4. Moments of the Maxwell distribution function:

The Maxwell distribution function is given by

$$f(\mathbf{v}) = n_0 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right)$$

By integrating over velocity space compute the particle number density n, the kinetic energy density $\frac{m}{2}\langle v^2\rangle$, and the integral used to derive the collision frequency $\langle v^3\rangle$, where the brackets indicate averages determined by the velocity space integration.

Help: It is useful to represent the velocity in the integration in spherical coordinates.

Solution:

(a) Particle density:

$$\begin{aligned}
n &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(v) v^2 dv \sin\theta d\theta d\phi \\
&= 4\pi n_0 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^{\infty} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv \\
&= 4\pi n_0 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \left(\frac{2k_B T}{m}\right)^{3/2} \int_0^{\infty} x^2 \exp\left(-x^2\right) dx \\
&= 4n_0 \left(\frac{1}{\pi}\right)^{1/2} \left\{-\left[\frac{x}{2}\exp\left(-x^2\right)\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} \exp\left(-x^2\right) dx\right\} \\
&= 4n_0 \left(\frac{1}{\pi}\right)^{1/2} \frac{\sqrt{\pi}}{4} = n_0
\end{aligned}$$

(b) Kinetic energyb density:

$$\begin{aligned}
e_{kin} &= \frac{1}{n_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{m}{2} v^2 f(v) v^2 dv \sin\theta d\theta d\phi \\
&= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^{\infty} \frac{m}{2} v^4 \exp\left(-\frac{mv^2}{2k_B T}\right) dv \\
&= 4\pi \frac{m}{2} \left(\frac{m}{2\pi k_B T}\right)^{3/2} \left(\frac{2k_B T}{m}\right)^{5/2} \int_0^{\infty} x^4 \exp\left(-x^2\right) dx \\
&= 2m \left(\frac{1}{\pi}\right)^{1/2} \frac{2k_B T}{m} \left\{-\left[\frac{x^3}{2}\exp\left(-x^2\right)\right]_0^{\infty} + \frac{3}{2}\int_0^{\infty} x^2 \exp\left(-x^2\right) dx\right\} \\
&= 4 \left(\frac{1}{\pi}\right)^{1/2} k_B T \left\{-\left[\frac{x^3}{2}\exp\left(-x^2\right)\right]_0^{\infty} - \left[\frac{3}{4} x \exp\left(-x^2\right)\right]_0^{\infty} + \frac{3}{4}\int_0^{\infty} \exp\left(-x^2\right) dx\right\} \\
&= 4 \left(\frac{1}{\pi}\right)^{1/2} k_B T \frac{3}{4}\frac{\sqrt{\pi}}{2} = \frac{3}{2} k_B T
\end{aligned}$$

(c) Average of $\langle v^3 \rangle$:

$$\begin{aligned}
\left\langle v^3 \right\rangle &= \frac{1}{n_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v^3 f(v) v^2 dv \sin\theta d\theta d\phi \\
&= 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int_0^{\infty} v^5 \exp\left(-\frac{mv^2}{2k_B T}\right) dv \\
&= 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2}\left(\frac{2k_B T}{m}\right)^{6/2}\int_0^{\infty} x^5 \exp\left(-x^2\right) dx \\
&= 4\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{2k_B T}{m}\right)^{3/2}\int_0^{\infty} x^5 \exp\left(-x^2\right) dx \\
&= 4\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{2k_B T}{m}\right)^{3/2}\left\{-\left[\frac{x^4}{2}\exp\left(-x^2\right)\right]_0^{\infty} - \left[x^2\exp\left(-x^2\right)\right]_0^{\infty} - \left[\exp\left(-x^2\right)\right]_0^{\infty}\right\} \\
&= 4\left(\frac{1}{\pi}\right)^{1/2}\left(\frac{2k_B T}{m}\right)^{3/2} = 8\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{k_B T}{m}\right)^{3/2}
\end{aligned}$$

5. Plasma properties:

Consider an ordinary fluid with a continuity equation $\partial n/\partial t + \nabla\cdot(\mathbf{v}n) = 0$. The number of particles in an arbitrary volume is $N = \int_V n d^3 r$. Show that the number of particles changes only due to particle flux through the surface of the volume V.

Solution:

Number of particles in volume V:

$$N = \iiint\limits_V n d^3 r$$

Change of the number of particles in this volume (with fixed boundaries):

$$\begin{aligned}
\frac{dN}{dt} &= \iiint\limits_V \frac{\partial n}{\partial t} d^3 r \\
&= -\iiint\limits_V \nabla\cdot(\mathbf{v}n)\, d^3 r \\
&= -\iint\limits_V n\mathbf{v}\cdot d\mathbf{s}
\end{aligned}$$

where we have used Gauss's law and the last surface integral is over the total surface of the the volume V. Note that $d\mathbf{s}$ is the surface element with the outward normal such that $n\mathbf{v}\cdot d\mathbf{s}$ is the local flux outward (when $\mathbf{v}$ has a component along $d\mathbf{s}$.

6. Lorentz equation of motion:

A particle with the mass m and the electric charge e is moving in homogeneous magnetic and electric fields: $\mathbf{B} = B\mathbf{e}_z$ and $\mathbf{E} = E\mathbf{e}_y$. Determine a solution of the equations of motion for the initial conditions $\mathbf{r}\,(t=0) = 0$ and $\mathbf{v}\,(t=0) = v_0\mathbf{e}_x$. For what value of v_0 is the gyro-velocity 0?

Solution:

Lorentz force equation:

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Separation of the velocity into a steady $\mathbf{v}_E$ and a time varying portion$\mathbf{v} = \mathbf{v}_E + \mathbf{w}$ yields

$$\frac{d\mathbf{w}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v}_E \times \mathbf{B} + \mathbf{w} \times \mathbf{B})$$

leads to $\mathbf{E} + \mathbf{v}_E \times \mathbf{B} = 0$ or $\mathbf{v}_{E\perp} = \mathbf{E} \times \mathbf{B}/B^2$ or $\mathbf{v}_E = E/B\mathbf{e}_y \times \mathbf{e}_z = E/B\mathbf{e}_x$.

To solve the equations of motion let us consider a magnetic field of $\mathbf{B} = B\mathbf{e}_z$. The time dependent part can be split into a component parallel and perpendicular to $\mathbf{B}$:

$$\begin{aligned} \frac{dw_z}{dt} &= 0 \\ \frac{d\mathbf{w}_\perp}{dt} &= \frac{qB}{m}\mathbf{w}_\perp \times \mathbf{e}_z \end{aligned}$$

the z component has the solution $w_z = w_{z0} = \text{const}$. With $\omega_g = qB/m$ the perpendicular equations are

$$\begin{aligned} \frac{d}{dt}w_x &= \omega_g w_y \\ \frac{d}{dt}w_y &= -\omega_g w_x \end{aligned}$$

Substitution of the 2nd eqation into the time derivative of the first equation yields

$$\frac{d^2 w_x}{dt^2} + \omega_g^2 w_x = 0$$

with the general solution $w_x = w_{\perp 0}\cos(\omega_g t + \phi)$. Integration or differentiation yields for the $y$component $w_y = -w_{\perp 0}\sin(\omega_g t + \phi)$. In summary the total velocity is

$$\begin{aligned} v_x &= E/B + w_{\perp 0}\cos(\omega_g t + \phi) \\ v_y &= -w_{\perp 0}\sin(\omega_g t + \phi) \\ v_z &= w_{z0} \end{aligned} \tag{1}$$

with $v_\perp^2 = v_x^2 + v_y^2$. The initial condition $\mathbf{v}\,(t=0) = v_0\mathbf{e}_x$ yields

$$\begin{aligned} E/B + w_{\perp 0}\cos(\omega_g t + \phi) &= v_0 \\ -w_{\perp 0}\sin(\omega_g t + \phi) &= 0 \\ w_{z0} &= 0 \end{aligned}$$

or $\phi = 0$ and $w_{\perp 0} = v_0 - E/B$. Integration of the velocities in (1) yields for the coordinates

$$\begin{aligned} x - x_0 &= \frac{E}{B} t + \frac{w_{\perp 0}}{\omega_g} \sin(\omega_g t) \\ y - y_0 &= \frac{w_{\perp 0}}{\omega_g} \cos(\omega_g t) \\ z - z_0 &= 0 \end{aligned} \tag{2}$$

Applying the initial condition $\mathbf{r}(t = 0) = 0$ yields $x_0 = 0$, $y_0 = -w_{\perp 0}/\omega_g$, and $z_0 = 0$. The complete solution is

$$\begin{aligned} x &= \frac{E}{B} t + \frac{v_0 - E/B}{\omega_g} \sin(\omega_g t) \\ y &= \frac{v_0 - E/B}{\omega_g} [\cos(\omega_g t) - 1] \\ z &= 0 \end{aligned}$$

Thus the gyro-motion vanishes entirely for $v_0 = E/B$.