Chapter 5

Instabilities

5.1 Maco-Instabilities - Ideal MHD

5.1.1 Rayleigh-Taylor Instability

Before we consider the so-caled normal mode approach let us first look at the result using th evariational approach:

The geometry: Magnetic field along the (horizontal) y direction, gravity pointing in the negative z direction with the potential $\psi = gz$. We also assume for simplicity a two-dimensional incompressible perturbations $\boldsymbol{\xi}$ with $\partial_y = 0$ and $\nabla \cdot \boldsymbol{\xi} = 0$.

The equilibrium condition for this configuration is given by

$$\frac{d}{dz}\left[p\left(z\right) + \frac{B_y\left(z\right)^2}{2\mu_0}\right] + \rho\left(z\right) g = 0$$

In this case the potential simplifies considerable resulting in

$$U_{2m} = -\frac{g}{2}\int_V \rho' \left|\xi_z\right|^2 dxdz$$

with $\rho' = d\rho/dz$. This expression leads to the immediate result that the configuration is unstable for $\rho' > 0$. The resulting change is a reconfiguration of straight magnetic field lines (=> interchange).

Normal mode analysis:

Note that MHD momentum equation with gravity can be written as

$$\rho\frac{\partial \mathbf{u}}{\partial t} = \rho\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \mathbf{j}\times\mathbf{B} - \rho g$$

We now linearize the ideal MHD equations around the equilibrium state.

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\mathbf{u}_1 \cdot \nabla \rho - \mathbf{u} \cdot \nabla \rho_1 - \rho_1 \nabla \cdot \mathbf{u} - \rho \nabla \cdot \mathbf{u}_1 \\
\rho \frac{\partial \mathbf{u}_1}{\partial t} &= -\rho_1 \mathbf{u} \cdot \nabla \mathbf{u} - \rho \mathbf{u}_1 \cdot \nabla \mathbf{u} - \rho \mathbf{u} \cdot \nabla \mathbf{u}_1 \\
&\quad -\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}_1 - \rho_1 g \mathbf{e}_z \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{u}_1 \times \mathbf{B}) + \nabla \times (\mathbf{u} \times \mathbf{B}_1) \\
\frac{\partial p_1}{\partial t} &= -\mathbf{u}_1 \cdot \nabla p - \mathbf{u} \cdot \nabla p_1 - \gamma p_1 \nabla \cdot \mathbf{u} - \gamma p \nabla \cdot \mathbf{u}_1
\end{aligned}$$

Here we omitted the index 0 for steady state variables. To simplify the problem further we assume that

- All perturbed quantities assume the form $\mathbf{u}_1 (x, z, t) = \tilde{\mathbf{u}}_1 (z) \exp [ikx + qt]$
- Perturbations are incompressible $\nabla \cdot \mathbf{u}_1 = 0 \Longrightarrow$
- Gravity is in the z direction.
- All variations of the equilibrium (steady state) are in the z direcion - $\mathbf{B} = \mathbf{B} (z)$, $\mathbf{u} = \mathbf{u} (z)$, ...
- Plasma flow is in the x direction.
- The magnetic field has only x and y components.

$$\begin{aligned}
q\rho_1 &= -u_{1z} \partial_z \rho - iku_x \rho_1 \\
\rho q \mathbf{u}_1 &= -\rho u_{1z} \partial_z u_x \mathbf{e}_x - ik\rho u_x \mathbf{u}_1 \\
&\quad -\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B} - \rho_1 g \mathbf{e}_z \\
q\mathbf{B}_1 &= \nabla \times (\mathbf{u}_1 \times \mathbf{B}) + \nabla \times (\mathbf{u} \times \mathbf{B}_1) \\
qp_1 &= -u_{1z} \partial_z p - iku_x p_1
\end{aligned}$$

$\mathbf{j} \times \mathbf{B}$ term:

$$\begin{aligned}
[(\nabla \times \mathbf{B}_1) \times \mathbf{B}]_x &= (\partial_z B_{1x} - \partial_x B_{1z}) B_z - (\partial_x B_{1y} - \partial_y B_{1x}) B_y \\
&= (\partial_z B_{1x} - ikB_{1z}) B_z - ikB_{1y} B_y \\
[(\nabla \times \mathbf{B}_1) \times \mathbf{B}]_y &= (\partial_x B_{1y} - \partial_y B_{1x}) B_x - (\partial_y B_{1z} - \partial_z B_{1y}) B_z \\
&= ikB_{1y} B_x + \partial_z B_{1y} B_z \\
[(\nabla \times \mathbf{B}_1) \times \mathbf{B}]_z &= \partial_z B_{1y} B_y - (\partial_z B_{1x} - ikB_{1z}) B_x
\end{aligned}$$

Induction equation:

$$
\begin{aligned}
qB_{1x} &= -\partial_z \left(u_{1z}B_x - u_{1x}B_z - u_x B_{1z}\right) \\
qB_{1y} &= \partial_z \left(u_{1y}B_z - u_{1z}B_y\right) - ik\left(u_{1x}B_y - u_{1y}B_x + u_x B_{1y}\right) \\
qB_{1z} &= ik\left(u_{1z}B_x - u_{1x}B_z - u_x B_{1z}\right)
\end{aligned}
$$

Conditions $\nabla \cdot \mathbf{u}_1 = 0$ and $\nabla \cdot \mathbf{B}_1 = 0$

$$
\begin{aligned}
iku_{1x} + \partial_z u_{1z} &= 0 \\
ikB_{1x} + \partial_z B_{1z} &= 0
\end{aligned}
$$

Consider $B_z = 0$:

$$
\begin{aligned}
\rho q u_{1x} &= -\rho u_{1z}\partial_z u_x - ik\rho u_x u_{1x} - ikp_1 - \frac{1}{\mu_0} ikB_y B_{1y} \\
\rho q u_{1y} &= -ik\rho u_x u_{1y} + \frac{1}{\mu_0} ikB_x B_{1y} \\
\rho q u_{1z} &= -ik\rho u_x u_{1z} - \partial_z p_1 + \frac{1}{\mu_0}\left[B_y \partial_z B_{1y} - B_x\left(\partial_z B_{1x} - ikB_{1z}\right)\right] - \rho_1 g \\
qB_{1x} &= -\partial_z\left(u_{1z}B_x - u_x B_{1z}\right) \\
qB_{1y} &= -ik\left(-u_{1y}B_x + u_x B_{1y}\right) \\
qB_{1z} &= ik\left(u_{1z}B_x - u_x B_{1z}\right)
\end{aligned}
$$

Where in the qB_{1y} term $\nabla \cdot \mathbf{u}_1 = 0$ has been used.

Considering only the y components

$$
\begin{aligned}
\rho\left(q + iku_x\right) u_{1y} &= \frac{1}{\mu_0} ikB_x B_{1y} \\
\left(q + iku_x\right) B_{1y} &= iku_{1y}B_x \\
\Longrightarrow \qquad & q = -iku_x \pm ik\left[\frac{B_x^2}{\mu_0 \rho}\right]^{1/2}
\end{aligned}
$$

such that the y components of the perturbation decouple fom the rest of the equations.

$$
\begin{aligned}
\rho q u_{1x} &= -\rho u_{1z}\partial_z u_x - ik\rho u_x u_{1x} - ikp_1 \\
\rho q u_{1z} &= -ik\rho u_x u_{1z} - \partial_z p_1 - \frac{1}{\mu_0} B_x\left(\partial_z B_{1x} - ikB_{1z}\right) - \rho_1 g \\
qB_{1x} &= -\partial_z\left(u_{1z}B_x - u_x B_{1z}\right) \\
qB_{1z} &= ik\left(u_{1z}B_x - u_x B_{1z}\right)
\end{aligned}
$$

Rayleigh-Taylor Instability: Assuming $u_x = 0$:

$$\begin{aligned}
q\rho_1 &= -u_{1z}\partial_z\rho \\
\rho q u_{1x} &= -ikp_1 \\
\rho q u_{1z} &= -\partial_z p_1 - \frac{1}{\mu_0}B_x\left(\partial_z B_{1x} - ikB_{1z}\right) - \rho_1 g \\
qB_{1x} &= -B_x\partial_z u_{1z} = B_x iku_{1x} \\
qB_{1z} &= B_x iku_{1z} \\
0 &= iku_{1x} + \partial_z u_{1z}
\end{aligned}$$

$$\begin{aligned}
\rho q u_{1z} &= -\partial_z\frac{\rho q u_{1x}}{-ik} - \frac{1}{\mu_0}B_x\left(\partial_z\frac{B_x iku_{1x}}{q} - ik\frac{B_x iku_{1z}}{q}\right) + g\frac{u_{1z}\partial_z\rho}{q} \\
\rho q u_{1z} &= \partial_z\frac{\rho q\partial_z u_{1z}}{-\left(ik\right)^2} + \frac{1}{\mu_0}B_x\left(\partial_z\frac{B_x\partial_z u_{1z}}{q} + ik\frac{B_x iku_{1z}}{q}\right) + g\frac{u_{1z}\partial_z\rho}{q} \\
k^2\rho u_{1z} &= \partial_z\left(\rho\partial_z u_{1z}\right) + \frac{k^2}{q^2}\frac{B_x^2}{\mu_0}\left(\partial_z^2 u_{1z} - k^2 u_{1z}\right) + \frac{gk^2}{q^2}\left(\partial_z\rho\right)u_{1z}
\end{aligned}$$

$$\partial_z\left(\rho\partial_z u_{1z}\right) + \frac{k^2}{q^2}\frac{B_x^2}{\mu_0}\left(\partial_z^2 u_{1z} - k^2 u_{1z}\right) - k^2\rho u_{1z} = -\frac{gk^2}{q^2}\left(\partial_z\rho\right)u_{1z} \tag{5.1}$$

For $\rho = const$ solutions of the equation are $u_{1z} = a\exp\left(\pm kz\right)\exp\left[ikx + qt\right]$. This particularly implies that $u_{1z}\left(z \to +\infty\right) \sim \exp\left(-kz\right)$ and $u_{1z}\left(z \to -\infty\right) \sim \exp\left(+kz\right)$. Let us consider a situation where we have a boundary at $z = 0$ where $\rho = \rho_1$ for $z < 0$ and $\rho = \rho_2$ for $z > 0$ such that

$$\begin{aligned}
\tilde{u}_{1z}\left(z\right) &= a\exp\left(kz\right) \qquad for \qquad z < 0 \\
\tilde{u}_{1z}\left(z\right) &= a\exp\left(-kz\right) \qquad for \qquad z \geq 0
\end{aligned}$$

where the choice of the same coefficient a insures that the velocity perturbation is continuous at $z = 0$. However we also have to consider the discontinuity in ρ for (5.1). This can be resolved by integrating the equation from $-\varepsilon$ to $+\varepsilon$ equation and considering the limit of $\varepsilon \to 0$.

$$\begin{aligned}
\Delta_\varepsilon\left(\rho\partial_z\tilde{u}_{1z}\right) + \frac{k^2}{q^2}\frac{B_x^2}{\mu_0}\Delta_\varepsilon\partial_z\tilde{u}_{1z} &= -\frac{gk^2}{q^2}\left(\Delta_\varepsilon\rho\right)\tilde{u}_{1z} \\
where \qquad & \Delta_\varepsilon f\left(z\right) = \lim_{\varepsilon\to 0}\left[f\left(\varepsilon\right) - f\left(-\varepsilon\right)\right]
\end{aligned}$$

Substitution of the solution $\tilde{u}_{1z}\left(z\right)$ into this condition yields

$$\begin{aligned}
\left(-ka\rho_2 - ka\rho_1\right) + \frac{k^2}{q^2}\frac{B_x^2}{\mu_0}\left(-ka - ka\right) &= -\frac{gk^2}{q^2}\left(\rho_2 - \rho_1\right)a \\
q^2 &= gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} - \frac{2k}{g}\frac{B_x^2}{\mu_0\left(\rho_2 + \rho_1\right)}\right)
\end{aligned}$$

or considering that we also included a B_y component in the equilibrium:

$$q^2 = gk\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1} - \frac{2}{gk}\frac{(\mathbf{k}\cdot\mathbf{B})^2}{\mu_0(\rho_2+\rho_1)}\right)$$

Discussion:

- magnetic field reduces growth rate if $\mathbf{k}\cdot\mathbf{B} \neq 0$.
- Instability stabilized for

$$k \geq \frac{\mu_0 g(\rho_2-\rho_1)}{2B_x^2}$$

- Maximum growth rate for

$$k = \frac{\mu_0 g(\rho_2-\rho_1)}{4B_x^2}$$

- Magnetic field B_y has no influence on the instability
- Magnetic field acts similar to an effective surface tension force $T_{eff} = 2B_x^2/\mu_0 k$

Figure...

5.1.2 Kelvin-Helmholtz Instability

Conditions $\nabla\cdot\mathbf{u}_1 = 0$ and $\nabla\cdot\mathbf{B}_1 = 0$

$$\begin{aligned} iku_{1x} + \partial_z u_{1z} &= 0 \\ ikB_{1x} + \partial_z B_{1z} &= 0 \end{aligned}$$

such that the y components of the perturbation decouple fom the rest of the equations.

$$\begin{aligned} q\rho_1 &= -u_{1z}\partial_z\rho - iku_x\rho_1 \\ \rho q u_{1x} &= -\rho u_{1z}\partial_z u_x - ik\rho u_x u_{1x} - ikp_1 \\ \rho q u_{1z} &= -ik\rho u_x u_{1z} - \partial_z p_1 - \frac{1}{\mu_0}B_x(\partial_z B_{1x} - ikB_{1z}) - \rho_1 g \\ qB_{1x} &= -\partial_z(u_{1z}B_x - u_x B_{1z}) \\ qB_{1z} &= ik(u_{1z}B_x - u_x B_{1z}) \end{aligned}$$

Manipulating equations:

$$\partial_z \left[\rho \left(q+iku_x\right)\partial_z u_{1z} - ik\rho\left(\partial_z u_x\right)u_{1z}\right] = k^2\rho\left(q+iku_x\right)u_{1z} + k^2\frac{B_x^2}{\mu_0}\left(\partial_z \frac{\partial_z u_{1z}}{q+iku_x} - \frac{k^2 u_{1z}}{q+iku_x}\right) + ik^3\frac{B_x^2}{\mu_0}\partial_z\left(\frac{\partial_z u_x}{\left(q+iku_x\right)^2}u_{1z}\right) + \frac{gk^2}{q+iku_x}\left(\partial_z\rho\right)u_{1z} \quad (5.2)$$

Important to note that continuity at a boundarys z_s is determined by

$$\partial_t z_{s1} + u_x \partial_x z_{s1} = u_{1z}\left(z_s\right)$$

and since the displacement of the boundary has to be unique this implies that $u_{1z}\left(z_s\right)/\left(q+iku_x\right)$ must be continuous at the boundary. Let us consider a steady state with constant u_x and ρ on the two sides of a plane boundary with a jump across the boundary. The solutions again must be exponential $\pm kz$ are chosen as

$$\begin{aligned} \tilde{u}_{1z}(z) &= a\left(q+ikU_1\right)\exp\left(kz\right) \qquad with \qquad U_1 = u_x \qquad for \qquad z<0 \\ \tilde{u}_{1z}(z) &= a\left(q+ikU_2\right)\exp\left(-kz\right) \qquad with \qquad U_2 = u_x \qquad for \qquad z\geq 0 \end{aligned}$$

to provide the correct boundary and continuity conditions. By integrating (5.2) over an ε vicinity of the boundary we obtain

$$\rho_2\left(q+ikU_2\right)^2 + \rho_1\left(q+ikU_1\right)^2 = gk\left(\rho_1-\rho_2\right) + 2k^2 B_x^2/\mu_0$$

With $\omega = iq$ the roots of the dispersion relation are

$$\begin{aligned} \omega &= k\left(\alpha_1 U_1 + \alpha_2 U_2\right) \\ &\pm \left[gk\left(\alpha_1-\alpha_2\right) + 2k^2\frac{B_x^2}{\mu_0\left(\rho_2+\rho_1\right)} - k^2\alpha_1\alpha_1\left(U_2-U_1\right)^2\right]^{1/2} \\ with \qquad & \alpha_i = \frac{\rho_i}{\rho_1+\rho_2} \end{aligned}$$

A negative argument in the squareroot generates an imaginary part of ω (or a real part of q) and represents instability. The gravitational term leads to the same Rayleigh-Taylor mode that has been discussed before. Ignoring this term illustrates that any difference in the velocity can lead to instability.

Discussion in absence of gravity:

- magnetic field reduces growth rate if $\mathbf{k}\cdot\mathbf{B}\neq 0$.
- Instability stabilized for

$$\frac{2B_x^2}{\mu_0\left(\rho_2+\rho_1\right)} \geq \alpha_1\alpha_1\left(U_2-U_1\right)^2$$

- Instability is suppressed if relative velocity does not exceed the root-mean-square Alfven speed.
- Magnetic field B_y has no influence on the instability

Other Macro-instabilities: Firehose, Mirror

5.1.3 Resistive Tearing Mode

This mode is an instability that occurs only in current sheet. For this derivation we assume the equilibrium to be the simple Harris sheet with $\partial_z = 0$ and normalized variables such that $p(A) = \exp(-2A)$, $\Delta A = -0.5 dp/dA$, and

$$\begin{aligned} A = A_z &= A_0 \ln\cosh y \\ p &= \cosh^{-2} y \\ B_x &= \tanh y \\ B_y &= 0 \\ j_z &= -\cosh^{-2} y \end{aligned}$$

Different from our approach for the Kelvin Helmholtz and Rayleigh-Taylor modes we linearize the resistive MHD equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{u} &= 0 \\ \frac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) &= -\nabla p + \mathbf{j}\times\mathbf{B} \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla\times(\mathbf{u}\times\mathbf{B} - \eta\mathbf{j}) \\ \frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p + \nabla\cdot p\mathbf{u}\right) &= -p\nabla\cdot\mathbf{u} + \eta\mathbf{j}^2 \\ \nabla\times\mathbf{B} &= \mathbf{j} \end{aligned}$$

with the resistivity η but we are still using the assumption of incompressibility $\nabla\cdot\mathbf{u} = 0$. In the normalized equations the resistivity is also normalized and is given by $\eta = \eta_p/(\mu_0 L_0 v_A)$, with the physical resistivity η_p, typical length scale L_0 and typical Alfven speed v_A. The normalized resistivity is equal to the inverse Lundguist number (magnetic Reynoldsnumber) $S = 1/\eta = \tau_{diff}/\tau_A$. Here $\tau)diff$is the diffusion time and τ_A is the Alfven time. The linearization is performed by reformulating the basic equations using the z component of the vector potential (or flux function) A and a stream function v to express magnetic field and velocity as

$$\begin{aligned} \mathbf{B} &= \nabla A\times\mathbf{e}_z + B_z\mathbf{e}_z \\ \mathbf{u} &= \nabla v\times\mathbf{e}_z + u_z\mathbf{e}_z \end{aligned}$$

This choice alway satisfies $\nabla\cdot\mathbf{B} = 0$ and $\nabla\cdot\mathbf{u} = 0$. Inserting the expressions for $\mathbf{B}$ and $\mathbf{u}$ into the MHD equations (momentum and Ohm's law) leads to

$$\begin{aligned} \nabla\cdot[\rho d_t \nabla v] &= \mathbf{e}_z\cdot(\nabla\Delta A\times\nabla A) \\ \rho d_t u_z &= \mathbf{e}_z\cdot(\nabla B_z\times\nabla A) \\ d_t A &= \eta\Delta A \\ d_t B_z &= \mathbf{e}_z\cdot(\nabla u_z\times\nabla a) + \nabla\cdot(\eta\nabla b_z) \end{aligned}$$

where we abbreviated $d_t f = df/dt = \partial_t f + \mathbf{u}_\perp \cdot \nabla f = \partial_t f + \mathbf{e}_z \cdot (\nabla f \times \nabla v)$ which is the total or convective derivative. This form is also convenient because it demonstrates that the solution does not depend on the pressure gradient because of $\nabla \cdot \mathbf{u} = 0$. Further it illustrates that the solution for u_z and B_z decouples from the equations for the magnetic flux and for the stream function, i.e., choosing $u_z = 0$ and B_z=0 does not alter the solution for A and v. The above set of equations is easy to linearize (noting that the velocity is 0 for the equilibrium, $v = u_z = 0$). Using perturbations of the type

$$f(x, y, t) = \tilde{f}(y) \exp(ikx + qt)$$

we obtain for the equations for v and A (for constant η):

$$q\nabla \cdot \rho \nabla v_1 = \frac{dj}{dA}\mathbf{B} \cdot \nabla A_1 + \mathbf{B} \cdot \nabla \Delta A_1 \tag{5.3}$$

$$qA_1 = \mathbf{B} \cdot \nabla v_1 + \eta \Delta A_1 \tag{5.4}$$

Since the equilibrium solution is smoothly varying with y we cannot use the simple approach used for the Rayleigh-Taylor instability. A single analytic solution is difficult to obtain such that the typical approach is to scale the 2 ordinary differential equation for typical values of y. Thus we will obtain an outer solution similar to the prior treatment that is valid for $y \geq O(1)$ and an inner solution that applies to a small ε vicinity of $y = 0$. We than match the value and derivatives of these solutions to obtain the dispersion relation $q(k)$. The scaling used is given by

$$\begin{aligned} q^2 &\ll k^2 \leq O(1) \\ |q| &\gg \eta = 1/S \end{aligned}$$

In explicit form the equations (5.3) and (5.4) for $\rho = 1$ are

$$qv_1'' - qk^2 v_1 + ikB_x'' A_1 + ik^3 B_x A_1 - ikB_x A_1'' = 0 \tag{5.5}$$

$$qA_1 - ikB_x v_1 - \eta A_1'' + \eta k^2 A_1 = 0 \tag{5.6}$$

Here we ommitted the ~ in $\tilde{A}_1$ for convenience.

Outer solution:

We note that the Lundquist number is a very large number for most space plasma. For the outer solutions $y \geq O(1)$ and equation (5.6) implies that $v_1 = O(qA_1/kB_x)$. Substitution in (5.5) yields to lowest order

$$A_1'' + \left(\frac{1}{\cosh^2 y} - k^2\right) A_1 = 0$$

The general solution for this equation is

$$A_1 = c_1 \exp(-ky)(\tanh y + k) + c_2 \exp(ky)(\tanh y - k)$$

For the matching to the inner solution we need the logarithmic derivative in the limit to $y \to 0$:

$$\frac{A_1'(0)}{A_1(0)} = \frac{1-k^2}{k}$$

Inner Solution:

Similar to the outer solution we need to identify the significant terms in the differential equations that determine the inner solution. This solution is determined in a small ε vicinity of $y = 0$. Here we can use the Taylor expansion of the magnetic field and scaling y to the new coordinate $\zeta = y/\varepsilon$ such the $y = \varepsilon\zeta$ and $d/dy = \varepsilon^{-1} d/d\zeta$ is used to identify the leading order terms in (5.5) and (5.6). The resulting equations are again expressed in y because the scaling is only used to identify the leading order terms.

$$\begin{aligned} qv_1'' - ikyA_1'' &= 0 \\ qA_1 - ikyv_1 - \eta A_1'' &= 0 \end{aligned}$$

Integrating the first of these equations to eleiminate v_1 leads to

$$\begin{aligned} zA_1''' - A_1'' - \left(\kappa^2 y^3 + \lambda\kappa y\right) A_{1'} + \left(\lambda\kappa + \kappa^2 y^2\right) A_1 &= \kappa^2 y^2 \tilde{c} \\ with \quad & \kappa = kS^{1/2}/q^{1/2} \\ & \lambda = q^{3/2} S^{1/2}/k \end{aligned} \tag{5.7}$$

The general solution of (5.7) is given in a closed analytic form:

$$\begin{aligned} A_1(y) &= \tilde{c} + c_0 y + c_1 y \int_0^{\kappa y^2} m_1(w)\, dw + +c_2 y \int_{\kappa y^2}^{\infty} m_2(w)\, dw + +c_p y \int_{\kappa y^2}^{\infty} m_p(w)\, dw \\ with \quad & m_1(w) = \exp(-w/2)\, M\left(\frac{\lambda+5}{4}, \frac{5}{2}, w\right) \\ & m_2(w) = \exp(-w/2)\, U\left(\frac{\lambda+5}{4}, \frac{5}{2}, w\right) \\ m_p(w) &= m_2(w) \int_0^w m_1(w')\, dw' + m_1(w) \int_w^{\infty} m_2(w')\, dw' \\ & c_p = \frac{\lambda\kappa^{1/2}\Gamma\left(\frac{\lambda+5}{4}\right)}{8\Gamma\left(\frac{5}{2}\right)} \tilde{c} \end{aligned}$$

Where M and U are the confluent hypergeometric functions (Kummer functions). The 4 integration constants are subject to the the condition remaining finite for $z \to \infty$ and symmetry implies $A_1' = A_1''' = 0$. The logarithmic derivative of the inner solution is

$$\lim_{y\to\infty} \frac{A_1'(y)}{A_1(y)} = \frac{\pi\lambda\kappa^{1/2}\Gamma\left(\frac{\lambda+5}{4}\right)}{(1-\lambda^2)\,\Gamma\left(\frac{\lambda+1}{4}\right)}$$

Dispersion relation:

equalizing the asymptotic logarithmic derivativesleads to the dipersion relation:

$$\begin{aligned}\lim_{y\to\infty}\frac{A'_{1a}(y)}{A_{1a}(y)} &= \frac{A'_{1o}(0)}{A_{1o}(0)}\\ &or\\ \tilde{q}^2 &= \frac{\sqrt{\lambda}}{\pi}\left(1-k^2\right)\left(1-\lambda^2\right)\frac{\Gamma\left(\frac{\lambda+1}{4}\right)}{\Gamma\left(\frac{\lambda+1}{4}\right)}\\ with &\qquad \lambda = q^{3/2}S^{1/2}/k = \tilde{q}^{3/2}/\tilde{k}\\ &\qquad \tilde{q} = qS^{1/2} \qquad \tilde{k} = kS^{1/4}\end{aligned}$$

In the scaling of $\tilde{q}$ and $\tilde{k}$ the maximum growth rate occurs if these are of order unity, i.e., $q_{max} = O\left(S^{-1/2}\right)$. More precisely the maximum is given by $\lambda_0 = 0.36$ and $\tilde{q}_0 = 0.62$ leading to

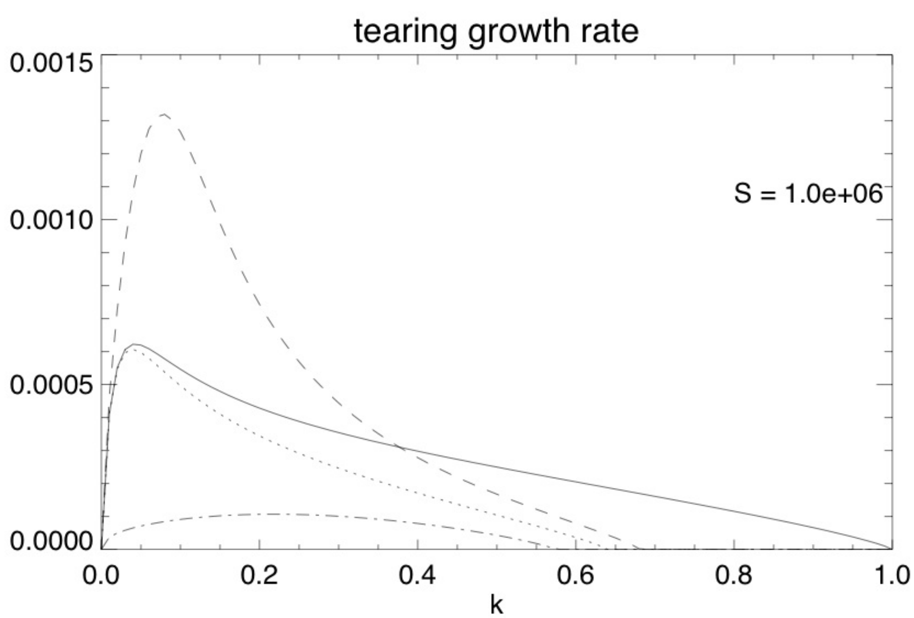
$$\begin{aligned}k_{max} &= 1.36 \cdot S^{-1/4}\\ q_{max} &= 0.62 \cdot S^{-1/2}\end{aligned}$$

We note that the width of the ε layer is given by

$$\varepsilon = \frac{1}{\sqrt{\kappa}} = \frac{q^{1/4}}{k^{1/2}S^{1/4}} = 0.7S^{-1/4}$$

finally we can expand this dispersion relation for $S^{-1/4} \ll k \leq 1$ to obtain

$$q = 0.95\left(1-k^2\right)^{4/5} S^{-3/5}k^{-2/5}$$



5.1.4 Collisionless Tearing Mode

Microscopic kinetic instabilities can be studied in the same framework that we derived for collisionless plasma waves. The only difference to the wave application is the a different sign in the imaginary part of ω. These instabilities require a source of free energy tha can drive the instability. For many micro-instbilities this source is a relative drift of different particle populations. These drifts can be also within one population for instance if there are two electron beams which move with different velocities. Other common sources are particle anisotropies which can generate so-called mirror and firhose modes. Most of these instabilities can be considered in a two fluid or anisotropic fluid approximation, however, the inclusion of the full kinetic effects require to start with a kinetic plasma description.

Many of the macro instabilities also have a kinetic counterpart. Ususally these lead to significant modification only if the wave length of the underlying mode is comparable to kinetic scales ()gyro scale or Debye length). A significant difference in the growth is present for the tearing mode. Note, that the resistive tearing mode is stricly applicable only in the case of sufficient collisions whil many space plasmas are collisionless.

To examine the propoerties of the collisionless tearing mode we start again from the Harris sheet where we assume $\partial/\partial z = 0$ and exact neutrality $\phi = 0$. Here the constants of motion for the particle species s are

$$H_s = m_s v^2/2 + q_s\phi \quad \text{and} \quad P_s = m_s v_z + q_s A_0$$

where P_s and A_0 are the z components of the conjugate momentum and of the vector potential. Any function of the constants of motion $f_s(\mathbf{r}, \mathbf{v}) = F_s(H_s, P_s)$ solves the collisionless Boltzmann equation. As outlined before the equilibrium distributions

$$F_s(H_s, P_s) = c_s \exp(-\alpha_s H_s - \beta_s P_s)$$

The solution for this configuration is

$$\begin{aligned} A_0 &= \hat{A}_0 \ln\cosh\frac{y}{L} \\ \mathbf{B}_0 &= \hat{B}_0 \tanh\frac{y}{L}\mathbf{e}_x \\ \mathbf{j}_0 &= -\hat{j}_0 \cosh^{-2}\frac{y}{L} \end{aligned}$$

with

$$\begin{aligned} L &= \lambda_i \frac{v_{thi}}{|w_i|}\sqrt{\frac{2T_i}{T_i + T_e}} \\ \hat{A}_0 &= -2\frac{k_B T_i}{e w_i} \\ \hat{B}_0 &= -\mathrm{sgn}(w_i)\,\sqrt{2\mu_0 (p_{i0} + p_{e0})} \\ \hat{j}_0 &= \hat{B}_0 / L_0 \end{aligned}$$

and $p_{s0} = n_0 k_B T_s$ and $v_{thi} = \sqrt{k_B T_i / m_i}$. Following the derivation for kinetic wave we start from the collisionless Boltzmann equation for a distribution function of the form $f\left(x, y, v_x, v_y, P, t\right)$ where P is a constant of motion

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dv_x}{dt}\frac{\partial f}{\partial v_x} + \frac{dv_y}{dt}\frac{\partial f}{\partial v_y} = 0$$

for each particle species. Using the equations of motion for the particles

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{q}{m}\left(v_y B_z - \frac{P - qA}{m} B_y\right)\frac{\partial f}{\partial v_x} \\ &\quad + \frac{q}{m}\left(\frac{P - qA}{m} B_x - v_x B_z\right)\frac{\partial f}{\partial v_y} = 0 \end{aligned}$$

where we replaced $m w_z = P - qA$. We now linearize this equation using $f = F\left(H, P\right) + f_1$

$$\begin{aligned} \frac{df_1}{dt} &= \frac{q}{m}\frac{\partial F}{\partial H}\frac{P - qA}{m}\left(v_y B_{x1} - v_x B_{y1}\right) - \frac{q^2}{m}\frac{\partial F}{\partial H} B_x v_y A_1 \\ &= -\frac{d}{dt}\left[\frac{q}{m}\frac{\partial F}{\partial H}\left(P - qA\right) A_1\right] + \frac{q}{m}\frac{\partial F}{\partial H}\left(P - qA\right)\frac{\partial A_1}{\partial t} \end{aligned}$$

Similar to the electromagnetic wave discussion we can formally integrate this to obtain:

$$f_1 = -\frac{q}{m}\frac{\partial F}{\partial H}\left(P - qA\right) A_1 + \frac{q}{m}\frac{\partial F}{\partial H}\int_{-\infty}^{t}\left(P - qA\right)'\left(\frac{\partial A_1}{\partial t}\right)' dt$$

This integral has to be integrated over the unperturbed particle orbits. A critical parameter for these orbits is given by the gyromotion inside the Harris sheet close to the center of the current sheet one has to distinguish between the regular drifting particle motion and a chaotic particle motion for particles which cross the center of the current sheet. It is relatively straightforward to particles located outside of the center by a distance $d = \sqrt{2 L r_g}$ undergo just the regular ∇B drift while those inside of this carry out a chaotic motion.

Following we assume $d \ll L$. For particles with a regular dift the variation of the integrant is averaging out and we can neglect this integrant for the perturbaed distribution function. For $|y| < d$ the average velocity is approximately the thermal velocity, in which case one can approximate the integral by taking $P - qA$ in front of the integral. Using perturbations of the form $A_1 = \hat{A}_1\left(y\right)\exp\left(\gamma t + ikx\right)$ where $x' = x + v_x\left(t' - t\right)$. Now the perturbed distribution is

$$f_1 = -\frac{q}{m}\frac{\partial F}{\partial H}\left(P - qA\right) A_1 + \gamma\frac{q}{m}\frac{\partial F}{\partial H}\frac{P - qA}{\gamma + ikv_x} A_1$$

where it was also assumed that $\hat{A}_1 \approx constant$ inside of d. It is also reminded that we have a perturbed distribution for each particle species where we had ommitted the index s for convenience. Now the perturbed current density is

$$j_{z1}=\sum_s q_s\int\frac{P-q_sA}{m_s}f_{s1}dv_xdv_yd\frac{P}{m}-\sum_s\frac{q_s^2}{m_s}A_1\int F_sd\tau dv_xdv_yd\frac{P}{m}$$

Substituting the perturbed distribution function yields

$$j_{1z}=\begin{cases}\frac{\partial j_0}{\partial A}A_1+\gamma A_1\sum_s\frac{q_s}{m_s}\int\frac{\partial F_s}{\partial H_s}\frac{(P-q_sA)^2}{\gamma+ikv_x}dv_xdv_yd\frac{P}{m} & y<d\\ \frac{\partial j_0}{\partial A}A_1 & y\geq d\end{cases}$$

This equation is the same for the outer solution from the resistive tearing mode such that the logaritmic derivative of the outer solution is

$$\frac{A_1'(0)}{A_1(0)}=\frac{1-k^2}{k}$$

The integral for the inner solution is

$$\int\frac{\partial F_s}{\partial H_s}\frac{(P-q_sA)^2}{\gamma+ikv_x}dv_xdv_yd\frac{P}{m}=-\left(\frac{\pi}{2k_BTs}\right)^{1/2}\frac{m_s^{3/2}n_0}{k\cosh^2(y/L)}\left(1+2\frac{r_g^2}{L^2}\right)$$

In this evaluation we assumed the small growth rate limit for the plasma dispersion function. Now the perturbed current density is

$$j_{z1}=\frac{2A_1}{\mu_0L^2\cosh^2(y/L)}\left[1-\gamma M\left(1+2\frac{r_g^2}{L^2}\right)\right],\qquad M=\frac{\pi^{1/2}L^2}{2r_g^2kv_t}$$

To obtain the dispersion relation we need to solve Ampere's law

$$\frac{d^2A_1}{dy^2}-\left(k^2-\frac{2-2\gamma M\left(1+2r_g^2/L^2\right)}{L^2\cosh^2(y/L)}\right)A_1=0$$

The dominant term in this equation is the γM term such that this is again a boundary layer problem where the inner solution is important on the scale $\vartheta=y\sqrt{d/L}$. In this expansion we need to solve

$$\frac{d^2A_1}{dy^2}=\frac{2\gamma M}{L^2}A_1$$

This has a straightforward solution and keeping in mind the symmetry condition $dA_1/dz=0$ we can match the logarithmic derivatives to obtain

$$\gamma=c_0\Omega_0\left(\frac{r_g}{L}\right)^{5/2}\left(1-k^2L^2\right)$$

where c_0 is a constant close to 1. This result has been derived for the condition of $d\ll L$. For a slightly modified problem one can actully drop this approximation to obtain the growth rate for a thin current sheet with $d\approx L$

$$\gamma = \frac{\Omega_0}{\sqrt{\pi}} \left(\frac{r_g}{L}\right)^3 \frac{kL\,(2+kL)\,(1-kL)}{1+2r_g^2/L^2}$$

which yields a much larger growth rate than the thick current sheet with $d/L \ll 1$.

5.2 Two-Stream Instability

Starting from the equations for electrons, ions and Poisson equation and using the plane wave approach one obtains

$$\begin{aligned}
(\omega - kV_0)\, n_{e1} &= kn_0 u_e \\
(\omega - kV_0)\, u_e &= kc_{se}^2 \frac{n_{e1}}{n_0} - i\frac{e}{m_e}E \\
\omega n_{i1} &= kn_0 u_i \\
\omega u_i &= kc_{si}^2 \frac{n_{i1}}{n_0} + i\frac{e}{m_i}E \\
ikE &= \frac{e}{\epsilon_0}\,(n_{i1} - n_{e1})
\end{aligned}$$

From the continuity equations we obtain

$$\begin{aligned}
u_e &= \frac{\omega - kV_0}{kn_0} n_{e1} \\
u_i &= \frac{\omega}{kn_0} n_{i1}
\end{aligned}$$

which can be substituted in the momentum equations

$$\begin{aligned}
\left(\frac{(\omega - kV_0)^2}{k^2} - c_{se}^2\right) k\frac{n_{e1}}{n_0} &= -i\frac{e}{m_e}E \\
\left(\frac{\omega^2}{k^2} - c_{si}^2\right) k\frac{n_{i1}}{n_0} &= i\frac{e}{m_i}E
\end{aligned}$$

or

$$\begin{aligned}
n_{e1} &= -i\frac{en_0}{m_e}\left((\omega - kV_0)^2 - k^2 c_{se}^2\right)^{-1} kE \\
n_{i1} &= i\frac{en_0}{m_i}\left(\omega^2 - k^2 c_{si}^2\right)^{-1} kE
\end{aligned}$$

Substitution into Poisson's equation yields

$$ikE = ikE\left[\frac{e^2 n_0}{\epsilon_0 m_i}\left(\omega^2 - k^2 c_{si}^2\right)^{-1} + \frac{e^2 n_0}{\epsilon_0 m_e}\left((\omega - kV_0)^2 - k^2 c_{se}^2\right)^{-1}\right]$$

or

$$1 = \omega_{pi}^2 \left(\omega^2 - k^2 c_{si}^2\right)^{-1} + \omega_{pe}^2 \left((\omega - kV_0)^2 - k^2 c_{se}^2\right)^{-1}$$

Other micro-instabilities