},t)$ characterises the state of the two particle system.

Generalisation to N_0 particles => Phase space has 6N coordinates

Density:

$$N(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,...\mathbf{r}_{N_0},\mathbf{v}_{N_0},t) = \prod_{i=0}^{N_0}\delta(\mathbf{r}_i - \mathbf{R}_i(t))\,\delta(\mathbf{v}_i - \mathbf{V}_i(t))$$

2.2.2 Liouville equation

Interested in the time evolution of N and with

$$\frac{\partial\delta(\mathbf{r}_i - \mathbf{R}_i(t))}{\partial t} = -\frac{d\mathbf{X}\mathbf{R}_i(t)}{dt}\cdot\nabla_{\mathbf{r}_i}\delta(\mathbf{r}_i - \mathbf{R}_i(t))$$

=>

$$\begin{aligned}\frac{\partial N(\mathbf{r},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_0}\dot{\mathbf{R}}_i\cdot\nabla_{\mathbf{r}_i}\prod_{i=0}^{N_0}\delta(\mathbf{r}_i - \mathbf{R}_i(t))\,\delta(\mathbf{v}_i - \mathbf{V}_i(t))\\ &\quad -\sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}\prod_{i=0}^{N_0}\delta(\mathbf{r}_i - \mathbf{R}_i(t))\,\delta(\mathbf{v}_i - \mathbf{V}_i(t))\end{aligned}$$

As before we can substitute the phase space coordinates $\mathbf{r}_i$ instead of $\mathbf{R}_i(t)$ and $\mathbf{v}_i$ instead of $\mathbf{V}_i(t)$ in the Lorentz force:

$$\begin{aligned}\dot{\mathbf{R}}_i &= \mathbf{V}_i(t)\\ \dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\left[\mathbf{E}^m(\mathbf{r}_i,t) + \mathbf{v}_i\times\mathbf{E}^m(\mathbf{r}_i,t)\right]\end{aligned}$$

Exercise: Demonstrate that this is correct!

Such that

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i}N + \sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}N = 0 \tag{2.3}$$

which is **Liouville's equation.**

Properties:

(a)

$$\frac{D}{Dt}N\left(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,...\mathbf{r}_{N_0},\mathbf{v}_{N_0},t\right)=0$$

with

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\sum_{i=1}^{N_0}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i}+\sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}$$

=> incompressibility

(b) Continuity

$$\begin{aligned}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i}N &= \nabla_{\mathbf{r}_i}\cdot(\mathbf{v}_iN)\\ \dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i} &= \nabla_{\mathbf{v}_i}\cdot\left(\dot{\mathbf{V}}_iN\right)\end{aligned}$$

because

$$\nabla_{\mathbf{v}_i}\cdot\dot{\mathbf{V}}_i=\nabla_{\mathbf{v}_i}\cdot\left\{\frac{q_s}{m_s}\left[\mathbf{E}^m\left(\mathbf{r}_i,t\right)+\mathbf{v}_i\times\mathbf{E}^m\left(\mathbf{r}_i,t\right)\right]\right\}=0$$

=>

$$\frac{\partial N}{\partial t}+\sum_{i=1}^{N_0}\nabla_{\mathbf{r}_i}\cdot(\mathbf{v}_iN)+\sum_{i=1}^{N_0}\nabla_{\mathbf{v}_i}\cdot\left(\dot{\mathbf{V}}_iN\right)=0 \tag{2.4}$$

Exercise: Demonstrate the relations $\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i}N=\nabla_{\mathbf{r}_i}\cdot(\mathbf{v}_iN)$ and $\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}=\nabla_{\mathbf{v}_i}\cdot\left(\dot{\mathbf{V}}_iN\right)$.

Probability density:

Ensemble of systems N:

Def.:

$$f_{N_0}\left(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,...\mathbf{r}_{N_0},\mathbf{v}_{N_0},t\right)d\mathbf{r}_1d\mathbf{v}_1d\mathbf{r}_2d\mathbf{v}_2...d\mathbf{r}_{N_0}d\mathbf{v}_{N_0}$$

is the probability that

$\mathbf{R}_1(t)$ is in the interval $[\mathbf{r}_1,\mathbf{r}_1+d\mathbf{r}_1]$, $\mathbf{R}_2(t)$ is in the interval $[\mathbf{r}_2,\mathbf{r}_2+d\mathbf{r}_2]$, ..

$\mathbf{V}_1(t)$ is in the interval $[\mathbf{v}_1,\mathbf{v}_1+d\mathbf{v}_1]$, $\mathbf{V}_2(t)$ is in the interval $[\mathbf{v}_2,\mathbf{v}_2+d\mathbf{v}_2]$, ..

Probability is conserved along trajectory:

each fluid element moves along the trajectory as a probability

With $\nabla_{\mathbf{r}_i}\cdot\mathbf{v}_i=0$ and $\nabla_{\mathbf{v}_i}\cdot\dot{\mathbf{V}}_i=0$ =>

$$\frac{\partial f_{N_0}}{\partial t}+\sum_{i=1}^{N_0}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i}f_{N_0}+\sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}f_{N_0}=0$$

or

$$\frac{D}{Dt} f_{N_0}\left(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, ...\mathbf{r}_{N_0}, \mathbf{v}_{N_0}, t\right) = 0$$

We have now a probability distribution function which determines the kinetic evolution exactly, however, this distribution is in a $6N_0$ dimensional space. Thus there is now reduction in complexity compared to Klimontovich equation!

2.2.3 BBGKY Hierarchy

BBGKY -Bogoliubov, Born, Green, Kirkwood, and Yvon

Motivation: Reduction of complexity

Probability density:

$$f_{N_0}\left(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, ...\mathbf{r}_{N_0}, \mathbf{v}_{N_0}, t\right) d\mathbf{r}_1 d\mathbf{v}_1 d\mathbf{r}_2 d\mathbf{v}_2 ... d\mathbf{r}_{N_0} d\mathbf{v}_{N_0}$$

is the probability to find

particle 1 in $I_1 = \left[\left(\mathbf{r}_1, \mathbf{v}_1\right), \left(\mathbf{r}_1 + d\mathbf{r}_1 \mathbf{v}_1 + d\mathbf{v}_1\right)\right]$,

particle 2 in $I_2 = \left[\left(\mathbf{r}_2, \mathbf{v}_2\right), \left(\mathbf{r}_2 + d\mathbf{r}_2 \mathbf{v}_2 + d\mathbf{v}_2\right)\right]$,

etc.

Reduced probability distributions:

$$f_k\left(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, ...\mathbf{r}_k, \mathbf{v}_k, t\right) \equiv V^k \int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1} d\mathbf{r}_{k+2} d\mathbf{v}_{k+2} ... d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} f_{N_0} \tag{2.5}$$

= prob. to find particle 1 in I_1, ...particle k in I_k irrespective of the locations for p_{k+1} to p_{N_0}.

Note: dimension of $f_k \sim l^{-6N} v^{-6k}$

- V spatial volume occupied by particles
- V^k needed for normalization -> later
- f_{N_0} is a probability => $\int_{-\infty}^{\infty} d\mathbf{r}_1 d\mathbf{v}_1 ... d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} f_{N_0} = 1$
- boundary conditions: $f_{N_0} \rightarrow 0$ for $|\mathbf{r}_i| \rightarrow \pm\infty$
  $f_{N_0} \rightarrow 0$ for $|\mathbf{v}_i| \rightarrow \pm\infty$
- Symmetry regarding particle labels
  $f_{N_0}\left(\mathbf{r}_1, \mathbf{v}_1, ...\mathbf{r}_k, \mathbf{v}_k, ...\mathbf{r}_{N_0}, \mathbf{v}_{N_0}, t\right) = f_{N_0}\left(\mathbf{r}_k, \mathbf{v}_k, ...\mathbf{r}_1, \mathbf{v}_1, ...\mathbf{r}_{N_0}, \mathbf{v}_{N_0}, t\right)$
- Assume Coulomb model:
  $\mathbf{V}_i(t) = \sum_{j=1}^{N_0} \mathbf{a}_{ij}$ with $\mathbf{a}_{ij} = \frac{q^2}{4\pi\varepsilon_0 m \left|\mathbf{r}_i - \mathbf{x}_j\right|^3}\left(\mathbf{r}_i - \mathbf{r}_j\right)$ and $\mathbf{a}_{ii} = 0$

Liouville equation

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{N_0} + \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} = 0 \tag{2.6}$$

(a) Reduced distribution function f_{N_0-1}**:**

Integrate over $d\mathbf{r}_{N_0} d\mathbf{v}_{N_0}$

$$\begin{aligned} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \frac{\partial f_{N_0}}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{N_0} & \\ + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} &= 0 \end{aligned}$$

First term

$$T_1 = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} f_{N_0} = V^{1-N_0} \frac{\partial f_{N_0-1}}{\partial t}$$

Second term

$$\begin{aligned} T_2 &= \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} f_{N_0} + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \mathbf{v}_{N_0} \cdot \nabla_{\mathbf{r}_{N_0}} f_{N_0} \\ &= V^{1-N_0} \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{N_0-1} + \int_{-\infty}^{\infty} dy_{N_0} dz_{N_0} d\mathbf{v}_{N_0} v_{xN_0} \cdot f_{N_0} \Big|_{x_{N_0}=-\infty}^{x_{N_0}=\infty} + \ldots \\ &= V^{1-N_0} \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{N_0-1} \end{aligned}$$

Third term: Split

$$\sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \mathbf{a}_{ij} = \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} \mathbf{a}_{ij} + \sum_{j=1}^{N_0-1} \mathbf{a}_{N_0 j} + \sum_{i=1}^{N_0-1} \mathbf{a}_{iN_0} + \mathbf{a}_{N_0 N_0}$$

such that

$$\begin{aligned} T_3 &= \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \\ &= \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \\ &\quad + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \sum_{j=1}^{N_0-1} \mathbf{a}_{N_0 j} \cdot \nabla_{\mathbf{v}_{N_0}} f_{N_0} + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \sum_{i=1}^{N_0-1} \mathbf{a}_{iN_0} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \\ &= V^{1-N_0} \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} + \sum_{i=1}^{N_0-1} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{iN_0} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \end{aligned}$$

Collect terms and multiply with V^{N_0-1} to obtain the equation for the reduced distributions function f_{N_0-1}

$$\frac{\partial f_{N_0-1}}{\partial t} + \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{N_0-1} + \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} + V^{N_0-1} \sum_{i=1}^{N_0-1} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{iN_0} \cdot \nabla_{\mathbf{v}_i} f_{N_0} = 0 \qquad (2.7)$$

This is the equation for the reduced distribution function f_{N_0-1}.

(b) Reduced distribution function f_{N_0-2}:

Note recurrence relation

$$f_{k-1} = V^{-1} \int_{-\infty}^{\infty} d\mathbf{r}_k d\mathbf{v}_k f_k$$

Integrate the equation for f_{N_0-1} over $d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1}$

Term 1:

$$T_1 = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1} f_{N_0-1} = V \frac{\partial}{\partial t} f_{N_0-2}$$

Term 2:

$$T_2 = V \sum_{i=1}^{N_0-2} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_{N_0-2}$$

Term 3:

$$\begin{aligned} T_3 &= V \sum_{i=1}^{N_0-2} \sum_{j=1}^{N_0-2} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-2} \\ &\quad + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1} \sum_{i=1}^{N_0-2} \mathbf{a}_{i,N_0-1} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} \\ &\quad + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1} \sum_{j=1}^{N_0-2} \mathbf{a}_{N_0-1,j} \cdot \nabla_{\mathbf{v}_{N_0-1}} f_{N_0-1} \\ &= V \sum_{i=1}^{N_0-2} \sum_{j=1}^{N_0-2} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-2} + \int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1} \sum_{i=1}^{N_0-2} \mathbf{a}_{i,N_0-1} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} \end{aligned}$$

Term 4:

$$\begin{aligned} T_4 &= V^{N_0-1}\sum_{i=1}^{N_0-1}\int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1}\mathbf{a}_{iN_0}\cdot\nabla_{\mathbf{v}_i} f_{N_0} \\ &= V^{N_0-1}\sum_{i=1}^{N_0-2}\int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1}\mathbf{a}_{iN_0}\cdot\nabla_{\mathbf{v}_i} f_{N_0} \\ &= V^{N_0-1}\sum_{i=1}^{N_0-2}\int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1}\mathbf{a}_{iN_0-1}\cdot\nabla_{\mathbf{v}_i}\int_{-\infty}^{\infty} d\mathbf{r}_{N_0} d\mathbf{v}_{N_0} f_{N_0} \\ &= \sum_{i=1}^{N_0-2}\int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1}\mathbf{a}_{iN_0-1}\cdot\nabla_{\mathbf{v}_i} f_{N_0-1} \end{aligned}$$

Collect terms and divide by V to obtain the equation

$$\begin{aligned} &\frac{\partial f_{N_0-2}}{\partial t}+\sum_{i=1}^{N_0-2}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i} f_{N_0-2}+\sum_{i=1}^{N_0-2}\sum_{j=1}^{N_0-2}\mathbf{a}_{ij}\cdot\nabla_{\mathbf{v}_i} f_{N_0-2} \\ &\qquad +\frac{2}{V}\sum_{i=1}^{N_0-2}\int_{-\infty}^{\infty} d\mathbf{r}_{N_0-1} d\mathbf{v}_{N_0-1}\mathbf{a}_{iN_0-1}\cdot\nabla_{\mathbf{v}_i} f_{N_0-1} = 0 \end{aligned} \tag{2.8}$$

for the reduced distributions function f_{N_0-2}.

=> **Recurrence relation for** f_k

$$\begin{aligned} &\frac{\partial f_k}{\partial t}+\sum_{i=1}^{k}\mathbf{v}_i\cdot\nabla_{\mathbf{r}_i} f_k+\sum_{i=1}^{k}\sum_{j=1}^{k}\mathbf{a}_{ij}\cdot\nabla_{\mathbf{v}_i} f_k \\ &+\frac{N_0-k}{V}\sum_{i=1}^{k}\int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1}\mathbf{a}_{i,k+1}\cdot\nabla_{\mathbf{v}_i} f_{k+1} = 0 \end{aligned} \tag{2.9}$$

This is the **BBGKY Hierarchy** of kinetic equations.

This is a complete description without any reduction in the physics because each reduced distribution function couples to the next level.

Example $k=1$**:**

$$\frac{\partial f_1}{\partial t}+\mathbf{v}_1\cdot\nabla_{\mathbf{r}_1} f_1+\frac{N_0-1}{V}\int_{-\infty}^{\infty} d\mathbf{r}_2 d\mathbf{v}_2\mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1} f_2(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t)=0$$

where

- $f_1(\mathbf{r}_1,\mathbf{v}_1,t)\,d\mathbf{r}_1 d\mathbf{v}_1$ is the probability to find particle 1 in $I_1=[(\mathbf{r}_1,\mathbf{v}_1),(\mathbf{r}_1+d\mathbf{r}_1,\mathbf{v}_1+d\mathbf{v}_1)]$,
- $f_2(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t)\,d\mathbf{r}_1 d\mathbf{v}_1$ is the probability to find particles 1 and 2 in $I_2=[(\mathbf{r}_1,\mathbf{r}_2,\mathbf{v}_1,\mathbf{v}_2),(\mathbf{r}_1+d\mathbf{r}_1,\mathbf{r}_2+d\mathbf{r}_2,\mathbf{v}_1+d\mathbf{v}_1,\mathbf{v}_2+d\mathbf{v}_2)]$,

Example:

Single loaded dice which throws a 5 always: $P_1(x) = \delta(x-5)$ is the probability to throw a 5.

Two dice: $P_2(x,y) = \delta(x-5)\,\delta(y-5)$ is the joint probability to throw two 5's. No correlation => $P_2(x,y) = P_1(x)\,P_2(y)$

However, if the first throw sets a constraint for the second then

$$P_2(x,y) = P_1(x)\,P_2(y) + \delta P(x,y)$$

=> introduce **correlation function** $g(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t)$ with

$$f_2(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t) = f_1(\mathbf{r}_1,\mathbf{v}_1,t)\,f_1(\mathbf{r}_2,\mathbf{v}_2,t) + g(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t)$$

or symbolic:

$$f_2(1,2,t) = f_1(1,t)\,f_1(2,t) + g(1,2,t)$$

Substitute f_2 in the first equation of the hierarchy:

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1\cdot\nabla_{\mathbf{x}_1} f_1 + n_0\int_{-\infty}^{\infty} d2\,\mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1}\left[f_1(1,t)\,f_1(2,t) + g(1,2,t)\right] = 0$$

where $1 \equiv (\mathbf{r}_1,\mathbf{v}_1)$. With

$$\left\{n_0\int_{-\infty}^{\infty} d2\, f_1(2,t)\,\mathbf{a}_{1,2}\right\}\cdot\nabla_{\mathbf{v}_1} f_1(1,t) = \left\langle \mathbf{a}_{1,2}\right\rangle_{f_1}\cdot\nabla_{\mathbf{v}_1} f_1(1,t)$$

we obtain

$$\frac{\partial f_1(1,t)}{\partial t} + \mathbf{v}_1\cdot\nabla_{\mathbf{r}_1} f_1(1,t) + \mathbf{a}\cdot\nabla_{\mathbf{v}_1} f_1(1,t) = -n_0\int_{-\infty}^{\infty} d2\,\mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1} g(1,2,t)$$

with $\mathbf{a} = \left\langle \mathbf{a}_{1,2}\right\rangle_{f_1}$.

Collisionless Boltzmann equation for $g = 0$ => single particle distribution function.

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1\cdot\nabla_{\mathbf{r}_1} f_1 + \mathbf{a}\cdot\nabla_{\mathbf{v}_1} f_1 = 0$$

+ Maxwell's equations => Vlasov equations!

Collisions are determined only by the term $-n_0\int_{-\infty}^{\infty} d2\,\mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1} g(1,2,t)$ that is by the correlation $g(1,2,t)$. Thus there is need to determine $g = f_2 - f_1 f_1$.

=> $k = 2$ in BBGKY hierarchy:

$$\begin{aligned}\frac{\partial f_2}{\partial t} + (\mathbf{v}_1\cdot\nabla_{\mathbf{r}_1} + \mathbf{v}_2\cdot\nabla_{\mathbf{r}_2})\,f_2 + (\mathbf{a}_{12}\cdot\nabla_{\mathbf{v}_1} + \mathbf{a}_{21}\cdot\nabla_{\mathbf{v}_2})\,f_2 & \\ + n_0\int_{-\infty}^{\infty} d3\,(\mathbf{a}_{1,3}\cdot\nabla_{\mathbf{v}_1} + \mathbf{a}_{2,3}\cdot\nabla_{\mathbf{v}_2})\,f_3(1,2,3,t) &= 0\end{aligned}$$

Now one can factor f_3 similar to $f_2(1,2,t) = f_1(1) f_1(2) + g(1,2)$ (Mayer cluster expansion):

$$\begin{aligned} f_3(1,2,3) &= f_1(1) f_1(2) f_1(3) + f_1(1) g(2,3) \\ &\quad + f_1(2) g(1,3) + f_1(3) g(1,2) + h(1,2,3) \end{aligned}$$

where h represents the so-called three-particle correlations. Note that although not explicitly stated all of these are also time dependent.

$$\frac{\partial f_1(1,t)}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} f_1(1,t) + \mathbf{a} \cdot \nabla_{\mathbf{v}_1} f_1(1,t) = -n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g(1,2,t)$$

Assume that three-particle correlations are negligible, i.e., $h = 0$ and replace f_2 and f_3 in the equation for $k = 2$:

Term 1:

$$\begin{aligned} T_1 &= \frac{\partial}{\partial t}\left[f_1(1) f_1(2) + g(1,2)\right] \\ &= f_1(1)\, \partial_t f_1(2) + f_1(2)\, \partial_t f_1(1) + \partial_t g(1,2) \end{aligned}$$

Term 2:

$$\begin{aligned} T_2 &= f_1(2)\, \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} f_1(1) + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} g(1,2) + \{1 \longleftrightarrow 2\} \\ &\quad + f_1(1)\, \mathbf{v}_2 \cdot \nabla_{\mathbf{r}_2} f_1(2) + \mathbf{v}_2 \cdot \nabla_{\mathbf{r}_2} g(1,2) \end{aligned}$$

Term 3:

$$\begin{aligned} T_3 &= \mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1}\left[f_1(1) f_1(2) + g(1,2)\right] \\ &\quad + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2}\left[f_1(2) f_1(1) + g(1,2)\right] \end{aligned}$$

Term 4:

$$\begin{aligned} T_4 &= n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} \{f_1(1) f_1(2) f_1(3) + f_1(1) g(2,3) \\ &\quad + f_1(2) g(1,3) + f_1(3) g(1,2)\} \\ &\quad n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2} \{f_1(1) f_1(2) f_1(3) + f_1(1) g(2,3) \\ &\quad + f_1(2) g(1,3) + f_1(3) g(1,2)\} \\ &= f_1(2)\, n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1}\left[f_1(1) f_1(3) + g(1,3)\right] \\ &\quad + n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1}\left[f_1(1) g(2,3) + f_1(3) g(1,2)\right] \\ &\quad f_1(1)\, n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2}\left[f_1(2) f_1(3) + g(2,3)\right] \\ &\quad + n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2}\left[f_1(2) g(1,3) + f_1(3) g(1,2)\right] \end{aligned}$$

where indicates that the same terms as before should be added with the index 1 and 2 interchanged.

Most terms in the equation for $k = 2$ cancel each other. For instance

$$f_1(2)\left[\partial_t f_1(1) + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} f_1(1) + n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} \left(f_1(1) f_1(3) + g(1,3)\right)\right] = 0$$

The remaining terms yield:

$$\begin{aligned}
\frac{\partial g(1,2)}{\partial t} &+ (\mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} + \mathbf{v}_2 \cdot \nabla_{\mathbf{r}_2})\, g(1,2) \\
&= -(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2})\left[f_1(1) f_1(2) + g(1,2)\right] \\
&\quad -n_0 \int_{-\infty}^{\infty} d3\, (\mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1})\left[f_1(1) g(2,3) + f_1(3) g(1,2)\right] \\
&\quad -n_0 \int_{-\infty}^{\infty} d3\, (\mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2})\left[f_1(2) g(1,3) + f_1(3) g(1,2)\right]
\end{aligned}$$

Re-organizing the terms in this equation the two lowest order equations of the BBGKY hierarchy are summorized below

$$\begin{aligned}
\frac{\partial f_1(1)}{\partial t} &+ \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} f_1(1) + \mathbf{a} \cdot \nabla_{\mathbf{v}_1} f_1(1) \\
&= -n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g(1,2) \\
\frac{\partial g(1,2)}{\partial t} &+ (\mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} + \mathbf{v}_2 \cdot \nabla_{\mathbf{r}_2})\, g(1,2) \\
&+ \left((\mathbf{a} + \mathbf{a}_{12}) \cdot \nabla_{\mathbf{v}_1} + (\mathbf{a} + \mathbf{a}_{21}) \cdot \nabla_{\mathbf{v}_2}\right) g(1,2) \\
&= -(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2})\, f_1(1) f_1(2) \\
&\quad -n_0 \int_{-\infty}^{\infty} d3\, \left[\mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1}\left[f_1(1) g(2,3)\right] + \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2}\left[f_1(2) g(1,3)\right]\right]
\end{aligned}$$

- => 2 equations for f_1 and g
- Truncation ignores three-particle correlations (3 body collisions)
- Derivation requires $N_0 \gg 1$
- Equations are almost exact (approximations are usually very well satisfied.

2.2.4 Scaling of the BBGKY hierarchy

Recall

$$\begin{aligned}
\frac{\partial f_k}{\partial t} + \sum_{i=1}^{k} \mathbf{v}_i \cdot \nabla_{\mathbf{r}_i} f_k + \sum_{i=1}^{k} \sum_{j=1}^{k} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_k & \\
+ \frac{N_0 - k}{V} \sum_{i=1}^{k} \int_{-\infty}^{\infty} d\mathbf{r}_{k+1} d\mathbf{v}_{k+1} \mathbf{a}_{i,k+1} \cdot \nabla_{\mathbf{v}_i} f_{k+1} &= 0
\end{aligned}$$

Assume Yukawa type potential for individual charges with minimum at ϕ_0 at distance R

Normalization

$$\begin{aligned} \phi(r) &= \phi_0 \varphi(\xi) \qquad r = R\xi \\ v &= v_{th} u \qquad t = \tau_0 \tau \\ \tau_0 &= \frac{R}{v_{th}} \qquad a_{ij} = -\frac{1}{m}\nabla_{\mathbf{r}} \phi_{ij} = -\frac{\phi_0}{mR}\nabla_\xi \varphi = \frac{\phi_0}{mR}\widetilde{a}_{ij} \end{aligned}$$

Multiply equation with τ_0 (and note $f_{k+1} \sim f_k / v_{th}^3$

$$\begin{aligned} \frac{\partial f_k}{\partial t} + \sum_i \mathbf{u}_i \cdot \nabla_{\xi_i} f_k + \frac{R}{v_{th}^2}\frac{e\phi_0}{mR}\sum_{i,j} \widetilde{\mathbf{a}}_{ij} \cdot \nabla_{\mathbf{v}_i} f_k & \\ + n_0 \frac{R}{v_{th}^2}\frac{e\phi_0}{mR} R^3 \sum_i \int_{-\infty}^{\infty} d\xi_{k+1} d\mathbf{u}_{k+1} \widetilde{\mathbf{a}}_{i,k+1} \cdot \nabla_{\mathbf{u}_i} \widetilde{f}_{k+1} &= 0 \end{aligned}$$

- where $k \ll N_0$ is assumed.
- Coefficient for the third term: $c_3 = e\phi_0 / \left(m v_{th}^2\right) = \alpha \left(= \Lambda^{-2/3}\right)$
- Coefficient for the fourth term: $c_4 = n_0 R^3 \phi_0 / \left(m v_{th}^2\right) = \alpha\beta$
- scaling of the equations for f_1 and g

$$\begin{aligned} \frac{\partial f_1(1)}{\partial t} &+ \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} f_1(1) + \left\{ \frac{e\phi_0}{m v_{th}^2} = \Lambda^{-2/3} \right\} \mathbf{a} \cdot \nabla_{\mathbf{v}_1} f_1(1) \\ &= \left\{ n R^3 \frac{e\phi_0}{m v_{th}^2} \frac{g_0}{f_0} = \Lambda^{1/3} \frac{g_0}{f_0} \right\} - n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g(1,2) \\ \frac{\partial g(1,2)}{\partial t} &+ \left(\mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} + \mathbf{v}_2 \cdot \nabla_{\mathbf{r}_2}\right) g(1,2) \\ &+ \left\{ \frac{e\phi_0}{m v_{th}^2} = \Lambda^{-2/3} \right\} \left((\mathbf{a} + \mathbf{a}_{12}) \cdot \nabla_{\mathbf{v}_1} + (\mathbf{a} + \mathbf{a}_{21}) \cdot \nabla_{\mathbf{v}_2} \right) g(1,2) \\ &= -\left\{ \frac{e\phi_0}{m v_{th}^2} \frac{f_0^2}{g_0} = \Lambda^{-2/3} \frac{f_0^2}{g_0} \right\} \left(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2} \right) f_1(1) f_1(2) \\ &\quad - n_0 \left\{ n R^3 \frac{e\phi_0}{m v_{th}^2} f_0 = \Lambda^{1/3} f_0 \right\} \int_{-\infty}^{\infty} d3 \left[\mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} \left[f_1(1) g(2,3) \right] + \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2} \left[f_1(2) g(1,3) \right] \right] \end{aligned}$$

Special cases:

a) Knudsen gas (rarefied gas)

$\alpha = \phi_0 / (k_B T) = O(1)$

$\beta = n_0 R^3 \ll 1$ => Boltzmann equation

b) Weak interaction

$\alpha \ll 1$, $\beta \leq O(1)$ => Landau equation

c) Plasma case

$\alpha = e\phi_0 / (k_B T) = \Lambda^{-2/3} \ll 1$, $\beta \sim \Lambda \gg 1$ => Lenard-Balescu equation.

2.3 Lenard Balescu Equation

2.3.1 Bogoliubov's Hypothesis

Motivation: Simplify equation for $f = f_1$ and two particle correlations g.

Consider the basic case of a homogeneous plasma:

$$\begin{aligned} f(\mathbf{r},\mathbf{v},t) &= f(\mathbf{v},t) \\ g(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2,t) &= g(|\mathbf{r}_1-\mathbf{r}_2|,\mathbf{v}_1,\mathbf{v}_2,t) \\ \text{and} \quad \langle \mathbf{a}_{1,2}\rangle_f &= n_0\int_{-\infty}^{\infty} d\mathbf{r}_2 d\mathbf{v}_2\, f(2,t)\,\mathbf{a}_{1,2} = 0 \end{aligned}$$

which yields

$$\frac{\partial f_1(\mathbf{v}_1,t)}{\partial t} = -n_0\int_{-\infty}^{\infty} d\mathbf{r}_2 d\mathbf{v}_2\, \mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1} g(|\mathbf{r}_1-\mathbf{r}_2|,\mathbf{v}_1,\mathbf{v}_2,t)$$

$$\begin{aligned} \frac{\partial g(\Delta\mathbf{r}_{12},\mathbf{v}_1,\mathbf{v}_2,t)}{\partial t} &+ (\mathbf{v}_1\cdot\nabla_{\mathbf{r}_1}+\mathbf{v}_2\cdot\nabla_{\mathbf{r}_2})\,g(1,2) \\ &+ (\mathbf{a}_{12}\cdot\nabla_{\mathbf{v}_1}+\mathbf{a}_{21}\cdot\nabla_{\mathbf{v}_2})\,g(1,2) \\ &+ n_0\int_{-\infty}^{\infty} d3\,\mathbf{a}_{1,3}\cdot\nabla_{\mathbf{v}_1} f(1)\,g(2,3) + n_0\int_{-\infty}^{\infty} d3\,\mathbf{a}_{2,3}\cdot\nabla_{\mathbf{v}_2} f(2)\,g(1,3) \\ &= -(\mathbf{a}_{12}\cdot\nabla_{\mathbf{v}_1}+\mathbf{a}_{21}\cdot\nabla_{\mathbf{v}_2})\,f(1)\,f(2) \end{aligned}$$

Note terms: $\int_{-\infty}^{\infty} d3\,\mathbf{a}_{1,3}\cdot\nabla_{\mathbf{v}_1} f_1(3)\,g(1,2) = 0$ because of $\langle \mathbf{a}_{1,2}\rangle_f = 0$

Pulverisation $g/f_1 \sim \Lambda^{-1}$, $a_{12} \sim e^2/m \sim \Lambda^{-1}$ => term 3 $= O(\Lambda^{-2})$; other terms $= O(\Lambda^{-1})$

Or normalization: Term 3 $= O(\Lambda^{-1})$

=>

$$\begin{aligned} \partial_t f_1(\mathbf{v}_1,t) &= -n_0\int_{-\infty}^{\infty} d\mathbf{r}_2 d\mathbf{v}_2\, \mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1} g(|\mathbf{r}_1-\mathbf{r}_2|,\mathbf{v}_1,\mathbf{v}_2,t) \\ \partial_t g(1,2) + V_1 g + V_2 g &= S \end{aligned}$$

with

$$\begin{aligned} V_1 g &= \mathbf{v}_1\cdot\nabla_{\mathbf{r}_1} g(1,2) + \left(n_0\int_{-\infty}^{\infty} d3\, g(2,3)\,\mathbf{a}_{1,3}\right)\cdot\nabla_{\mathbf{v}_1} f(1) \\ V_2 g &= \mathbf{v}_2\cdot\nabla_{\mathbf{r}_2} g(1,2) + \left(n_0\int_{-\infty}^{\infty} d3\, g(1,3)\,\mathbf{a}_{2,3}\right)\cdot\nabla_{\mathbf{v}_2} f(2) \\ S(\Delta\mathbf{r}_{12},\mathbf{v}_1,\mathbf{v}_2,t) &= (\mathbf{a}_{12}\cdot\nabla_{\mathbf{v}_1}+\mathbf{a}_{21}\cdot\nabla_{\mathbf{v}_2})\,f(1)\,f(2) \end{aligned}$$

Bogoliubov's hierarchy of time scales:

τ_{diff} - Diffusion time (macroscopic),e.g., heat conduction or magnetic diffusion

$\ll \tau_{hydr}$ - Macroscopic dynamical time $\tau_{hydr} = L/c_s$

$\ll \tau_{kin}$ Relaxation time of the one-particle distribution $\tau_{kin} \approx L_c/v_{the} = 1/\nu_{coll}$

$\ll \tau_{corr}$ Relaxation time of the correlation function, $\tau \approx \lambda_D/v_{th} \approx 1/\omega_{pe}$

Hypothesis: => $f_1(1,t)$ varies slow compared to g

(g relaxes fast)

=> Source term S can be treated time independent

Procedure to derive the Lenard-Balescu equation:

Equation for $g(1,2,t)$ is a linear equation

=> Fourier transformation to solve for $g(1,2,t)$.

Need $g(t \to \infty)$ for the equation for f because g relaxes fast

=> Laplace transformation and substitution in equation

Fourier transform.:

$$\widetilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int d^3x \exp(-i\mathbf{k}\cdot\mathbf{r}) f(\mathbf{r})$$
$$f(\mathbf{r}) = \int d^3k \exp(i\mathbf{k}\cdot\mathbf{r}) \widetilde{f}(\mathbf{k})$$

Laplace transform.:

$$\overline{f}(\omega) = \int_0^\infty dt \exp(-i\omega t) f(t)$$
$$f(t) = \int_L \frac{d\omega}{2\pi} \exp(-i\omega t) \overline{f}(\omega)$$

with $\mathbf{r}$, $\mathbf{k}$, and t along the real axis and ω along a suitable contour L.

Example: Acceleration

$$\mathbf{a}_{12}(\mathbf{r}) = \frac{e^2}{4\pi\varepsilon_0 m_e |\mathbf{r}|^3}\mathbf{r}$$

has the Fourier transform.:

$$\widetilde{\mathbf{a}}_{12}(\mathbf{k}) = -\frac{i\mathbf{k}}{m_e}\varphi(\mathbf{k})$$
$$\varphi(\mathbf{k}) = \frac{e^2}{8\pi^3\varepsilon_0\mathbf{k}^2}$$

Solution to the kinetic equation: **Lenard-Balescu equation**

$$\frac{\partial f(\mathbf{v},t)}{\partial t} = -\frac{8\pi^4 n_0}{m_e^2}\nabla_{\mathbf{v}}\cdot\int d^3k d^3v' \mathbf{k}\mathbf{k}\cdot\frac{\varphi^2(\mathbf{k})}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} \delta\left(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')\right)\left[(\nabla_{\mathbf{v}'}-\nabla_{\mathbf{v}}) f(\mathbf{v}) f(\mathbf{v}')\right]$$

with the dielectric function

$$\varepsilon(\mathbf{k},\omega) = 1 + \frac{\omega_e^2}{\mathbf{k}^2}\int d^3v \frac{\mathbf{k}\cdot\nabla_{\mathbf{v}} f(\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v}}$$

Assumptions:

- homogeneous plasma
- 3 particle correlations negligible
- 2 particle correlation relaxes much faster than f
- "stable plasma", dynamic processes with frequency $\approx \omega_{pe}$ excluded.

2.3.2 Discussion of the Lenard-Balescu equation

(a) Problem with the equation

Large k (LB) $\sim \int \frac{dk}{k} \sim \ln k$ => (LB) diverges for $k \to \infty$

=> $k < 2\pi/\lambda_L$ λ_L- Landau length $\frac{e^2}{\lambda_L} = k_B T$

or $k < 2\pi/\lambda_{dB}$ λ_{dB}– deBroglie length $\lambda_{dB} = \frac{h}{mv_{th}}$

Why does (LB) diverge?

Assumption $|g| \ll |f_1 f_1|$ => simplification of (B2)

But: if two electron are very close => a_{12} very large

=> Incorrect to assume $|g| \ll |f_1 f_1|$ for small $|\mathbf{r}_1 - \mathbf{r}_2|$

(b) Properties

- $f \geq 0$ at $t = 0$ => $f \geq 0$ at all times
- Particle are conserved:

$$\frac{d}{dt} \int d^3v f(\mathbf{v}, t) = 0$$

- Momentum is conserved:

$$\frac{d}{dt} \int d^3v \mathbf{v} f(\mathbf{v}, t) = 0$$

- Kinetic energy is conserved:

$$\frac{d}{dt} \int d^3v \mathbf{v}^2 f(\mathbf{v}, t) = 0$$

- Any Maxwellian is a time independent solution
- As $t \to \infty$ any f satisfying (a) approaches a Maxwellian.

(c) Further simplification of the LB equation

Re-write LB:

$$\begin{aligned}\frac{\partial f(\mathbf{v},t)}{\partial t} &= -\nabla_{\mathbf{v}}\cdot\int d^3v'\underline{\underline{\mathbf{Q}}}(\mathbf{v},\mathbf{v}')\cdot[(\nabla_{\mathbf{v}}-\nabla_{\mathbf{v}'})]f(\mathbf{v})f(\mathbf{v}')\\ &\text{with}\\ \underline{\underline{\mathbf{Q}}}(\mathbf{v},\mathbf{v}') &= -\frac{8\pi^4 n_0}{m_e^2}\int d^3k\frac{\mathbf{k}\mathbf{k}\,\varphi^2(\mathbf{k})}{|\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}\delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))\end{aligned}$$

with

$$\varphi(\mathbf{k})=\frac{e^2}{8\pi^3\mathbf{k}^2}$$

and the Dielectric function

$$\varepsilon(\mathbf{k},\omega)=1+\frac{\omega_e^2}{k^2}\int d^3\tilde{v}\frac{\mathbf{k}\cdot\nabla_{\tilde{\mathbf{v}}}f(\tilde{\mathbf{v}})}{\omega-\mathbf{k}\cdot\tilde{\mathbf{v}}}$$

or

$$\begin{aligned}\varepsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v}) &= 1+\frac{\psi}{k^2\lambda_{de}^2}\\ &\text{with}\\ \psi(\mathbf{k},\mathbf{k}\cdot\mathbf{v}) &= v_{the}^2\int d^3\tilde{v}\frac{\mathbf{k}\cdot\nabla_{\tilde{\mathbf{v}}}f(\tilde{\mathbf{v}})}{\mathbf{k}\cdot(\mathbf{v}-\tilde{\mathbf{v}})}\end{aligned}$$

such that

$$\underline{\underline{\mathbf{Q}}}(\mathbf{v},\mathbf{v}')=-\frac{n_0e^4}{8\pi^2\varepsilon_0^2m_e^2}\int d^3k\frac{\mathbf{k}\mathbf{k}}{k^4}\frac{\delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))}{\left|1+\frac{\psi}{k^2\lambda_{de}^2}\right|^2}$$

Notes:

- ψ depends only on the direction not on the magnitude of k
- Max value of $|k|$given by impact parameter $\frac{1}{b}=\frac{4\pi\varepsilon_0 mv^2}{e^2}$

Evaluation of $\underline{\underline{\mathbf{Q}}}(\mathbf{v},\mathbf{v}')$:

Notes on the δ function

$$\begin{aligned}\delta(f(x)) &= \sum_i\left[\frac{df}{dx}(x_i)\right]^{-1}\delta(x-x_i)\\ x_i &= \text{zeros of } f(x)\\ \int f(x)\frac{d\delta(x-a)}{dx}dx &= -\frac{df}{da}(a)\end{aligned}$$

Assume: $k_1\,\|\mathbf{v}-\mathbf{v}'$ =>

$$Q_{ij}=-\frac{2n_0e^4}{m_e^2}\int dk_1dk_2dk_3\frac{k_ik_j}{k^4}\frac{1}{|\mathbf{v}-\mathbf{v}'|}\frac{\delta(k_1)}{\left|1+\frac{\psi}{k^2\lambda_{de}^2}\right|^2}$$

If $i = 1$ or $j = 1$ => $Q_{ij} = 0$

Since k_1 integral is trivial assume: $k_2 = k\cos\vartheta$ and $k_3 = k\sin\vartheta$

E.g.:

$$\begin{aligned} Q_{33} &= -\frac{n_0 e^4}{8\pi^2\varepsilon_0^2 m_e^2 |\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi} d\vartheta \sin^2\vartheta \int_0^{k_0} \frac{dk}{k} \frac{1}{\left|1+\psi/\left(k^2\lambda_{de}^2\right)\right|^2} \\ &= -\frac{n_0 e^4}{16\pi^2\varepsilon_0^2 m_e^2 |\mathbf{v}-\mathbf{v}'|}\int_0^{2\pi} d\vartheta \sin^2\vartheta \frac{\mathrm{Im}\left[\psi \ln\left(1+k_0\lambda_{de}^2/\psi\right)\right]}{\mathrm{Im}\,\psi} \end{aligned}$$

ψ - order unity, ignore Im => next chapter

$$1+k_0\lambda_{de}^2/\psi = 1+\lambda_{de}^2/\left(b^2\psi\right) = 1+\Lambda^2/\psi \approx \Lambda^2$$

=>

$$Q_{33}\left(\mathbf{v},\mathbf{v}'\right) = Q_{22}\left(\mathbf{v},\mathbf{v}'\right) = -\frac{n_0 e^4}{8\pi\varepsilon_0^2 m_e^2 |\mathbf{v}-\mathbf{v}'|}\ln\Lambda$$

Define $\mathbf{g} = \mathbf{v}-\mathbf{v}'$, $g = |\mathbf{v}-\mathbf{v}'|$

with $\mathbf{g}\|\mathbf{k}_1$ direction:

$$\underline{\underline{\mathbf{Q}}}\left(\mathbf{v},\mathbf{v}'\right) = -\frac{n_0 e^4}{8\pi\varepsilon_0^2 m_e^2}\ln\Lambda \frac{g^2\underline{\underline{1}}-\mathbf{gg}}{g^3}$$

=> Landau form of $\underline{\underline{\mathbf{Q}}}$

Derivation of a Fokker-Planck equation

General Fokker-Planck equation -> Nicholson

Note: $\nabla_{\mathbf{v}}\nabla_{\mathbf{v}} g = \frac{g^2\underline{\underline{1}}-\mathbf{gg}}{g^3}$ and $\nabla_{\mathbf{v}} g = -\nabla_{\mathbf{v}'} g$ and $\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{v}} g = 2/g$ =>

$$\begin{aligned} \frac{\partial f(\mathbf{v},t)}{\partial t} &= \frac{n_0 e^4}{8\pi\varepsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\cdot\int d^3v' \frac{g^2\underline{\underline{1}}-\mathbf{gg}}{g^3}\cdot\left(\nabla_{\mathbf{v}}-\nabla_{\mathbf{v}'}\right) f(\mathbf{v}) f\left(\mathbf{v}'\right) \\ &= \frac{n_0 e^4\ln\Lambda}{8\pi\varepsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\cdot\int d^3v' \left(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}} g\right)\cdot\left(\nabla_{\mathbf{v}}-\nabla_{\mathbf{v}'}\right) f(\mathbf{v}) f\left(\mathbf{v}'\right) \\ &= \frac{n_0 e^4\ln\Lambda}{8\pi\varepsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\cdot\left\{\left(\nabla_{\mathbf{v}} f(\mathbf{v})\right)\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' g f\left(\mathbf{v}'\right) - f(\mathbf{v})\int d^3v'\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}} g\right)\cdot\nabla_{\mathbf{v}'} f\left(\mathbf{v}'\right)\right\} \\ &= \frac{n_0 e^4\ln\Lambda}{8\pi\varepsilon_0^2 m_e^2}\left\{\nabla_{\mathbf{v}}\nabla_{\mathbf{v}} f(\mathbf{v})\cdot\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' g f\left(\mathbf{v}'\right) - 2\left(\nabla_{\mathbf{v}} f(\mathbf{v})\right)\cdot\int d^3v'\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{v}} g\right) f\left(\mathbf{v}'\right)\right\} \\ &= \frac{n_0 e^4\ln\Lambda}{8\pi\varepsilon_0^2 m_e^2}\left\{-4\nabla_{\mathbf{v}}\cdot f(\mathbf{v})\nabla_{\mathbf{v}}\int d^3v' \frac{f\left(\mathbf{v}'\right)}{g} + \nabla_{\mathbf{v}}\nabla_{\mathbf{v}} f(\mathbf{v})\cdot\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' g f\left(\mathbf{v}'\right)\right\} \end{aligned}$$

which yields

$$\frac{\partial f(\mathbf{v},t)}{\partial t} = -\nabla_{\mathbf{v}}\cdot\left[\mathbf{A} f(\mathbf{v})\right] + \frac{1}{2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\cdot\cdot\left[\underline{\underline{\mathbf{B}}} f(\mathbf{v})\right]$$

with

$$\mathbf{A}(\mathbf{v},t) = \frac{n_0 e^4 \ln\Lambda}{2\pi\varepsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\int d^3v' \frac{f(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|}$$

$$\underline{\underline{\mathbf{B}}}(\mathbf{v},t) = \frac{n_0 e^4 \ln\Lambda}{4\pi\varepsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' \left|\mathbf{v}-\mathbf{v}'\right| f\left(\mathbf{v}'\right)$$

this is the Landau form of the Fokker-Planck equation.

Term $\mathbf{A} \hookrightarrow$ slowing of particles by many small angle collisions

Term $\underline{\underline{\mathbf{B}}} \hookrightarrow$ increase of the perpendicular velocity from small angle collisions

..Figure..

Approximation to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \nu\nabla_{\mathbf{v}}\left[(\mathbf{v}-\mathbf{v}_0) f + v_{the}^2 \nabla_{\mathbf{v}} f\right]$$

where ν is a collision frequency and v_0 is a constant velocity. This is usually rather crude and only provides a rough idea of collisional effects.

Even more basic: Krook model

$$\frac{\partial f}{\partial t} = \nu\left(f - f_0\right)$$

which only describes the relaxation of any distribution function to f_0.