7. Particle dynamics

Consider an electron and an oxygen ion in the Earth's ionosphere at a magnetic pole and 300 km altitude.

a) Determine the electric field which is necessary to balance the gravitational force on the particles.

b) Assuming that the parallel velocity of the particles is 0, determine the perpendicular velocity for the two particles which is necessary to balance the gravitational force by the mirror force.

Comment your results in a) and b).

Solution:

(a) Since the acceleration is only along the parallel direction force balance is $qE = mg$ or

$$E = \frac{mg}{q}$$

$$\begin{aligned}\text{Electron :}\quad & E = \frac{9.11\cdot 10^{-31}\cdot 9.81}{1.60\cdot 10^{-19}}\mathrm{Vm^{-1}} = 5.59\cdot 10^{-11}\mathrm{Vm^{-1}} \\ \text{Oxygen ion :}\quad & E = \frac{16\cdot 1.67\cdot 10^{-27}\cdot 9.81}{1.60\cdot 10^{-19}}\mathrm{Vm^{-1}} = 1.64\cdot 10^{-6}\mathrm{Vm^{-1}}\end{aligned}$$

Here the electric field should point up for the oxygen ion and down for the electron. The electric field to balance gravity is very small which is expected because electric forces are much larger than gravitational forces.

(b) Mirror force: $\mu\nabla_{\|}B$ with $\mu = mv_{\perp}^2/(2B)$. Total force balance

$$m\frac{dv_{\|}}{dt} = mg - \mu\nabla_{\|}B = 0$$

Solving for $v_{\perp}$:

$$v_{\perp} = \sqrt{\frac{2gB}{\nabla_{\|}B}}$$

The magnetic field magnitude is

$$B = \frac{B_E R_E^3}{r^3}\left(1 + 3\cos^2\theta\right)^{1/2} \quad \text{with} \quad B_E = 3.11\cdot 10^{-5}\mathrm{T}$$

At the magnetic pole we have

$$\nabla_{\|}B = \left.\frac{\partial B}{\partial r}\right|_{\lambda=90^\circ} = -6\frac{B_E R_E^3}{r^4} \quad \textit{and} \quad \frac{1}{B}\nabla_{\|}B = -\frac{3}{r}$$

such that the perpendicular velocity is $v_{\perp} = \sqrt{-2gr/3}$. Note that the minus sign comes from the mirror force and is compensated by the gravity term because $g = -9.81\,\mathrm{m/s^2}$.

Thus the perpendicular velocity required to compensate gravity through the mirror force is $v_\perp = \sqrt{2 \cdot 9.81 \cdot 6.4 \cdot 10^6/}$
$6.47\,\mathrm{km/s}$ irrespective of the particle mass.

There are two intersting aspects: First the perpendicular particle velocity necessary to balance gravity through the mirror force is independent of particle mass because both gravitational and mirror force are proportional to particle mass.

Second, all particles that reach ionospheric heights and have energies smaller than that corresponding to the calculated velocity for force balance are lost in the ionosphere. Note that the enrgy cut off for these distributions is much smaller for electrons than for ions (by a factor of m_e/m_i).

8. Magnetic gradient drift:

Compute the gradient $\mathbf{B}$ drift velocity in the Earth's dipole field at 6 R_E radial distance for ions with energies of 1, 10 100, and 1000 keV. Assume that the particle energy is entirely in the perpendicular velocity.

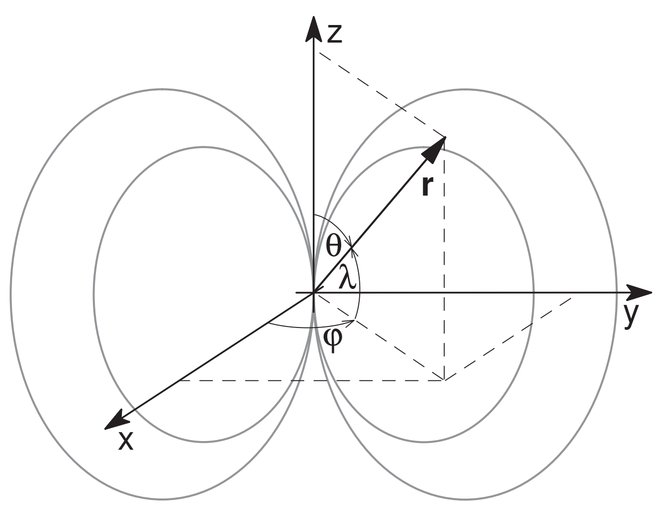
Help:

The Earth's dipole field is given in spherical coordinates by

$$B_r = -2B_E R_E^3 \frac{\sin\lambda}{r^3} \quad (1)$$

$$B_\lambda = B_E R_E^3 \frac{\cos\lambda}{r^3} \quad (2)$$

with $B_E = 3.11 \cdot 10^{-5}$ T as illustrated below.



Solution:

(a) Gradient and curvature drifts in the absence of electric currents (dipole field):

$$\mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3} \left(\mathbf{B} \times \nabla B\right)$$

With the magnetic field ($B_E = 3.11 \cdot 10^{-5}\mathrm{T}$)

$$
\begin{aligned}
B^2 &= B_r^2 + B_\lambda^2 = \left(\frac{B_E R_E^3}{r^3}\right)^2 \left(\cos^2\lambda + 4\sin^2\lambda\right) \\
B &= \frac{B_E R_E^3}{r^3}\left(1 + 3\sin^2\lambda\right)^{1/2}
\end{aligned}
$$

with $k_B = 1.38 \cdot 10^{-23}$ JK^{-1}, $\epsilon_0 = 8.85 \cdot 10^{-12}$ F/m, $m_e = 9.11 \cdot 10^{-31}$ kg, $m_p = 1.673 \cdot 10^{-27}$ kg, and $e = 1.60 \cdot 10^{-19}$ C, and $B_E = 3.11 \cdot 10^{-5}$ T

1 eV particle: $\frac{m}{2}v^2 = 1eV = 1.60 \cdot 10^{-19}J$

In the equatorial plane ∇B has only a radial component (and $\lambda = 0$, $\cos\lambda = 1$):

$$
\nabla B = \partial B/\partial r\, \mathbf{e}_r = -3\frac{B_E R_E^3}{r^4}\,\mathbf{e}_r = -3\frac{B}{r}\,\mathbf{e}_r
$$

such that

$$
\begin{aligned}
\frac{1}{B^3}\left(\mathbf{B} \times \nabla B\right)_{\lambda=0} &= -3\frac{B}{r}\frac{B_\lambda}{B^3}\left(\mathbf{e}_\lambda \times \mathbf{e}_r\right) \\
&= -\frac{3}{r}\frac{B_E R_E^3}{r^3}\left(\frac{r^3}{B_E R_E^3}\right)^2 \mathbf{e}_\phi = -\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi
\end{aligned}
$$

Gradient Drift velocity:

$$
\begin{aligned}
\mathbf{v}_\nabla &= \frac{m v_\perp^2}{2qB^3}\left(\mathbf{B} \times \nabla B\right) \\
&= -\frac{m v_\perp^2}{2q}\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi
\end{aligned}
$$

For a 1 keV particle

$$
\begin{aligned}
\mathbf{v}_\nabla &= -10^3 \frac{3 \cdot 6^2}{3.11 \cdot 10^{-5} \cdot 6.4 \cdot 10^6}\mathbf{e}_\phi \mathrm{m\,s^{-1}} \\
&= -10^3 \cdot 0.543 \mathbf{e}_\phi \mathrm{m\,s^{-1}} = 0.543\,\mathrm{km\,s^{-1}}
\end{aligned}
$$

Since the drift velocity is proportional to particle energy we obtain $\mathbf{v}_\nabla = 5.43\,\mathrm{km\,s^{-1}}$ for a 10keV, $\mathbf{v}_\nabla = 54.3\,\mathrm{km\,s^{-1}}$ for a 100keV, and $\mathbf{v}_\nabla = 543\,\mathrm{km\,s^{-1}}$ for a MeV particle.