1. Debye shielding:

Consider a test charge q_t and compute the net charge of the Debye shielding cloud (Yukawa potential) as a function of radial distance measured in units of the Debye length.

Compute the net charge of the shielding cloud for the Yukawa potential.

$$\Phi_D = \frac{q_t}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right) \tag{1}$$

with

$$\lambda_D^{-2} = \frac{n_0 e^2}{\epsilon_0 k_B}\left(\frac{1}{T_e} + \frac{1}{T_i}\right)$$

Poisson's equations:

$$\nabla^2\Phi = -\frac{1}{\epsilon_0} q_t \delta(\mathbf{r}) + \frac{1}{\lambda_D^2}\Phi(\mathbf{r}) = -\frac{1}{\epsilon_0}\rho_c$$

This yields for the charge density:

$$\begin{aligned}
\rho_c &= q_t\delta(\mathbf{r}) - \frac{\epsilon_0}{\lambda_D^2}\Phi(\mathbf{r}) \\
&= q_t\delta(\mathbf{r}) - \frac{q_t}{4\pi\lambda_D^2}\frac{1}{r}\exp\left(-\frac{r}{\lambda_D}\right)
\end{aligned}$$

Integration over total space yields the total charge:

$$\begin{aligned}
Q_{net} &= q_t - \frac{q_t}{4\pi\lambda_D^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{1}{r}\exp\left(-\frac{r}{\lambda_D}\right) r^2 dr \sin\theta d\theta d\phi \\
&= q_t - \frac{q_t}{\lambda_D^2}\int_0^R r\exp\left(-\frac{r}{\lambda_D}\right) dr \\
&= q_t - \frac{q_t}{\lambda_D^2}\left[-\lambda_D r\exp\left(-\frac{r}{\lambda_D}\right) - \lambda_D^2\exp\left(-\frac{r}{\lambda_D}\right)\right]_0^R \\
&= q_t - \frac{q_t}{\lambda_D^2}\left[-\lambda_D R\exp\left(-\frac{R}{\lambda_D}\right) - \lambda_D^2\exp\left(-\frac{R}{\lambda_D}\right) + \lambda_D^2\right] \\
&= q_t\left[\frac{R}{\lambda_D} + 1\right]\exp\left(-\frac{R}{\lambda_D}\right)
\end{aligned}$$

where the integral over r is solved through integration by parts.

Note, an alternative derivation of the charge density can be obtained by computing $\nabla^2\Phi_D$ as indicated in class (with the radial electric field given by

$$E_r = -\frac{\partial \Phi}{\partial r} = \left(\frac{r}{\lambda_D} + 1\right) \frac{q_t}{4\pi\epsilon_0 r^2} \exp\left(-\frac{r}{\lambda_D}\right)$$

$$\begin{aligned} Q_{net} &= \int_{sphere}^{R} \rho_c d^3 r = \epsilon_0 \int_{sphere}^{R} \nabla \cdot \mathbf{E} d^3 r \\ &= \epsilon_0 \int_{surface}^{R} \mathbf{E} \cdot d^2 s = 4\pi R^2 \epsilon_0 E_r(R) \\ &= q_t \left(\frac{R}{\lambda_D} + 1\right) \exp\left(-\frac{R}{\lambda_D}\right) \end{aligned}$$

Note be care full to consider that $\nabla(1/r) = \mathbf{r}/r^3$ and $\nabla \cdot (\mathbf{r}/r^3) = 4\pi\delta(\mathbf{r})$.

2. Plasma properties:

(a) Calculate the electron thermal speed, Debye length, and the plasma parameter for

- a tokamak plasma with $T_e = 10^8$ K, $n_0 = 10^{19}$ m^{-3}
- the tail magnetosphere with $T_e = 10^7$ K, $n_0 = 10^6$ m^{-3}
- the ionosphere with $T_e = 10^3$ K, $n_0 = 10^{12}$ m^{-3}
- the solar atmosphere with $T_e = 10^4$ K, $n_0 = 10^{20}$ m^{-3}

with $k_B = 1.38 \cdot 10^{-23}$ JK^{-1}, $\epsilon_0 = 8.85 \cdot 10^{-12}$ F/m, $m_e = 9.11 \cdot 10^{-31}$ kg, and $e = 1.60 \cdot 10^{-19}$ C

Solution: Electron thermal speed:

$$v_{the} = \left(\frac{k_B T_e}{m_e}\right)^{1/2} = 3.9 \cdot 10^3 T_e^{1/2}\, m/s$$

Electron Debye length:

$$\begin{aligned} \lambda_{De} &= \left(\frac{\epsilon_0 k_B T_e}{n_0 e^2}\right)^{1/2} \\ &= \left(\frac{8.85 \cdot 10^{-12} \cdot 1.38 \cdot 10^{-23}}{1.6^2 \cdot 10^{-38}}\right)^{1/2} \left(\frac{T_e}{n_0}\right)^{1/2} m \\ &= 69.1 \cdot \left(\frac{T_e}{n_0}\right)^{1/2} m \end{aligned}$$

Electron plasma parameter:

$$\Lambda_e = n_0 \lambda_{De}^3 = 3.3 \cdot 10^5 \frac{T_e^{3/2}}{n_0^{1/2}}$$

	T_e / K	n_0 / m^{-3}	v_{the} / m/s	λ_{De} / m	Λ_e
Tokamak	10^8	10^{19}	$3.9 \cdot 10^7$	$2.2 \cdot 10^{-4}$	$1.04 \cdot 10^8$
Magnetotail	10^7	10^6	$1.2 \cdot 10^7$	220	$1.04 \cdot 10^{13}$
Ionosphere	10^3	10^{12}	$1.2 \cdot 10^5$	$2.2 \cdot 10^{-3}$	$1.04 \cdot 10^4$
Solar atmosphere	10^4	10^{20}	$3.9 \cdot 10^5$	$6.9 \cdot 10^{-7}$	33

(b) Compute collision frequency and mean free path for these plasmas.

Solution: Plasma frequency:

$$\omega_{pe} = \left(\frac{n_0 e^2}{m_e \epsilon_0}\right)^{1/2} \tag{2}$$

$$\begin{aligned} \omega_{pe} &= \left(\frac{n_0 e^2}{m_e \epsilon_0}\right)^{1/2} \\ &= 1.60 \cdot 10^{-19} \left(\frac{n_0}{9.11 \cdot 10^{-31} 8.85 \cdot 10^{-12}}\right)^{1/2} = \frac{1.60 \cdot 10^{-19}}{2.84 \cdot 10^{-21}} n_0^{1/2} = 56.34 n_0^{1/2} \end{aligned}$$

Collision frequency:

$$\begin{aligned} \nu_c &= \sqrt{\frac{\pi}{2}} \frac{1}{32\pi} \frac{\omega_p}{\Lambda} \ln\left[12\pi\Lambda\right] \\ &= 1.247 \cdot 10^{-2} \frac{\omega_p}{\Lambda} \ln\left(37.7\Lambda\right) \end{aligned}$$

Mean free path:

$$\begin{aligned} L_c &= v_{the}/\nu_c \\ &= \sqrt{\frac{2}{\pi}} 32\pi \left(\frac{k_B T_e}{m_e}\right)^{1/2} \left(\frac{m_e \epsilon_0}{n_0 e^2}\right)^{1/2} \frac{\Lambda}{\ln\left[12\pi\Lambda\right]} = \sqrt{\frac{2}{\pi}} 32\pi \frac{\lambda_{De}\Lambda}{\ln\left[12\pi\Lambda\right]} \end{aligned}$$

	n_0 / m^{-3}	Λ_e	v_{the} / m/s	ω_p / s^{-1}	ν_c / s^{-1}	$\ln(37.7\Lambda)$	L_c / m
Tokamak	10^{19}	$1.04 \cdot 10^8$	$3.9 \cdot 10^7$	$1.78 \cdot 10^{11}$	$4.71 \cdot 10^2$	22.09	$8.28 \cdot 10^4$
Magnetotail	10^6	$1.04 \cdot 10^{13}$	$1.2 \cdot 10^7$	$5.63 \cdot 10^4$	$2.27 \cdot 10^{-9}$	33.60	$5.29 \cdot 10^{15}$
Ionosphere	10^{12}	$1.04 \cdot 10^4$	$1.2 \cdot 10^5$	$5.63 \cdot 10^7$	$8.69 \cdot 10^2$	12.88	$1.38 \cdot 10^2$
Solar Atmosph.	10^{20}	33	$3.9 \cdot 10^5$	$5.63 \cdot 10^{11}$	$1.52 \cdot 10^9$	7.13	$2.5 \cdot 10^{-4}$

3. Plasma Definition

Can a fully ionized plasma be maintained at temperatures of $T_e = 100$ K (Hint: derive a condition for the density in relation to the temperature). How important is recombination for such a cold plasma?

Solution:

Plasma definition:

$$\Lambda = n\lambda_D^3 = n^{-1/2}\left(\frac{\epsilon_0 k_B T_e}{e^2}\right)^{3/2} \gg 1$$

Taking the square

$$\begin{aligned} n^{-1}\left(\frac{\epsilon_0 k_B T_e}{e^2}\right)^3 &= n^{-1}\left(\frac{8.85\cdot 10^{-12}\cdot 1.38\cdot 10^{-23}\cdot 10^2}{1.6^2\cdot 10^{-38}}\right)^3\left[m^{-3}\right] \\ &= n^{-1}\left(4.8\cdot 10^5\right)^3\left[m^{-3}\right] = n^{-1}\cdot 1.1\cdot 10^{17}\left[m^{-3}\right] \end{aligned}$$

or taking number density as particle per cm^{-3}

$$\sqrt{n} \ll 3.3\cdot 10^5$$

which implies that n could be as small as 10^7or even 10^8 cm^{-3} to satisfy the plasma definition, i.e., that the potential energy is much smaller than the kinetic energy of the charged particles.

While this is true there is the additional aspect of how long such a plasma could be maintained. Even tho a sufficiently low density gas of charged particles can satisfy the plasma definition, recombination caused by collisions can remove the the charged particles. For long life time of such a plasma it would be necessary to have a very low level of collisions that could result in recombinations. In other words the collision frequency must be sufficiently low and with a comparatively small value of the plasma parameter of for instance 10^2 the collision frequency is still quite large and would lead to relatively fast recombination leading to a loss of this plasma within a short time frame.