Chapter 3

Vlasov Equations

3.1 Collisionless Boltzmann Equation

The Vlasov equations have been introduced in section 1.3.3 and has theoretically motivated by the considerations in 2. Here we want to explore properties of the Vlasov equations, specifically with regard to kinetic waves in a so-called warm plasma. The full set of equations that has to be solved is from (1.18)

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s = 0 \tag{3.1}$$

for each particle species in combination with Maxwell's equations (1.9) - (1.12)

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0} \rho_c \\ \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j} \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

where we need the charge density and current density

$$\rho_c = \sum_s \rho_{cs} \quad and \quad \mathbf{j} = \sum_s \mathbf{j}_s$$

$$with$$

$$\begin{aligned} \rho_{cs} &= q_s \int_{-\infty}^{\infty} d^3 v f_s (\mathbf{x}, \mathbf{v}, t) \\ \mathbf{j}_s &= q_s \int_{-\infty}^{\infty} d^3 v \mathbf{v} f_s (\mathbf{x}, \mathbf{v}, t) \end{aligned}$$

The collisionless Boltzmann equation has a few noteworthy properties. Electromagnetic forces can be derived from a Hamiltonian

$$H_s(\mathbf{r},\mathbf{P}) = \frac{1}{2m_s}(\mathbf{P} - q_s\mathbf{A}(\mathbf{r},t))^2 + q_s\phi(\mathbf{r},t)$$

with the canonical coordinates $\mathbf{r}$ and the conjugate momentum $\mathbf{P} = m_s\mathbf{v} + q_s\mathbf{A}(\mathbf{r},t)$. In this case the collisionless Boltzmann equation (3.1) can be expressed through

$$\frac{\partial f_s}{\partial t} + \sum_i \dot{r}_i \frac{\partial f_s}{\partial r_i} + \sum_i \dot{P}_i \frac{\partial f_s}{\partial P_i} = 0$$

or with the definition of Poisson brackets

$$\{g,h\}_{\mathbf{r},\mathbf{P}} \equiv \sum_i \left(\frac{\partial g}{\partial P_i}\frac{\partial h}{\partial r_i} - \frac{\partial g}{\partial r_i}\frac{\partial h}{\partial P_i} \right)$$

the collisionless Boltzmann equation is

$$\frac{\partial f_s}{\partial t} + \{H_s, f_s\}_{\mathbf{r},\mathbf{P}} = 0$$

which is clear from the canonical equations

$$\begin{aligned} \frac{dr_i}{dt} &= \frac{\partial H_s}{\partial P_i} \\ \frac{dP_i}{dt} &= -\frac{\partial H_s}{\partial r_i} \end{aligned}$$

Considering the 6-dimensional space of the conjugate variables $\mathbf{R}_6 = (\mathbf{r},\mathbf{P})$ with the corresponding 6-dimensional velocity $\dot{\mathbf{R}}_6 = \left(\dot{\mathbf{r}},\dot{\mathbf{P}}\right)$, it follows immediately that the 6-dimensional divergence of this velocity

$$\nabla_6 \cdot \dot{\mathbf{R}}_6 = \sum_i \left(\frac{\partial \dot{r}_i}{\partial r_i} + \frac{\partial \dot{P}_i}{\partial P_i} \right) = 0$$

such that the collisionless Boltzmann equation is also equivalent to

$$\frac{\partial f_s}{\partial t} + \nabla_6 \cdot \left(\dot{\mathbf{R}}_6 f_s\right) = 0$$

These considerations demonstrate that the collisionless Boltzmann equation describes a continuity equation with an incompressible flow in phase space. This is specifically helpful to identify stationary solutions. In this case the Hamiltonian is a constant of motion and we need to determine solutions of

$$\{H_s, f_s\}_{\mathbf{r},\mathbf{P}} = 0$$

It is obvious that any distribution function that is a function of the Hamiltonian $f_s = F_s(H_s)$ represents a stationary solution of the collisionless Boltzmann equation

$$\{H_s, F_s(H_s)\}_{\mathbf{r},\mathbf{P}} = \frac{\partial F_s}{\partial H_s}\{H_s, H_s\}_{\mathbf{r},\mathbf{P}} = 0$$

Specifically, this implies that a Maxwell distribution $F_s = c\exp[-H_s/kT_s]$ represents a steady state solution. This approach can be generalized to other constants of motion for instance if the system has ignorable variable (translational invariance etc.)

3.1.1 Simple Electrostatic Waves

To introduce warm plasma waves let us first consider the case where the magnetic field vanishes $\mathbf{B} = 0$ and the electric field has only an x component. In this case the $\mathbf{k}$ vector of the waves is along the x direction.

Exercise: Demonstrate that the Maxwell equations predict no magnetic field for waves with $\mathbf{k} \parallel \mathbf{E}$.

All perturbations in this case are $\sim \exp(ikx - i\omega t) + c.c.$ In the equilibrium the electric field is 0 and the distribution function is $f_s = f_{s0} + \delta f_s$ with the equilibrium distribution f_{s0} and the perturbation δf_s such that the collisionless Boltzmann equation becomes

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v}\cdot\nabla(f_{s0} + \delta f_s) + \frac{q_s}{m_s}\mathbf{E}\cdot\nabla_{\mathbf{v}}(f_{s0} + \delta f_s) = 0$$

Since the equilibrium is time independent the 0th order terms in this $\mathbf{v}\cdot\nabla f_{s0} = 0$ such that any function $f_{s0} = F_s(\mathbf{v})$ is a solution. Here we assume the equilibrium solution to be the Maxwellian with a Temperature T_s.

$$f_{s0} = \frac{n_0}{(2\pi k_B T_s/m_s)^{3/2}}\exp\left[-\frac{m_s v^2}{2k_B T_s}\right] = \frac{n_0}{(2\pi)^{3/2} u_s^3}\exp\left[-\frac{v^2}{2u_s^2}\right] \tag{3.2}$$

The first order terms of the Boltzmann equation are

$$\frac{\partial \delta f_s}{\partial t} + v_x\partial_x\delta f_s + \frac{q_s}{m_s}E\partial_{v_x}f_{s0} = 0$$

where $E = E_x$. For plane wave solutions this equation is

$$\begin{aligned} -i\omega\delta f_s + ikv_x\delta f_s + \frac{q_s}{m_s}E\partial_{v_x}f_{s0} &= 0 \\ &or \\ \delta f_s &= \frac{-iq_s/m_s}{\omega - kv_x}E\partial_{v_x}f_{s0} \end{aligned}$$

Using Maxwell's equations we only need to solve the Poisson equation for this case of electrostatic waves. Using a proton and an electron plasma and noting that the equilibrium densities must satisfy neutrality $n_{i0} = n_{e0}$ we obtain

$$
\begin{aligned}
\partial_x E &= \frac{e}{\varepsilon_0}\left(\int_{-\infty}^{\infty} d^3v f_i - \int_{-\infty}^{\infty} d^3v f_e\right) \\
ikE &= \frac{e}{\varepsilon_0} E\left(-i\frac{e}{m_p}\int_{-\infty}^{\infty} d^3v \frac{\partial_{v_x} f_{i0}}{\omega - kv_x} - i\frac{e}{m_e}\int_{-\infty}^{\infty} d^3v \frac{\partial_{v_x} f_{e0}}{\omega - kv_x}\right) \\
&= -i\frac{e^2}{\varepsilon_0 m_e} E\left(\int_{-\infty}^{\infty} d^3v \frac{\partial_{v_x} f_{e0}}{\omega - kv_x} + \frac{m_e}{m_p}\int_{-\infty}^{\infty} d^3v \frac{\partial_{v_x} f_{i0}}{\omega - kv_x}\right)
\end{aligned}
$$

We can now carry out the integration over v_y and v_z separate from v_x and define

$$
\begin{aligned}
g(v_x) &= \frac{1}{n_0}\left[\int\int dv_y dv_z f_{e0} + \frac{m_e}{m_i}\int\int dv_y dv_z f_{i0}\right] \\
&= \frac{1}{(2\pi)^{1/2} u_e}\exp\left[-\frac{v_x^2}{2u_e^2}\right] + \frac{m_e}{m_i}\frac{1}{(2\pi)^{1/2} u_i}\exp\left[-\frac{v_x^2}{2u_i^2}\right]
\end{aligned}
$$

where we have defined $u_s = k_B T_s / m_s$ and $u = v_x$ for convenience such that the Poisson equation reduces to

$$
1 - \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{\infty} du \frac{dg(u)/du}{u - \omega/k} = 0 \tag{3.3}
$$

There are different ways to solve this dispersion relation. An exact solution is difficult such that the typical approach is an expansion depending on the wave modes and plasma parameters under consideration. The dispersion relation has to be integrated over the pole determined by the denominator of integrand which needs carefully consideration. A simple case is to ignore the ion contribution to g and to restrict waves to $\omega/k \gg u_{typ}$ such that $g(u)$ and $dg(u)/du$ are negligible at $u = \omega/k$. Then we can integrate the integral by parts and expand the denominator in a series of uk/ω.

$$
\begin{aligned}
1 - \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{\infty} du \frac{g(u)}{(\omega/k - u)^2} &= \\
1 - \frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{\infty} du g(u)\left(1 + \frac{2uk}{\omega} + \frac{3u^2k^2}{\omega^2}\right) &= 0
\end{aligned}
$$

The individual integrals are $\int du g(u) = 1$, $\int du g(u) u = 0$, and $\int du g(u) u^2 = k_B T_e = u_e^2$ with the result

$$
1 - \frac{\omega_e^2}{\omega^2} - \frac{3k^2 u_e^2 \omega_e^2}{\omega^4} = 0 \tag{3.4}
$$

This dispersion relation has two solutions

$$
\omega^2 = \frac{1}{2}\omega_e^2 \pm \frac{1}{2}\omega_e^2\left[1 + \frac{12k^2u_e^2}{\omega_e^2}\right]^{1/2} \tag{3.5}
$$

with the high frequency solution

$$\omega^2 = \omega_e^2 \left(1 + 3k^2\lambda_e^2\right) \tag{3.6}$$

which is the well known dispersion relation for Langmuir waves...

3.1.2 Laplace transform and Landau Damping

The problem treated in the previous section 3.1.1 produces a dispersion relation but it should have been treated as an initial value problem rather then with the simplified Fourier normal mode approach. The simple Fourier transformation assumes that the solution are periodic in space and in time.

$$\begin{aligned}\frac{\partial \delta f}{\partial t} + v\partial_x \delta f &= \frac{e}{m_e} E \partial_v f_0 \\ \partial_x E &= -\frac{e}{\varepsilon_0}\int_{-\infty}^{\infty} d^3v \delta f\end{aligned}$$

where we ignored the ion contribution and omit the index e for electrons to simplify the notation. Fourier transformation in space:

$$\begin{aligned}\delta f(x,v,t) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} dk \delta f(k,v,t)\exp(ikx) \\ E(x,t) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} dk E(k,t)\exp(ikx)\end{aligned}$$

changes the Vlasov equations to

$$\begin{aligned}\frac{\partial \delta f(k,v,t)}{\partial t} + ikv\delta f(k,v,t) &= \frac{e}{m_e} E(k,t)\partial_v f_0 \\ E(k,t) &= \frac{ie}{\varepsilon_0 k}\int_{-\infty}^{\infty} d^3v \delta f(k,v,t)\end{aligned}$$

As an initial value problem we use the Laplace transform

$$\begin{aligned}\delta f(k,v,p) &= \int_0^{\infty} dt \delta f(k,v,t)\exp(-pt) \\ E(k,p) &= \int_0^{\infty} dt E(k,t)\exp(-pt)\end{aligned}$$

Note that $p > 0$ (and actually sufficiently large) such that integral converges for $t \to \infty$. (or $E(k,t)\exp(-pt) \to 0$). With

$$\begin{aligned}\int_0^\infty dt\frac{\partial\delta f(k,v,t)}{\partial t}\exp(-pt) &= [\delta f(k,v,t)\exp(-pt)]_0^\infty + p\int_0^\infty dt\,\delta f(k,v,t)\exp(-pt)\\ &= p\delta f(k,v,p)-\delta f(k,v,0)\end{aligned}$$

the Laplace transforms of the Vlasov equations are

$$\begin{aligned}(p+ikv)\,\delta f(k,v,p) &= \frac{e}{m_e}E(k,p)\,\partial_v f_0+\delta f(k,v,0)\\ E(k,p) &= \frac{ie}{\varepsilon_0 k}\int_{-\infty}^{\infty}d^3v\,\delta f(k,v,p)\end{aligned}$$

Solution:

$$\delta f(k,v,p)=\frac{1}{p+ikv}\left[\frac{e}{m_e}E(k,p)\,\partial_v f_0+\delta f(k,v,0)\right] \tag{3.7}$$

and

$$\begin{aligned}E(k,p) &= \frac{ie}{\varepsilon_0 k}\int_{-\infty}^{\infty}d^3v\frac{1}{p+ikv}\left[\frac{e}{m_e}E(k,p)\,\partial_v f_0+\delta f(k,v,0)\right]\\ &= \frac{ie^2}{\varepsilon_0 m_e k}E(k,p)\int_{-\infty}^{\infty}d^3v\frac{\partial_v f_0}{p+ikv}-\frac{ie}{\varepsilon_0 k}\int_{-\infty}^{\infty}d^3v\frac{\delta f(k,v,0)}{p+ikv}\end{aligned}$$

or

$$E(k,p) = \frac{e}{\varepsilon_0 k^2\varepsilon(k,p)}\int_{-\infty}^{\infty}d^3v\frac{\delta f(k,v,0)}{v-ip/k} \tag{3.8}$$

with

$$\varepsilon(k,p) = 1-\frac{\omega_{pe}^2}{n_0k^2}\int_{-\infty}^{\infty}d^3v\frac{\partial_v f_0}{v-ip/k} \tag{3.9}$$

The inverse of the Laplace transform

$$E(k,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}dp\,E(k,p)\exp(pt)$$

yields

$$E(k,t)=\frac{e}{2\pi i\varepsilon_0 k^2}\int_{a-i\infty}^{a+i\infty}dp\exp(pt)\,\frac{\int_{-\infty}^{\infty}d^3v\frac{\delta f(k,v,0)}{v-ip/k}}{1-\frac{\omega_{pe}^2}{n_0k^2}\int_{-\infty}^{\infty}dv\frac{\partial_v g}{v-ip/k}} \tag{3.10}$$

Both the numerator and the denominator of the integral have poles at $v = ip/k$. In addition the Laplace integral has singularities at all points where $\varepsilon(k,p) = 0$. The integral needs to be carried out at $a > \mathrm{Max}(\mathrm{Re}\, p_i)$, i.e. to the right because this is where our Laplace transform was defined. However, since the functions in the numerator and denominator can be analytically continued into the part of the complex plane where $\mathrm{Re}p < 0$ we can shift the integration path to a value $p = a' < 0$. Similar to the procedure outlined in the appendix A.3 we can deform the integration path (Figure 3.1).

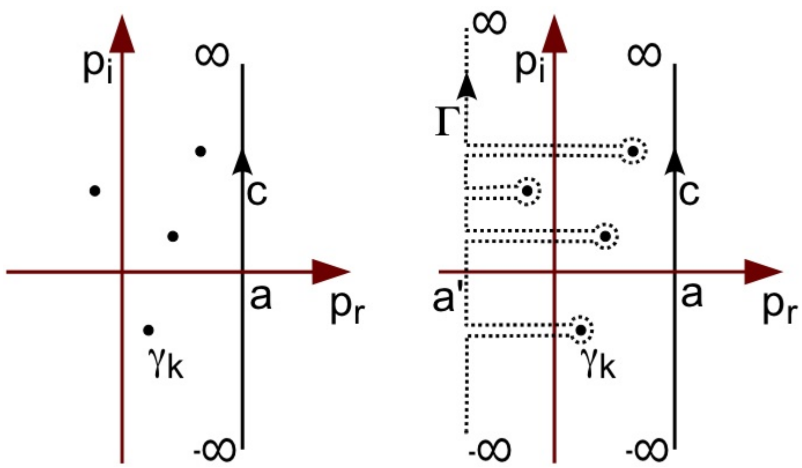

Figure 3.1: Illustration of the deformationof the integration path for the inverse Laplace transform.

In this case the solution is

$$E(k,t) = \sum_i r_i(p_i)\exp[p_i(k)t] + \frac{1}{2\pi i}\int_{a'-i\infty}^{a'+i\infty} dpE(k,p)\exp(pt) \tag{3.11}$$

where r_i are the residues defined by the single poles of ε. For $t \to \infty$ the integral approaches 0 and the solution is the sum of the residues.

In order to actually solve the dispersion relation we need to find the poles determined the zeroes of the dielectric function $\varepsilon(k,p) = 0$. The integration in (3.9) is along the real $v = v_x$ axis (For a Maxwellian we can perform the integration along the v_y, and v_z axes as demonstrated before). Assuming that the dispersion relation has a solution with $ip = \omega = \omega_r + i\gamma$ and that the real part of p is small. The pole in the integral of (3.9) is at $v = ip/k = (\omega_r + i\gamma)/k$. Depending on the sign of γ we need the analytic continuation of the integral in the dispersion relation such that the integral in (3.9) is

$$\int_{-\infty}^{\infty} dv \frac{\partial_v g}{v - \omega/k} = \begin{cases} \int_{-\infty}^{\infty} dv_r \frac{\partial_v g}{v_r - \omega_r/k - i\gamma/k} & \gamma > 0 \\ \mathsf{P}\int_{-\infty}^{\infty} dv_r \frac{\partial_v g}{v_r - \omega/k} + i\pi\partial_v g(v = \omega/k) & \gamma = 0 \\ \int_{-\infty}^{\infty} dv_r \frac{\partial_v g}{v_r - \omega_r/k - i\gamma/k} + 2\pi i\partial_v g(v = \omega/k) & \gamma < 0 \end{cases}$$

$$\text{with} \qquad \omega = \omega_r + i\gamma$$

where the case with $\gamma = 0$ is using the Plemelj formula. Note that $\gamma < 0$ corresponds to damping. The integration path for the different cases of γ is similar to the integration path for the analytic continuation discussed in the appendix (Figures A.4 and)

Solution of the Dispersion Relation

For the case of a weak damping we can split the dielectric function in it real and imaginary parts and expand ε in terms of γ

$$\varepsilon(k,\omega) = \varepsilon_r(k,\omega_r + i\gamma) + i\varepsilon_i(k,\omega_r + i\gamma)$$

Assuming weak damping $\gamma \ll \omega_r$ one can expand ε in a Taylor series at $\gamma = 0$

$$\varepsilon(k,\omega) = \varepsilon_r(k,\omega_r) + i\gamma\frac{\partial\varepsilon_r(k,\omega_r)}{\partial\omega_r} + i\varepsilon_i(k,\omega_r)$$

which yields

$$\gamma = -\frac{\varepsilon_i(k,\omega_r)}{\partial\varepsilon_r(k,\omega_r)/\partial\omega_r}$$

Using the Plemelj formula for $\gamma \ll \omega_r$ the dielectric function is

$$\varepsilon(k,\omega_r + i\gamma) = 1 - \frac{\omega_{pe}^2}{k^2}\mathsf{P}\int_{-\infty}^{\infty} dv\frac{\partial_v g(v)}{v - \omega/k} - i\pi\frac{\omega_{pe}^2}{k^2}\partial_v g(v = \omega/k) \tag{3.12}$$

Using the same reasoning $\omega/k \gg u_e$ we can integrate by parts and expand the integral the same way as for (3.4) to find the real part of the dielectric function:

$$\varepsilon_r(k,\omega_r) = 1 - \frac{\omega_e^2}{\omega_r^2} - \frac{3k^2u_e^2\omega_e^2}{\omega_r^4} = 0$$

With the high frequency (Langmuir wave solution of (3.6)). The imaginary part γ is determined by

$$\begin{aligned}
\frac{\partial\varepsilon_r(k,\omega_r)}{\partial\omega_r} &\approx 2\frac{\omega_e^2}{\omega_r^3} \approx \frac{2}{\omega_e} \\
\varepsilon_i(k,\omega_r) &= -\pi\frac{\omega_{pe}^2}{k^2}\,\partial_v g|_{v=\omega_r/k} \\
\gamma &= \frac{\pi\omega_{pe}^3}{2k^2}\,\partial_v g|_{v=\omega_r/k}
\end{aligned}$$

Such that the complete solution of the dispersion relation for Langmuir waves is

$$\omega_r + i\gamma = \omega_e\left(1 + \frac{3}{2}k^2\lambda_e^2\right) + i\frac{\pi\omega_{pe}^3}{2k^2}\,\partial_v g|_{v=\omega/k}$$

with a Maxwellian $g = (2\pi)^{-1/2}u_e^{-1}\exp\left[-v^2/2u_e^2\right]$ and $v = \omega_e/k$ the last term is

$$\begin{aligned}
\partial_v g| &= -\frac{v}{(2\pi)^{1/2} u_e^3} \exp\left[-\frac{v^2}{2u_e^2}\right] \quad \text{and} \\
\gamma &= -\frac{\pi \omega_{pe}^3}{2k^2} \frac{\omega_e/k}{(2\pi)^{1/2} u_e^3} \exp\left[-\frac{\omega_e^2\left(1+3k^2\lambda_e^2\right)}{2k^2 u_e^2}\right] \\
&= -\omega_e \frac{1}{k^3\lambda_e^3}\left(\frac{\pi}{2}\right)^{1/2} \exp\left[-\frac{1}{2k^2\lambda_e^2} - \frac{3}{2}\right]
\end{aligned}$$

Plasma Dispersion Function for electron distribution function

$$\begin{aligned}
\varepsilon(k,\omega) &= 1 - \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{\infty} dv \frac{\partial_v g}{v-\omega/k} \\
&\text{with} \\
g(v) &= \frac{1}{(2\pi)^{1/2} u_e} \exp\left[-\frac{v^2}{2u_e^2}\right]
\end{aligned}$$

with $\lambda_{De} = u_e/\omega_{pe}$ and

$$t = \frac{v}{2^{1/2} u_e}, \quad \zeta = \frac{\omega}{k}\frac{1}{2^{1/2} u_e}$$

$$\begin{aligned}
\varepsilon(k,p) &= 1 - \frac{\omega_{pe}^2}{k^2}\frac{1}{(2\pi)^{1/2} u_e}\frac{1}{2^{1/2} u_e}\int_{-\infty}^{\infty} dt \frac{\partial_t \exp\left(-t^2\right)}{t-\zeta} \\
&= 1 + \frac{2}{2k^2\lambda_{De}^2}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} dv \frac{t-\zeta+\zeta}{t-\zeta}\exp\left(-t^2\right) \\
&= 1 + \frac{1}{k^2\lambda_{De}^2}\left[1 + \frac{\zeta}{\pi^{1/2}}\int_{-\infty}^{\infty} dv \frac{\exp\left(-t^2\right)}{t-\zeta}\right] \\
&= 1 + \frac{1}{k^2\lambda_{De}^2}\left[1+\zeta Z(\zeta)\right]
\end{aligned}$$

where

$$Z(\zeta) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} \frac{\exp\left(-z^2\right) dz}{z-\zeta} \tag{3.13}$$

with the derivative

$$\frac{dZ(\zeta)}{d\zeta} = -2\left[1+\zeta Z(\zeta)\right]$$

such that for electrons ions:

$$\varepsilon(k,p) = 1 - \frac{1}{2k^2\lambda_{De}^2}\frac{dZ(\zeta_e)}{d\zeta_e} - -\frac{1}{2k^2\lambda_{Di}^2}\frac{dZ(\zeta_i)}{d\zeta_i}$$

is the so-called plasma dispersion function (see also Section A.4). This function can be expanded in series expansions for $\zeta < 1$ and $\zeta \gg 1$ where the same rule regarding integration contours apply. Specifically if the imaginary part of ζ is 0 the plasma dispersion function can be written as

$$Z(\zeta) = \frac{1}{\pi^{1/2}}\mathrm{P}\int_{-\infty}^{\infty}\frac{\exp\left(-z^2\right)dz}{z-\zeta} + i\pi^{1/2}\exp\left(-\zeta^2\right)$$

i.e. it is composed of the Principal part of the integral and half of the residue as outlined by the Plemlj formula.

Wave energy and Landau damping

Change in kinetic energy of the electron distribution in the Langmuir wave field:

$$\begin{aligned}\delta W_e &= \frac{m_e}{2}\int_{-\infty}^{\infty}dv\left\{(v+\langle\Delta v\rangle)^2 - v^2\right\}f(v)\\ &= m_e\int_{-\infty}^{\infty}dvv\langle\Delta v\rangle f(v)\end{aligned}$$

with $\langle\Delta v\rangle$ being the average change in velocity associated with the wave electric field $\delta E(x,t) = \delta E_0\sin[kx(t)-\omega t]$ in the frame of the phase velocity $v_0 = \omega/k$:

$$\delta E(x,t) = \delta E_0\sin[kx]$$

with the substitution $v = v' + v_0$, and assuming that contributions due to the velocity change are largest in the vicinity of $v = v_0 = \omega/k$

$$\begin{aligned}\delta W_e &= m_e\int_{-\infty}^{\infty}dv\left(v'+v_0\right)\langle\Delta v\rangle f\left(v'+v_0\right)\\ &= m_e\int_{-\infty}^{\infty}dv'\left(v'+v_0\right)\langle\Delta v\rangle\left[f(v_0) + v'\left.\frac{\partial f}{\partial v'}\right|_{v'=v_0}\right]\end{aligned}$$

Change of velocity Δv in the wave field (for electrons)averaged over one wavelength: Here we expand the integral up to 2nd order in δE_0

and choose the initial conditions $x(t=0) = x'$ and $v(t=0) = v'$

$$
\begin{aligned}
\frac{dv}{dt} &= \frac{q\delta E_0}{m}\sin\left[kx(t)\right] \\
\Delta v(t) &= a\int_0^t dt'\sin\left[kx\left(t'\right)\right] \\
x\left(t'\right) &= x' + \int_0^{t'} dt'' v\left(t''\right) \\
v\left(t''\right) &= v' + a\int_0^{t''} dt'''\sin\left[kx\left(t'''\right)\right] \\
&\approx v' + a\int_0^{t''} dt'''\sin\left[k\left(x' + v't'''\right)\right] \\
\text{with}: \quad & a = \frac{q\delta E_0}{m}
\end{aligned}
$$

This yields for $\tilde{v}(t'')$:

$$
v\left(t''\right) = v' - \frac{a}{kv'}\left[\cos\left(kx' + kv't''\right) - \cos kx'\right]
$$

and the integral for $\tilde{x}(t')$:

$$
\begin{aligned}
x\left(t'\right) &= x' + v't' + \frac{at'}{kv'}\cos kx' - \frac{a}{k^2v'^2}\left[\sin\left(kx' + kv't'\right) - \sin kx'\right] \\
&= x' + v't' + \Lambda\left(t'\right) \\
\mathit{with} \quad & \Lambda\left(t'\right) = \frac{at'}{kv'}\cos kx' - \frac{a}{k^2v'^2}\left[\sin\left(kx' + kv't'\right) - \sin kx'\right]
\end{aligned}
$$

Since $\Lambda \sim \delta E_0$ one can expand the integrant in

$$
\begin{aligned}
\Delta v(t) &= a\int_0^t dt'\sin\left[k\left(x' + v't' + \Lambda\left(t'\right)\right)\right] \\
&= a\int_0^t dt'\left[\sin\left(kx' + kv't'\right) + \Lambda\left(t'\right)\cos\left(kx' + kv't'\right)\right]
\end{aligned}
$$

We can now average over one wavelength and integrate over time with the result:

$$
\langle\Delta v(t)\rangle = \frac{\varepsilon_0}{2}\delta E_0^2\frac{\omega_{pe}^2}{n_0 m}\frac{1}{k^2v'^3}\left[\cos\left(kv't\right) - 1 + \frac{1}{2}\left(kv't\right)\sin\left(kv't\right)\right]
$$

Exercice: Compute the average velocity change $\langle\Delta v(t)\rangle$ over one wave length.

We can use this expression in the integral for the energy change of the distribution.

$$\begin{aligned}\delta W_e &= m_e \int_{-\infty}^{\infty} dv' \left(v' + \frac{\omega}{k}\right) \langle \Delta v \rangle \left[f(v_0) + v' \left.\frac{\partial f}{\partial v'}\right|_{v_0} \right] \\ &= m_e \int_{-\infty}^{\infty} dv' v' \langle \Delta v \rangle \frac{\omega}{k} \left.\frac{\partial f}{\partial v'}\right|_{v_0} \\ &= m_e \frac{\omega}{k} \left.\frac{\partial f}{\partial v'}\right|_{v_0} \int_{-\infty}^{\infty} dv' v' \langle \Delta v \rangle\end{aligned}$$

where only the $v' \langle \Delta v \rangle$ terms survive in the integral because $\langle \Delta v(t) \rangle$ is an odd function of v'. Also the term with $f(v_0)$ term can be shown to be much smaller then the term associated with $\partial f / \partial v'$. We can now use $\langle \Delta v(t) \rangle$ substitute $kv't = z$ and evaluate the integral.

$$\begin{aligned}\delta W_e &= -\frac{\varepsilon_0}{2} \delta E_0^2 \pi \frac{\omega_{pe}^2}{n_0 m_e} \frac{t}{k} m_e \frac{\omega}{k} \left.\frac{\partial f}{\partial v'}\right|_{v_0} \\ &= -W_w(0) \pi \omega_{pe} \frac{\omega_{pe}^2}{n_0 k^2} t \left.\frac{\partial f}{\partial v'}\right|_{v_0}\end{aligned} \tag{3.14}$$

where we have replaced ω with ω_{pe}. The average wave energy as a function of time is $W_w(t) = \varepsilon_0 \delta E(t) \cdot \delta E^\star(t)/2$. Since the sum of the changes of the wave and particle energy is 0 the initial change of the particle energy is

$$\frac{\delta W_e}{t} = -\frac{dW_w}{dt} = 2\gamma_L W_w(0) \exp(-2\gamma t)$$

Comparison of this with equation (3.14) yields a damping rate of

$$\gamma_L = \frac{\pi \omega_{pe}^3}{2 n_0 k^2} \left.\partial_v f\right|_{v=\omega/k}$$

This is the same damping rate as derived before because f is normalized to n_0 and g in our prior derivation of the Landau damping for Langmuir waves is normalized to 1. The derivation here assumes explicit that δE_0 is sufficiently small to do the expansion in the integral over the particle orbits. Note, the importance of the derivative of f for the damping rate. The velocity change $\langle \Delta v(t) \rangle$is also largest in a small vicinity of $v_0 = \omega/k$ as illustrated in Figure 3.2.

Now let us briefly consider nonlinear effects of Landau damping. In the frame moving with the pase velocity ω/k, the electric field is

$$\delta E(x,t) = \delta E_0(t) \sin[kx]$$

where δE_0 actually changes slowly in time according to the damping of the wave (Figure 3.2). The electric field can be associated with a potential $\delta E = -\partial_x \varphi$ or

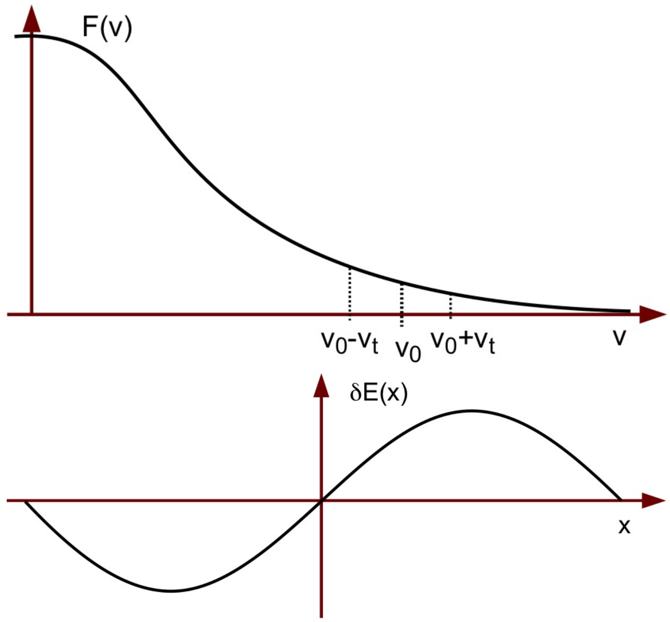

Figure 3.2: Distribution function indicating the resonant region in which particle velcities change most due to Landau damping and the electric field seen by particles moving with a velocity close to the wave speed.

$$\varphi(x,t) = \frac{\delta E_0(t)}{k}\cos[kx] + \text{const}$$

illustrated in Figure 3.3. If the electric field and the potential don't change in time (or change sufficiently slowly) the total energy of particles moving in this potential is

$$H = \frac{m}{2}v^2 + e\frac{\delta E_0(t)}{k}[1-\cos[kx]]$$

where we have chosen the integration constant of the potential such that H is always positive. The equation of motion for the individual particles is

$$m\ddot{x} = -e\delta E_0 \sin[kx]$$

and it is obvious that particles with a total energy $w \leq 2e\delta E_0(t)/k$ are trapped in the potential and carry out an oscillation with maximum velocity of $v = (2w/m)^{1/2}$ and the mirror points at $\varphi(x_m) = w$. The phase space trajectories of these particles are shown in Figure 3.3. Near the minimum of the potential we can expand the sin function and find for low energy particle a bounce frequency of

$$\omega_b = \left(\frac{e\delta E_0 k}{m}\right)^{1/2}$$

The physics of this particle motion is straightforward. Considering particles unifomly distributed throughout the potential, these particles will have altered their position and velocity after half a bounce time strongly (assume opposite values of the velocity). Thereby the distribution function will have significantly changed shape in the region where these resonant particles reside. To illustrate this consider the particles labelled with 1 and 2 in Figure 3.3 at time $t = 0$ both located at $x = 0$ but particle 1 with a positive velocity $v_1 (t = 0) = v_t = (2w_B/m)^{1/2}$ and particle 2 with a negative velocity $v_2 (t = 0) = -v_t$. After half a bounce time they have exchanged location in phase space and particle 1 has now a negative velocity of $v_1 (t = 0.5\tau_b) = -v_t$ and particle 2 a positive velocity of $v_2 (t = 0.5\tau_b) = v_t$. Assuming that the plot of the distribution function in Figure 3.2 is a cut at $x = 0$ and noting that $df/dt = 0$ (along the path in phase space!) this exchange of particles after half a bounce time implies that the value of the distribution function at $v_0 + v_t$ has changed to $f(v_0 + v_t, 0.5\tau_b) = f(v_0 - v_t, 0)$ and vice versa $f(v_0 - v_t, 0.5\tau_b) = f(v_0 + v_t, 0)$. In other words the average slope of $f$between $v_0 - v_t$ and $v_0 + v_t$ has reversed. Particle bounce time and frequency are a function of energy in the potential well but the overall tendency of the particle motion is to uniformly distribute the trapped particles effectively flattening the distribution in the vicinity of $v_0 = \omega/k$.

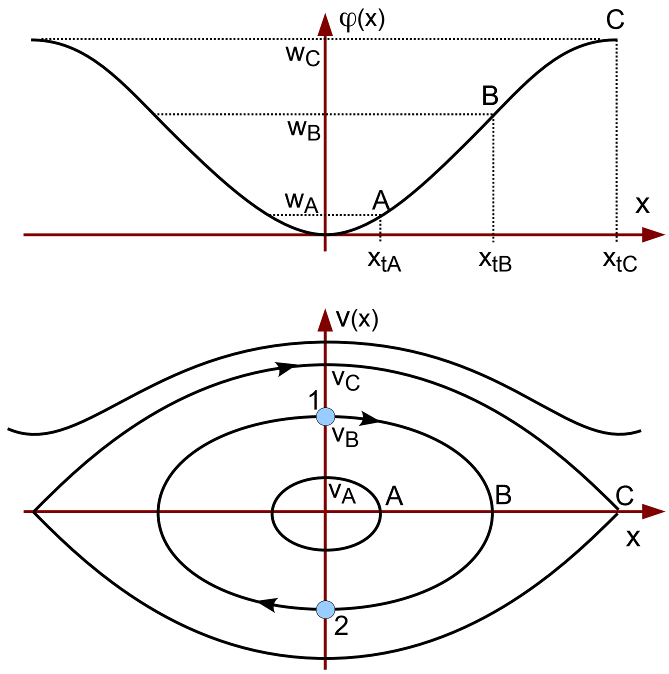

Figure 3.3: Potential associated with the wave electric field and phase space plot of particle trajectories in the wave potential.

In velocity space the resonant region is defined by the maximum value of the potential such that the maximum velocities of the trapped population are $v_{t0} = (4e\delta E_0(t)/km_e)^{1/2}$ and the flat region of the distribution is $v_0 - v_{t0} \leq v \leq v_0 + v_{t0}$. The bounce time appoaches ∞ for $v_t \longrightarrow v_{t0}$.

Clearly this represents such a strong change of the distribution that the linear theory (assuming a constant distribution) is not anymore applicable. The time scale for this linear approximation should satisfy $t \ll \tau_b = 1/\omega_b$. One could argue that after a full bounce time the original distribution is restored, however, this is only partially the case because the bounce period depends on the energy and trapped particles with higher energy have a longer bounce period such that these particles get out of phase.

This discussion also bears on the question of irreversibility of Landau damping. Note that Landau damping does not depend on collisions, and the collisionless Boltzmann equation is like any advective equation reversible in time. Clearly, for sufficiently short periods of time, Landau damping should be considered reversible. However, for long times the deformation of the distribution function approaches increasingly shorter scales which at some point in time go below any reasonable plasma physical lenth scale. At this point in time, changes must be considered irreversible. This behaviour is closely related to the topic of deterministic chaos and typical for a system with several degrees of freedom (complex system). Considering, for instance, the game of pool billiard with the errors in momentum and position only limited by Heisenberg's uncertainty relation. Seven consecutive collisions of a ball are typically suffcient to not being able to predict the outcome anymore (momentum and position of the ball).

Ion-Acoustic and Other Electrostatic Waves

Considering the ion contribution for electrostatic waves:

$$\varepsilon(k,p) = 1 + \frac{\omega_{pe}^2}{n_0 k^2}\int_{-\infty}^{\infty} dv \frac{f_{0e}(v)}{(v-ip/k)^2} + \frac{\omega_{pi}^2}{n_0 k^2}\int_{-\infty}^{\infty} dv \frac{f_{0i}(v)}{(v-ip/k)^2}$$

Expansion for $\omega/k \gg v_{ths}$

$$1 - \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega^2} - \frac{3k^2}{\omega^4}\left(u_e^2\omega_{pe}^2 + u_i^2\omega_{pi}^2\right) = 0$$

Solution

$$\omega^2 = \omega_{pe}^2\left(1 + \frac{m_e}{m_i}\right)\left[1 + \frac{3k^2\lambda_D^2}{1+m_e m_i}\left(1 + \frac{m_e^2}{m_i^2}\frac{T_i}{T_e}\right)\right]$$

Considering expansion for intermediate phase velocity

$$\frac{k_B T_i}{m_i} \ll \frac{\omega^2}{k^2} \ll \frac{k_B T_e}{m_e}$$

leads to

$$\varepsilon(k,\omega) = 1 + \frac{1}{k^2\lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\frac{k_B T_i}{m_i}\right) = 0$$

Solution

$$\omega_{ia}^2 = \frac{\omega_{pi}^2}{1 + 1/k^2\lambda_D^2}\left[1 + \frac{3T_i}{T_e}\left(1 + k^2\lambda_D^2\right)\right] = \frac{k^2 c_{ia}^2}{1 + k^2\lambda_D^2}\left[1 + \frac{3T_i}{T_e}\left(1 + k^2\lambda_D^2\right)\right]$$

with the ion sound speed $c_{ia}^2 = k_B T_e/m_i$ and $c'_{ia} = c_{ia}\left(1 + 3T_i/T_e\right)^{1/2}$. Considering the long wave length $k^2\lambda_D^2 \ll 1$ -> approximate solution

$$\omega = \pm k c'_{ia}$$

Short wavelength $k^2\lambda_D^2 > O(1)$ -> solution

$$\omega^2 = \omega_i^2 \left(1 + 3k^2\lambda_{Di}^2\right)$$

Similar: Electron acoustic waves where an additional high temperature (enrgetic) electron component is added.

3.2 Kinetic Waves in a Magnetized Plasma

3.2.1 Derivation of the general dispersion relation

Ampere's law and the induction equation

$$\begin{aligned} \nabla \times \mathbf{B} &= \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j} \\ \nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

yield

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) - \varepsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{j}}{\partial t}$$

Using a plane wave ansatz $\delta\mathbf{E}(\omega\mathbf{k}) = \delta\mathbf{E}_0(\omega\mathbf{k})\exp(i\mathbf{k}\mathbf{x} - i\omega t)$ yields

$$\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\underline{\underline{1}} - \mathbf{k}\mathbf{k}\right]\delta\mathbf{E}_0(\omega,\mathbf{k}) = i\omega\mu_0\mathbf{j}_0(\omega,\mathbf{k})$$

With Ohm's law $\delta\mathbf{j}(\omega\mathbf{k}) = \underline{\underline{\sigma}}(\omega\mathbf{k}) \cdot \delta\mathbf{E}_0(\omega\mathbf{k})$ the general dispersion relation given by

$$D(\omega\mathbf{k}) = \det\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\underline{\underline{1}} - \mathbf{k}\mathbf{k} - i\omega\mu_0\underline{\underline{\sigma}}(\omega,\mathbf{k})\right] = 0 \tag{3.15}$$

With the definition of the dielectric tensor

$$\underline{\underline{\varepsilon}}(\omega,\mathbf{k}) = \underline{\underline{1}} + \frac{i}{\omega\varepsilon_0}\underline{\underline{\sigma}}(\omega\mathbf{k}) \tag{3.16}$$

the general dispersion relation can be re-written as

$$\det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{k}\mathbf{k}}{k^2} - \underline{\underline{1}}\right) + \underline{\underline{\varepsilon}}(\omega,\mathbf{k})\right] = 0 \tag{3.17}$$

In order to solve this dispersion relation it is necessary to determin the dielectric tensor.

3.2.2 Magnetized Collisionless Plasma Waves

Here we assume that the theory behind kinetic waves in an unmagnetized plasma is known. We also assume that the equilibrium electric field is 0 and the magnetic field has only a z component. In a uniform medium the linearized Vlasov equations are

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\frac{q}{m}\mathbf{v}\times\mathbf{B}_0\cdot\frac{\partial}{\partial\mathbf{v}}\right)\delta f(\mathbf{r},\mathbf{v},t) &= -\frac{q}{m}(\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B})\cdot\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}}\\ \delta\mathbf{j} &= \sum_s q_s\int d^3v\mathbf{v}\delta f_s\\ \delta\rho_e &= \sum_s q_s\int d^3v\delta f_s\end{aligned}$$

where the rest frame electric field is 0, $\mathbf{B}_0 = B_0\mathbf{e}_z$, and we have omitted the species index s in the collisionless Boltzmann for convenience.

The lhs of the linearized Boltzmann equation represents the total time derivative along the six-dimensional path $(\mathbf{r}(t),\mathbf{v}(t))$ such that

$$\frac{d\delta f[\mathbf{v}(t),\mathbf{r}(t),t]}{dt} = -\frac{q}{m}(\delta\mathbf{E}[\mathbf{r}(t),t)]+\mathbf{v}\times\delta\mathbf{B}[\mathbf{r}(t),t)])\cdot\frac{\partial f_0[\mathbf{v}(t)]}{\partial\mathbf{v}(t)}$$

which can formally be integrated in time

$$\delta f[\mathbf{r}(t),\mathbf{v}(t),t] = -\frac{q}{m}\int_{-\infty}^{t}dt'\{\delta\mathbf{E}[\mathbf{r}(t),t)]+\mathbf{v}\times\delta\mathbf{B}[\mathbf{r}(t),t)]\}\cdot\frac{\partial f_0[\mathbf{v}(t)]}{\partial\mathbf{v}(t)}$$

This integral requires the knowledge of $(\mathbf{r}(t),\mathbf{v}(t))$ for all particles for $t' < t$ which is just the solution to the Lorentz force equations. For linear perturbations the particle trajectories are usually assumed to be determined by the equilibrium electric and magnetic fields such that the particle perform the usual gyromotion in the x and y plane and are moving with constant velocity along the z direction with the formal solution:

$$\begin{aligned}\mathbf{r}(t') &= \mathbf{r}+\int_t^{t'}\dot{\mathbf{r}}(t'')\,dt''\\ \mathbf{v}(t') &= \mathbf{v}+\int_t^{t'}\dot{\mathbf{v}}(t'')\,dt''\end{aligned}$$

such that $\mathbf{r}(t) = \mathbf{r}$ and $\mathbf{v}(t) = \mathbf{v}$. We can easily integrate the Lorentz force equation with the explcit result

$$\begin{aligned}v_x(t'-t) &= v_\perp\cos\left[\omega_g\left(t'-t\right)+\psi\right] && x(t'-t)-x = v_\perp/\omega_g\left\{\sin\left[\omega_g\left(t'-t\right)+\psi\right]-\sin[\psi]\right\}\\ v_y(t'-t) &= v_\perp\sin\left[\omega_g\left(t'-t\right)+\psi\right] && y(t'-t)-y = -v_\perp/\omega_g\left\{\cos\left[\omega_g\left(t'-t\right)+\psi\right]-\cos[\psi]\right\}\\ v_z(t'-t) &= v_\parallel && z(t'-t)z = v_\parallel(t'-t)\end{aligned}$$

Notes

- The perturbed electric and magnetic fields $\delta\mathbf{E}(\mathbf{r},t)$ and $\delta\mathbf{B}(\mathbf{r},t)$ with a plane wave approach are $\sim\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$.

- Substituting the particle orbits into the the fields and the distribution function generates an integrant that depends on;y on time.

With the transformation $\tau = t' - t$ the integration becomes

$$\delta f(\mathbf{v}) = -\frac{q}{m}\int_{-\infty}^{0} d\tau \{\delta\mathbf{E}(\tau) + \mathbf{v}(\tau) \times \delta\mathbf{B}(\tau)\} \cdot \frac{\partial f_0[\mathbf{v}(\tau)]}{\partial \mathbf{v}(\tau)}$$

Using the plane wave approach $\exp[i(\mathbf{k}\cdot\mathbf{r}(\tau) - \mathbf{r}) - i\omega\tau]$ yields for the perturbed distribution function

$$\begin{aligned}
\delta f(\mathbf{v}) &= -\frac{q}{m}\int_{-\infty}^{0} d\tau \left\{\delta\hat{\mathbf{E}} + \mathbf{v}(\tau) \times \delta\hat{\mathbf{B}}\right\} \exp[i(\mathbf{k}\cdot\mathbf{r}(\tau) - \mathbf{r}) - i\omega\tau] \cdot \frac{\partial f_0[\mathbf{v}(\tau)]}{\partial \mathbf{v}(\tau)} \\
&\mathit{or} \\
&= -\frac{q\delta\hat{\mathbf{E}}}{m\omega} \cdot \int_{-\infty}^{0} d\tau \left\{\mathbf{v}(\tau)\mathbf{k} + \underline{\underline{1}}(\omega - \mathbf{k}\cdot\mathbf{v}(\tau))\right\} \exp[i(\mathbf{k}\cdot\mathbf{r}(\tau) - \mathbf{r}) - i\omega\tau] \cdot \frac{\partial f_0[\mathbf{v}(\tau)]}{\partial \mathbf{v}(\tau)} \qquad (3.18)
\end{aligned}$$

where the induction equationis used to eliminate $\delta\mathbf{B}$ through $\mathbf{k}\times\delta\mathbf{E} = \omega\delta\mathbf{B}$. Here the operator $\nabla_{\mathbf{v}} f_0$ can be expressed as

$$\nabla_{\mathbf{v}} f_0 = 2(\mathbf{v} - v_z\mathbf{e}_z)\frac{\partial f_0}{\partial v_\perp^2} + 2v_z\frac{\partial f_0}{\partial v_z^2}\mathbf{e}_z$$

Since $v_\perp^2$ and v_z are constants of motion $\partial f_0/\partial v_\perp^2$ and $\partial f_0/\partial v_z^2$ are constant and can be removed from the integral. Similarly v_z is constant since the equilibrium magnetic field is in the z direction. All other integral are of the form

$$\int_{-\infty}^{0} d\tau \left\{v_x(\tau), v_y(\tau), 1\right\} \exp[i(\mathbf{k}\cdot(\mathbf{r}(\tau) - \mathbf{r}) - \omega\tau)]$$

Using $\mathbf{k} = k_\perp\mathbf{e}_x + \mathbf{k}_\parallel\mathbf{e}_z$ the exponent in this integral has the terms

$$\begin{aligned}
\mathbf{k}\cdot(\mathbf{r}(\tau) - \mathbf{r}) - \omega\tau &= k_\perp v_\perp/\omega_g \left\{\sin[\omega_g\tau + \psi] - \sin\psi\right\} \\
&\quad + k_\parallel v_\parallel \tau - \omega\tau
\end{aligned}$$

Here the sine function in the exponent can be integrated with the aid of

$$\exp[ix\sin(\phi - \omega\tau)] = \sum_{n=-\infty}^{\infty} J_n(x)\exp[in(\phi - \omega\tau)]$$

where J_n is the Bessel function of order n. With $a = k_\perp v_\perp/\omega_g$ we can for instance consider the integral

$$\begin{aligned}
I &= \int_{-\infty}^{0} d\tau \exp[i(\mathbf{k}\cdot(\mathbf{r}(\tau) - \mathbf{r}) - \omega\tau)] \\
&= \int_{-\infty}^{0} d\tau \exp[ia\sin(\omega_g\tau + \psi)]\exp(-ia\sin\psi)\exp\left[i\left(k_\parallel v_\parallel - \omega\right)\tau\right] \\
&= \int_{-\infty}^{0} d\tau \exp\left[i\left(k_\parallel v_\parallel - \omega\right)\tau\right] \sum_{l=-\infty}^{\infty} J_l(a)\exp[il(\omega_g\tau + \psi)] \sum_{n=-\infty}^{\infty} J_n(a)\exp[-in\psi] \\
&= \sum_{l=-\infty}^{\infty} J_l(a) \sum_{n=-\infty}^{\infty} J_n(a)\exp[i(l-n)\psi]\frac{1}{i\left(k_\parallel v_\parallel + l\omega_g - \omega\right)}
\end{aligned}$$

As it turns out this is the coefficient in equation (3.18) associated with the δE_z term.

Finally we nned to determine the current density associated with the perturbated distribution function.

$$\begin{aligned}
\delta\mathbf{j}(\mathbf{k},\omega) &= \sum_s q_s \int\int\int_{-\infty}^{\infty} d^3v\mathbf{v}\delta f_s \\
&= -\sum_s \frac{\varepsilon_0\omega_{ps}^2}{n_0\omega}\delta\mathbf{E}(\mathbf{k},\omega)\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi} v_\perp dv_\perp dv_\parallel d\psi \\
&\cdot\int_{-\infty}^{0} d\tau\left\{\mathbf{v}(\tau)\mathbf{k}+\underline{\underline{1}}(\omega-\mathbf{k}\cdot\mathbf{v}(\tau))\right\}\exp\left[i(\mathbf{k}\cdot\mathbf{r}(\tau)-\mathbf{r})-i\omega\tau\right]\cdot\frac{\partial f_0[\mathbf{v}(\tau)]}{\partial\mathbf{v}(\tau)}
\end{aligned}$$

With $j(\mathbf{k},\omega)=\underline{\underline{\sigma}}(\mathbf{k},\omega)\,\delta\mathbf{E}_0(\mathbf{k},\omega)$ we obtain $\underline{\underline{\sigma}}(\mathbf{k},\omega)$ and with the relations from section 3.2.1

$$\underline{\underline{\varepsilon}}(\mathbf{k},\omega)=\underline{\underline{1}}+\frac{i}{\omega\varepsilon_0}\underline{\underline{\sigma}}(\mathbf{k},\omega)$$

the dielectric function for a magnetized plasma becomes

$$\begin{aligned}
\underline{\underline{\varepsilon}}(\omega\mathbf{k}) &= \left(1-\sum_s\frac{\omega_{ps}^2}{\omega^2}\right)\underline{\underline{1}}-\sum_s\sum_{l=-\infty}^{\infty}\frac{2\pi\omega_{ps}^2}{n_{0s}\omega^2} \\
&\int_0^{\infty}\int_{-\infty}^{\infty} v_\perp dv_\perp dv_\parallel\left(k_\parallel\frac{\partial f_{0s}}{\partial v_\parallel}+\frac{l\omega_{gs}}{v_\perp}\frac{\partial f_{0s}}{\partial v_\perp}\right)\frac{\underline{\underline{\mathbf{S}}}_{ls}(v_\parallel,v_\perp)}{k_\parallel v_\parallel+l\omega_{gs}-\omega}
\end{aligned}$$

with the tensor $\underline{\underline{\mathbf{S}}}_{ls}$ of the form

$$\underline{\underline{\mathbf{S}}}_{ls}(v_\parallel,v_\perp)=\begin{bmatrix}
\frac{l^2\omega_{gs}^2}{k_\perp^2}J_l^2 & \frac{ilv_\perp\omega_{gs}}{k_\perp}J_lJ_{l,\xi} & \frac{lv_\parallel\omega_{gs}}{k_\perp}J_l^2 \\
\frac{ilv_\perp\omega_{gs}}{k_\perp}J_lJ_{l,\xi} & v_\perp^2 J_{l,\xi} & -iv_\parallel v_\perp J_lJ_{l,\xi} \\
\frac{lv_\parallel\omega_{gs}}{k_\perp}J_l^2 & iv_\parallel v_\perp J_lJ_{l,\xi} & v_\parallel^2 J_l^2
\end{bmatrix}$$

with the Bessel functions J_l, and $J_{l,\xi}=dJ_l/d\xi$ with the argument $\xi_s=k_\perp v_\perp/\omega_{gs}$.

This is the most general expression for the dielectric tensor in a plasma with a uniform magnetic field.

Simplifications and special cases:

- The purely electrostatic dispersion relation is obtained by taking the dot product of $\underline{\underline{\varepsilon}}(\omega\mathbf{k})$ with $\mathbf{k}$ from both sides.
- The condition $\omega-k_\parallel v_\parallel-l\omega_{gs}=0$ defines particle resonances. In the case of an unmagnetized plasma they reduce to the Landau resonance $\omega=k_\parallel v_\parallel$.
- Expansion for cold plasma $\xi_s=k_\perp v_\perp/\omega_{gs}$-> 0.

The general dispersion relation for magnetized plasma waves is obtained from

$$\det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{kk}}{k^2}-\underline{\underline{1}}\right)+\underline{\underline{\varepsilon}}(\mathbf{k},\omega)\right]=0$$

Electrostatic perturbations in a magnetized plasma are described by the dielectric response function

$$\varepsilon(\mathbf{k},\omega)=\frac{1}{k^2}\mathbf{k}\cdot\underline{\underline{\varepsilon}}(\mathbf{k},\omega)\cdot\mathbf{k}$$

3.2.3 Electrostatic Plasma Waves

with $\mathbf{k}=k_\perp\mathbf{e}_x+\mathbf{k}_\parallel\mathbf{e}_z$. This simplifies the dielectric tensor significantly:

$$\begin{aligned}\varepsilon(\mathbf{k},\omega) &= 1-\sum_s\sum_{l=-\infty}^{\infty}\frac{2\pi\omega_{ps}^2}{n_{0s}\omega^2}\\ &\int_0^\infty\int_{-\infty}^{\infty}v_\perp dv_\perp dv_\parallel\left(k_\parallel\frac{\partial f_{0s}}{\partial v_\parallel}+\frac{l\omega_{gs}}{v_\perp}\frac{\partial f_{0s}}{\partial v_\perp}\right)\frac{J_l^2(\xi_s)}{k_\parallel v_\parallel+l\omega_{gs}-\omega}\end{aligned} \tag{3.19}$$

with the Bessel functions J_l and $\xi_s=k_\perp v_\perp/\omega_{gs}$. In this case the dispersion relation reduces to

$$\varepsilon(\mathbf{k},\omega)=0 \tag{3.20}$$

To perform the integration we consider the gyrotropic Maxwellian distributions

$$f_{s0}\left(v_\perp,v_\parallel\right)=\frac{n_0}{\pi^{3/2}u_{s\parallel}u_{s\perp}^2}\exp\left[-\frac{v_\parallel^2}{u_{s\parallel}^2}-\frac{v_\perp^2}{u_{s\perp}^2}\right]$$

with $u_{s\perp}=2k_BT_\perp/m_s$ and $u_{s\parallel}=2k_BT_\parallel/m_s$.

$$\varepsilon(\mathbf{k},\omega)=1+\sum_s\sum_{l=-\infty}^{\infty}\frac{2\omega_{ps}^2\Lambda_l(\eta_s)}{\pi^{1/2}k^2u_{s\perp}^2}\int_{-\infty}^{\infty}\frac{dv_\parallel}{u_{s\parallel}}\left(\frac{T_{s\perp}}{T_{s\parallel}}k_\parallel v_\parallel+l\omega_{gs}\right)\frac{\exp\left(-v_\parallel^2/u_{s\parallel}^2\right)}{k_\parallel v_\parallel+l\omega_{gs}-\omega} \tag{3.21}$$

with the function

$$\begin{aligned}\Lambda_l(\eta_s) &= I_l(\eta_s)\exp(-\eta_s)\\ \eta_s &= \frac{k_\perp^2u_{s\perp}^2}{2\omega_{gs}^2}=\frac{k_\perp^2k_BT_{s\perp}}{m_s\omega_{gs}^2}=\frac{k_\perp^2r_{gs}^2}{2}\end{aligned}$$

Using the plasma dispersion function $Z(\zeta)=\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\exp\left(-z^2\right)dz}{z-\zeta}$ (Section A.4) the $v_\parallel$ integration is similar to the one using a Landau contour and leads to

$$\varepsilon(\mathbf{k},\omega) = 1 - \sum_s \sum_{l=-\infty}^{\infty} \frac{\omega_{ps}^2 \Lambda_l(\eta_s)}{k^2 u_{s\perp}^2} \left(\frac{T_{s\perp}}{T_{s\parallel}} Z'(\zeta_s) - \frac{2l\omega_{gs}}{k_\parallel u_{s\parallel}} Z(\zeta_s) \right) \tag{3.22}$$

with $Z'(\zeta) = dZ/d\zeta$ and $\zeta_s = (\omega - l\omega_{gs})/k_\parallel u_{s\parallel}$. The values with $\varepsilon = 0$ represent the so called electrostatic Eigenmodes of the magnetized warm plasma. Let us discuss a few examples in the following:

Magnetized Langmuir and Ion-Acoustic Waves

At high frequencies and for close to parallel propagation $k_\perp \to 0$ all $\Lambda_l(\eta_s) \to 0$ for $l \neq 0$. Considering only the lectron contribution:

$$\varepsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pe}^2 \Lambda_0(\eta_e)}{k^2 u_{e\parallel}^2} Z'(\zeta_s)$$

with $\zeta_e = \omega/k_\parallel u_{e\parallel}$ and $\eta_e = k_\perp^2 r_{ge}^2/2$. Expanding the plasma dispersion function for high frequencies $\zeta_e \gg 1$

$$Z(\zeta) = -\frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + ...\right) + 2i\pi^{1/2} i \exp\left(-\zeta^2\right)$$

yields

$$1 - \frac{\omega_{pe}^2}{k^2 u_{e\parallel}^2} \frac{\Lambda_0(\eta_e)}{\zeta_e^2} \left[1 + \frac{3}{2\zeta_e^2} - \frac{2i\pi^{1/2}}{\zeta_e} \exp\left(-\zeta_e^2\right)\right] = 0$$

With the solution

$$\omega_L^2(k,\theta) = \omega_{pe}^2 \Lambda_0(\eta_e) \cos^2\theta \left[1 + 3k^2 \lambda_{D\parallel}^2 \cos^2\theta\right]$$

and for weak damping we obtain with the method described before

$$\gamma_L(k,\theta) \approx -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_L(k,\theta)}{k^3 \lambda_e^3} \Lambda_0(\eta_e) \exp\left[-\frac{\Lambda_0(\eta_e)}{2k^2\lambda_e^2} - \frac{3}{2}\right]$$

where θ is the angle between the $\mathbf{k}$ vector of the wave and the magnetic field direction.

More discussion on the influence of the propagation direction on damping.

Magnetized ion acoustic wave can be treated the same way except that the ion dispersion function is expanded in the large argument limit.

$$\omega_{IA}^2 = \frac{\omega_{pi}^2 \Lambda_0(\eta_i) \cos^2\theta}{1 + \Lambda_0(\eta_e)/k^2\lambda_D^2} \left[1 + 3k^2 \lambda_{Di\parallel}^2 \cos^2\theta\right]$$

Electron Bernstein Waves

In the previous case of Langmuir and Ion Acoustic waves the influence of the magnetic was moderate and mostly associated with the dependence of the wave damping on the propagation direction. This is seen when we assumed $k_\perp \to 0$ which implied that all terms in the sum over the Bessel function vanished except for the $l = 0$ term. This also eleiminated the dependence on the gyrofrequency. Now, considering purely perpendicular propagation we assume $k_\parallel = 0$ such that $k = k_\perp$. Making this assumption in (3.19) the dispersion relation becomes

$$1 - \sum_s \frac{\omega_{ps}^2}{k_\perp^2} \frac{m_s^2}{k_B^2 T_s} \sum_{l=-\infty}^{\infty} \frac{l\omega_{gs}}{\omega - l\omega_{gs}} \int_{-\infty}^{\infty} v_\perp dv_\perp J_l^2(\eta_s) \exp\left(-v_\perp^2/u_{s\perp}^2\right)$$

Integration with the help of the Weber integral (A.10) yields

$$1 - \sum_s \frac{\omega_{ps}^2}{\omega_{gs}} \sum_{l=-\infty}^{\infty} \frac{l I_l(\eta_s)}{\omega - l\omega_{gs}} \frac{\exp(-\eta_s)}{\eta_s} = 0$$

Further, one can neglect ion contributions considering only high frequency modes $\omega \sim l\omega_{ge}$ such that

$$\begin{aligned} 1 - \frac{\omega_{pe}^2}{\omega_{ge}^2} \sum_{l=-\infty}^{\infty} \frac{\omega_{ge} l \Lambda_l(\eta_e)}{\eta_e(\omega - l\omega_{ge})} &= 0 \\ \text{with} \quad \eta_s = \frac{k_\perp^2 u_{e\perp}^2}{2\omega_{gse}^2} \end{aligned}$$

Using the symmetry $I_{-l} = I_l$ the dispersion relation is re-written to

$$1 - \frac{\omega_{pe}^2}{\omega_{ge}^2} \sum_{l=1}^{\infty} \frac{\omega_{ge} l^2 \Lambda_l(\eta_e)}{\eta_e(r_{\omega e}^2 - l^2)} = 0$$

with $r_{\omega e} = \omega/\omega_{ge}$. One can now discuss various limiting cases. Howeber it is clear that for larger values of l, $r_{\omega e} \approx l$, i.e., the solutions aer in bands with resonances at the harmonics of the electron cyclotron frequency. Further it can be shown that for small wave numbers $k_\perp \approx 0$ we have solutions either for $l \neq 0$ with $r_{\omega e} \approx l$ or if $r_{\omega e} \neq l$ only with

$$\omega^2(k=0) = \omega_{pe}^2 + \omega_{ge}^2$$

which is the upper hybrid frequency. Investigating damping it turns out that Bernstein modes which propagate exactly perpendicular show no Landau damping. However, more oblique propagating waves have stronger damping which implies that Bernstein modes should favor perpendicular propagation.

Ion waves:

Ion Bernstein modes,

lower-hybrid waves ($k_\perp^2 r_{gi}^2 \gg 1$)

3.3 Collisionless Waves in an Anisotropic Plasma

With the definition of the dielectric tensor

$$\underline{\underline{\varepsilon}}(\omega\mathbf{k}) = \underline{\underline{1}} + \frac{i}{\omega\varepsilon_0}\sigma(\omega\mathbf{k})$$

the general dispersion relation is

$$\det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{kk}}{k^2} - \underline{\underline{1}}\right) + \underline{\underline{\varepsilon}}(\omega\mathbf{k})\right] = 0$$

For an anisotropic (gyrotropic) distribution function the dielectric tensor is becomes

$$\underline{\underline{\varepsilon}}(\omega\mathbf{k}) = \underline{\underline{1}} + \sum_s \begin{pmatrix} \varepsilon_{s1} & \varepsilon_{s2} & \varepsilon_{s4} \\ -\varepsilon_{s2} & \varepsilon_{s1} - \varepsilon_{s0} & -\varepsilon_{s5} \\ \varepsilon_{s4} & \varepsilon_{s5} & \varepsilon_{s3} \end{pmatrix}$$

with

$$\begin{aligned}
\varepsilon_{s0} &= \frac{2\omega_{ps}^2}{\omega k_\| u_{s\|}}\sum_l \eta_s\Lambda_l'(\eta_s)\left[Z(\zeta_{s,l}) - \frac{k_\| u_{s\|}}{2\omega}A_s Z'(\zeta_{s,l})\right] \\
\varepsilon_{s1} &= \frac{\omega_{ps}^2}{\omega k_\| u_{s\|}}\sum_l \frac{l^2\Lambda_l(\eta_s)}{\eta_s}\left[Z(\zeta_{s,l}) - \frac{k_\| u_{s\|}}{2\omega}A_s Z'(\zeta_{s,l})\right] \\
\varepsilon_{s2} &= \frac{i\,\mathrm{sign}(q_s)\omega_{ps}^2}{\omega k_\| u_{s\|}}\sum_l l\Lambda_l'(\eta_s)\left[Z(\zeta_{s,l}) - \frac{k_\| u_{s\|}}{2\omega}A_s Z'(\zeta_{s,l})\right] \\
\varepsilon_{s3} &= -\frac{\omega_{ps}^2}{k_\|^2 u_{s\|}^2}\sum_l \left(1 - \frac{A_s}{A_s+1}\frac{l\omega_{gs}}{\omega}\right)\left(1 + \frac{l\omega_{gs}}{\omega}\right)\Lambda_l(\eta_s)Z'(\zeta_{s,l}) \\
\varepsilon_{s4} &= \frac{k_\perp}{2k_\|}\frac{\omega_{ps}^2}{\omega\omega_{gs}}\sum_l \left(A_s + 1 - \frac{l\omega_{gs}}{\omega}A_s\right)\frac{l\Lambda_l(\eta_s)}{\eta_s}Z'(\zeta_{s,l}) \\
\varepsilon_{s5} &= -\frac{i\,\mathrm{sign}(q_s)}{k_\perp k_\|}\frac{\omega_{ps}^2}{2\omega\omega_{gs}}\sum_l \left(A_s + 1 - \frac{l\omega_{gs}}{\omega}A_s\right)\Lambda_l'(\eta_s)Z'(\zeta_{s,l})
\end{aligned}$$

where the sum is take from $l = -\infty$ to $l = \infty$ and

$$\begin{aligned}
A_s &= \frac{T_{s\perp}}{T_{s\|}} - 1 \\
\zeta_{s,l} &= \frac{\omega - l\omega_{gs}}{k_\| v_{ths\|}} \\
\Lambda_l(\eta_s) &= I(\eta_s)\exp(-\eta_s) \\
\eta_s &= \frac{k_\perp^2 v_{ths\perp}^2}{2\omega_{gs}^2} = \frac{k_\perp^2 T_{s\perp}}{m_s\omega_{gs}^2} = \frac{k_\perp^2 r_{gs}^2}{2}
\end{aligned}$$

with the modified Bessel function $I(\eta_s)$ and the plasma dispersion function

$$Z(\zeta) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\exp\left(-z^2\right)dz}{z-\zeta}$$

Weak Damping and Weak Instability

We have in the previous parts assumed wave fields of the form

$$\delta A = \sum_k A_k \exp\left(i\mathbf{k}\cdot\mathbf{x} - i\omega t\right)$$

The amplitude of the waves will decrease in time if ω has an imaginary part which is negative $\omega = \omega_r + i\gamma$ with $\gamma < 0$. Weak damping assumes $|\gamma(\omega_r, \mathbf{k})| \ll \omega_r(\mathbf{k})$. In other words the damping proceeds slowly compared to the real wave period ω_r. In this case we can expand the dispersion relation

$$D(\omega, \gamma, \mathbf{k}) = D_r(\omega, \gamma, \mathbf{k}) + iD_i(\omega, \gamma, \mathbf{k}) = 0$$

around the real frequency $\omega = \omega_r$

$$D(\omega_r, \gamma, \mathbf{k}) = D_r(\omega_r, 0, \mathbf{k}) + i\gamma \left.\frac{\partial D(\omega_r, \gamma, \mathbf{k})}{\partial \omega_r}\right|_{\gamma=0} + iD_i(\omega_r, \gamma, \mathbf{k}) = 0$$

with the solutions

$$\begin{aligned} D_r(\omega_r, 0, \mathbf{k}) &= 0 \\ \gamma(\omega_r, \mathbf{k}) &= \frac{D_i(\omega_r, \gamma, \mathbf{k})}{\partial D(\omega_r, \gamma, \mathbf{k}) / \partial \omega_r|_{\gamma=0}} \end{aligned}$$

For instability one can follow the same line of arguments. Note that the exponential growth of the wave for $\gamma > 0$ implies that the amplitude of the wave assumes eventually that of the equilibrium plasma in which case the approach to linearize the equations is not anymore applicable. For instabilities with $\gamma(\omega_r, \mathbf{k}) \ll \omega_r(\mathbf{k})$ we can use the same expansion as in the case of weak damping and obtain the same solution for the real and imaginary parts of ω.

Electromagnetic Waves:

- Wistlers
- Electromagnetic Ion-Cyclotron Waves
- Kinetic Alfven waves

3.4 Stability: Nyquist Methods and Penrose Criterion

Information on damping and instability is contained in the dielectric function $\varepsilon(k, \omega)$. Consider k fixed such that ε is a function of ω only.

If ε has only simple zeros and if $\partial\varepsilon/\partial\omega$ has no poles inside of a contour:

$$\frac{1}{2\pi i}\oint_C \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega}d\omega = N \quad \left(\text{Number of } 0'\text{s of } \varepsilon \text{ inside } C\right) \tag{3.23}$$

This can be easily shown through a taylor expansion of ε and of $\partial\varepsilon/\partial\omega$ at any nullpoint of which yields

$$\begin{aligned}\varepsilon(k,\omega) &= 0 + \left.\frac{\partial\varepsilon}{\partial\omega}\right|_{\omega_0}(\omega-\omega_0)+..\\ \frac{\partial\varepsilon}{\partial\omega} &= \left.\frac{\partial\varepsilon}{\partial\omega}\right|_{\omega_0} + \left.\frac{\partial^2\varepsilon}{\partial\omega^2}\right|_{\omega_0}(\omega-\omega_0)+..\end{aligned}$$

such that near ω_0:

$$\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega} = \frac{1}{\omega-\omega_0}$$

such that the residue theorem just counts the number of zeroes of ε inside the contour C. Remembering that perturbation $\sim \exp(-i\omega t)$ any zero of ε with $\omega_i > 0$ corresponds to an instability which grows exponentially in time. The illustrations in Figure 3.4 would therefore result in $N = 0,1,2$ for the 3 illustrated examples.

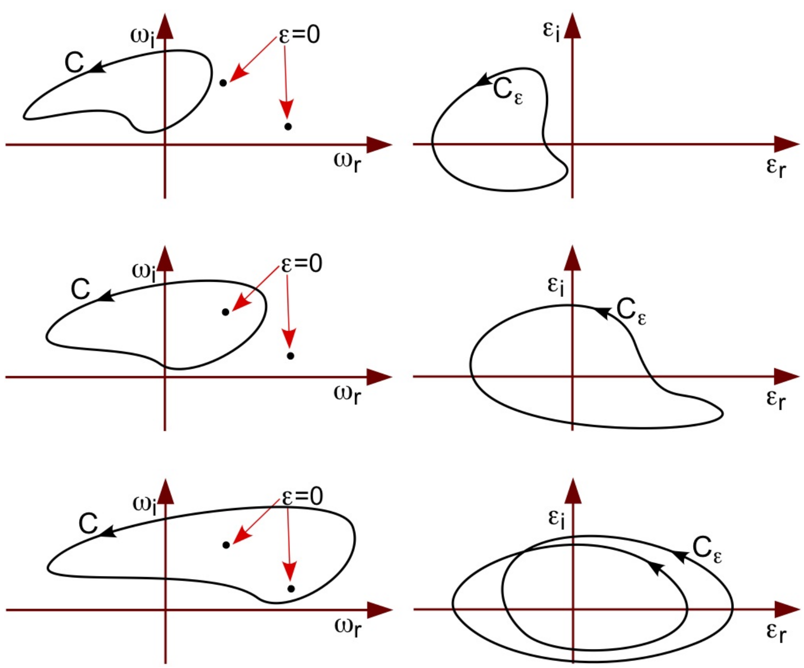

Figure 3.4: Illustration of a contour C inclosing no, one and two $\varepsilon = 0$ points in the complex ω plane and the corresponding contours C_ε in the complex ε plane.

Any value in ω produces $\varepsilon(\omega)$ such that that a contour C in the ω plane has a corresponding contour C_ε in the complex ε plane. For fixed k it follows that

$$\frac{1}{2\pi i}\oint_C \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\omega}d\omega = \frac{1}{2\pi i}\oint_{C_\varepsilon}\frac{d\varepsilon}{\varepsilon} = N \tag{3.24}$$

Consider the relation between the contour in the ω and ε plane for a contour C that incircles the total positive ω_i region.

- If there is no zero of ε with a positive ω_i: $N = 0$ and the corresponding closed contour C_ε in the complex ε plane does not incircle a pole, i.e., the point $\varepsilon = 0$.
- If there is one zero of ε with a positive ω_i: $N = 1$ and the corresponding closed contour C_ε in the ε plane does incircle the pole at $\varepsilon = 0$ once.
- If there are multiple zeros of ε with a positive ω_i: $N > 1$ and the corresponding closed contour C_ε in the ε plane incircles the pole at $\varepsilon = 0$ N time.

The situation is illustrated in Figure 3.4. The method has now provided a simple tool to identify instability. We can choose a contour in the complex ω plane which incloses the positive ω_i plane. We then map the with $\varepsilon(\omega)$ this contour into the complex ε plane. If the resulting contour incircles the origin $\varepsilon = 0$ at least once we know that the exists at least one ω with $\omega_i > 0$, i.e., the configuration is unstable. This approach to determine the stability of a distribution by examining the contour in the complex ε plane is called the Nyquist method.

Let consider this mapping in more detail in comparison with the dielectric function (3.3) or as a function of ω

$$\varepsilon(k,\omega) = 1 - \frac{\omega_{pe}^2}{n_0 k^2} \int_{-\infty}^{\infty} d^3v \frac{\partial_v f_0}{v - \omega/k} \tag{3.25}$$

We choose a contour along the real axis of the complex ω plane from $-\infty$ to $+\infty$ which is closed by the seim circle in the positive ω_i half of the plane as illustrated in Figure 3.5. Considering the semi circle at $|\omega| \to \infty$ the integral in (3.25) approaches 0 and therefore $\lim_{|\omega|\to\infty} \varepsilon = +1$. Considering the sign of the imaginary termin as derived in (3.12) for small values of $\gamma = \omega_i \ll \omega_r$

$$\varepsilon(k, \omega_r + i\gamma) = 1 - \frac{\omega_{pe}^2}{k^2} \mathrm{P} \int_{-\infty}^{\infty} dv \frac{\partial_v g(v)}{v - \omega/k} - i\pi \frac{\omega_{pe}^2}{k^2} \partial_v g(v = \omega/k) \tag{3.26}$$

it is clear that for $\omega_r \to +\infty$ we have $\varepsilon_i > 0$ and for $\omega_r \to -\infty$ we have $\varepsilon_i < 0$. Furthermore, $\varepsilon_r \approx 1 - \omega_e^2/\omega^2 < 1$ for sufficiently large ω_r. The path of C_ε in the vicinity of $\varepsilon = 1$ is shown in Figure 3.5.

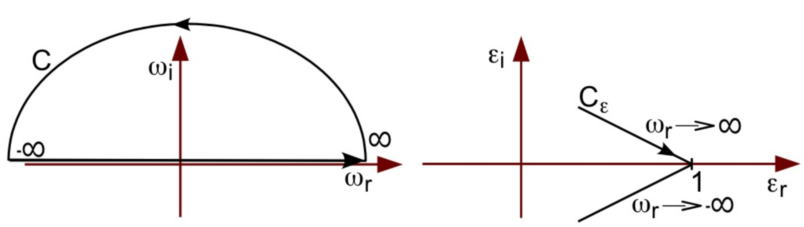

Figure 3.5: Path of C_ε in the vicinty of $\varepsilon = 1$

If there is no instability the remainder of the path C_ε should look like the one illustrate in the top of Figure 3.6. Note that the path in the complex ε plane should have the same orientation as in the complex ω plane. This also implies that the contour C_ε in the top plot of Figure 3.6cannot close by cutting the real ε axis at a point $\varepsilon_r < 0$ because it would imply the wrong orientation and imply the result $N = -1$.

The mainder of the path C_ε can be inferred from the imaginary part of the dispersion relation in (3.26), i.e., where and how many times the C_ε contour cuts the ε_r axis because each of these requires $\partial_v g\left(v = \omega_r/k\right) = 0$. Specifically a distribution function with a single maximum and no minimum has only one additional point along the ε_r axis where $\varepsilon_i = 0$.

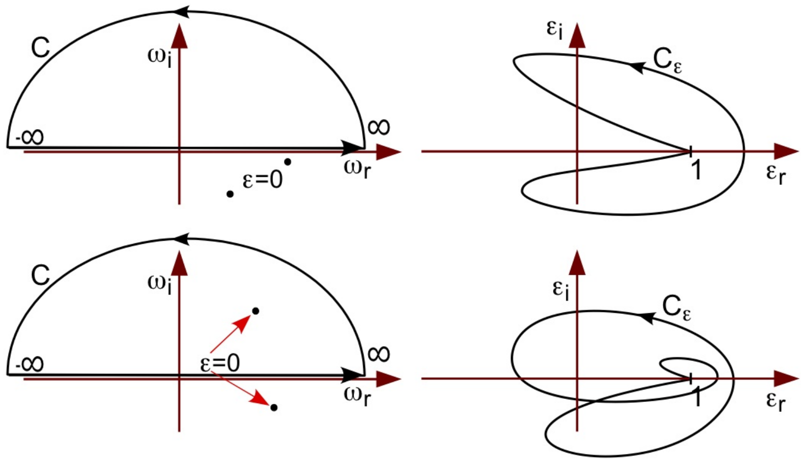

Figure 3.6: Illustration of the mapping of the upper semicircle with no and with one point $\varepsilon = 0$ inclosed in the complex ω plane and on the right, the corresponding contours C_ε in the complex ε plane.

In the case of $\varepsilon_i = 0$ we can use the information $\partial_v g\left(v = u_0 = \omega/k\right) = 0$ to compute ε_r through

$$\begin{aligned}\varepsilon_r\left(\omega_r = k u_0\right) &= 1 - \frac{\omega_{pe}^2}{k^2} \mathrm{P} \int_{-\infty}^{\infty} dv \frac{\partial_v g\left(v\right)}{v - \omega/k} = 1 - \frac{\omega_{pe}^2}{k^2} \mathrm{P} \int_{-\infty}^{\infty} dv \frac{\partial_v \left[g\left(v\right) - g\left(u_0\right)\right]}{v - u_0} \\ &= 1 + \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{\infty} dv \frac{\left[g\left(u_0\right) - g\left(v\right)\right]}{\left(v - u_0\right)^2}\end{aligned}$$

where the last step is obtained through integration by parts. If the distribution has only a single maximum as illustrated in the first example in Figure 3.7 $g\left(u_0\right) \geq g\left(v\right)$ and thus $\varepsilon_r > 1$.

The same method can be imployd for multiple peaked distribution functions. Considering the example oin Figure 3.7 of a dual peaked distribution which has maxima at $\omega/k = u_0$ and $\omega/k = u_2$ and a minimum at $\omega/k = u_2$ one can infer that the distribution is absolutely stabel at the absolute maximum of the distribution at $\omega/k = u_0$. We can now discuss the information on the path of C_ε based on the values of u_i and $g\left(u_i\right)$

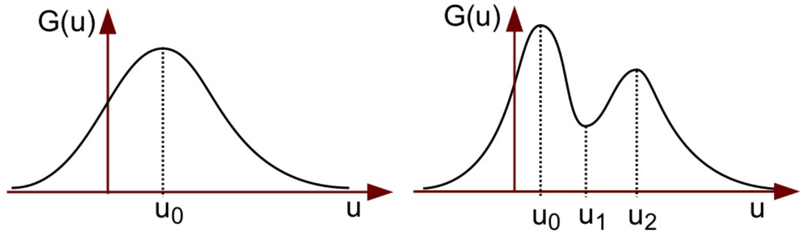

Figure 3.7: Examples of single and double peaked distributions $g(u)$

For the path of ω_r from $-\infty$ to $+\infty$, C_ε passes through $\varepsilon(u_0)$, $\varepsilon(u_1)$, $\varepsilon(u_2)$ in this order.

With

$$\varepsilon_r(\omega_r = ku_i) = 1 + \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{\infty} dv \frac{[g(u_i) - g(v)]}{(v-u_i)^2}$$

and $g(u_0) > g(u_2) > g(u_1)$ it is also clear that $\varepsilon_r(u_0) > \varepsilon_r(u_2) > \varepsilon_r(u_{01})$. A critical question is whether

$$I_g(u_1) = \int_{-\infty}^{\infty} dv \frac{[g(u_1) - g(v)]}{(v-u_1)^2} < 0$$

If $I_g(u_1) > 0$ it follows that $\varepsilon(u_1) > 1$ such that C_ε cannot cross the real ε axis for negative ε_i and the distribution is always stable.

However, if $I_g(u_1) < 0$ it is always possible to find a sufficiently small k such that $\varepsilon_r(\omega_r = ku_1) = 1 + I_g(u_1)\omega_{pe}^2/k^2 < 0$ implying instability. In fact we can determine th range of unstable k values to be $k^2 < -\omega_{pe}^2 I_g(u_1)$.

This is the Penrose criterion for instablityand itprovides a necessary and sufficient condition for the linear instability of the Poisson-Vlasov system.