1. Particle dynamics

Assume a plasma density of 1 cm^{-3}, temperature equivalent to 1 keV, and a magnetic field of 20 nT which are typical for the near Earth magnetotail.

Determine the temperature in degrees Kelvin. Determine the energy density in kW hours/m^3 and kW hours /R_E^3 (1 R_E=6400 km). For the sake of simplicity assume that the plasma sheet of the magnetosphere is represented by a cylinder with 15 R_E radius and 100 R_E length. How long could a power plant with an output of 1000 MW operate on the thermal energy stored in the plasma? How many power plants could operate on the respective power if the plasma sheet energy were restored every 30 minutes ?

Solution:

- Energy: $e_0 = 10^3 eV = k_B T$. Temperature in degrees Kelvin: $T = 10^3 eV/k_B = 10^3 \cdot 1.6 \cdot 10^{-19}/1.38 \cdot 10^{23} K = 1.16 \cdot 10^7 K$
- Energy density: $E_{th} = \frac{3}{2} n k_B T$:

$$\begin{aligned} \varepsilon &= 1.5 \cdot 10^6 \cdot 1.38 \cdot 10^{-23} \cdot 1.16 \cdot 10^7 \text{Jm}^{-3} = 2.4 \cdot 10^{-10} J/m^3 = \frac{2.4 \cdot 10^{-10}}{10^3 \cdot 3600} \text{kWhours/m}^3 \\ &= 6.7 \cdot 10^{-17} \text{kWhours/m}^3 \end{aligned}$$

- In kW hours /R_E^3: $\varepsilon = 6.7 \cdot 10^{-17} \cdot 6.4^3 \cdot 10^{18} = 1.7 \cdot 10^4$kW hours /$R_E^3$.
- Energy in 15 R_E radius and 100 R_E length cylinder: $W = \pi \cdot 2.25 \cdot 10^4 \cdot 1.7 \cdot 10^4 = 1.24 \cdot 10^9$kW hours
- Operation time for 1000 MW power plant $t = W/10^6 = 1240$ hours≈ 52 days
- Power into the plasma sheet: $P = 1.24 \cdot 10^9/0.5$ kW $= 2.5 \cdot 10^6$ MW.
- Number of 1000 MW power plants: 2500

2. Magnetic gradient drift:

Geosynchronous (6 R_E geocentric radial distance) satellites observe frequently sudden increases in energetic particle fluxes in the Magnetosphere. This observation is called energetic particle injection. Typically this increase is seen first in the most energetic particles and subsequently at lower energies. This property is called energy dispersion in the particle fluxes. At the satellite location the magnetic field is reasonably approximated by a dipole field.

(a) Assuming a point source for the energetic particles, why is it that more energetic particle fluxes increase first?

The gradient drift for higher energy particles is faster. Therefore higher particles that are injected at the same time in the same location as lower energy particles outrun these particles and arrive first at a remote observer (who has to be in the path).

(b) Consider that 0.5 MeV flux increases about a minute after the 1 MeV particle flux increase. What is the distance to the origin of the energetic particles?

(c) At what times after the increase of the 1 MeV particles do you expect that the 100 keV and the 10 keV fluxes increase?

(Help: Although particles are generally conducting a gradient and curvature drift assume for this problem that particles have only a perpendicular velocity and are drifting in the equatorial plane of the dipole field.

Solution:

a) The gradient drift for higher energy particles is faster. Therefore higher particles that are injected at the same time in the same location as lower energy particles outrun these particles and arrive first at a remote observer (who has to be in the path).

b) Magnetic gradient drift velocity:

$$\mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}\left(\mathbf{B} \times \nabla B\right)$$

Dipole magnetic field:

$$B_r = -2B_E R_E^3 \frac{\sin\lambda}{r^3} \quad (1)$$
$$B_\lambda = B_E R_E^3 \frac{\cos\lambda}{r^3} \quad (2)$$

with $B_E = 3.11 \cdot 10^{-5}$ T . Magnitude of the magnetic field

$$\begin{aligned} B^2 &= B_r^2 + B_\lambda^2 = \left(\frac{B_E R_E^3}{r^3}\right)^2 \left(\cos^2\lambda + 4\sin^2\lambda\right) \\ B &= \frac{B_E R_E^3}{r^3}\left(1 + 3\sin^2\lambda\right)^{1/2} \end{aligned}$$

In the equatorial plane ∇B has only a radial component (and $\lambda = 0$, $\cos\lambda = 1$)

$$\nabla B = \partial B/\partial r\, \mathbf{e}_r = -3\frac{B_E R_E^3}{r^4}\,\mathbf{e}_r = -3\frac{B}{r}\,\mathbf{e}_r$$

such that

$$
\begin{aligned}
\frac{1}{B^3}\left(\mathbf{B}\times\nabla B\right)_{\lambda=0} &= -3\frac{B}{r}\frac{B_\lambda}{B^3}\left(\mathbf{e}_\lambda\times\mathbf{e}_r\right) \\
&= -\frac{3}{r}\frac{B_E R_E^3}{r^3}\left(\frac{r^3}{B_E R_E^3}\right)^2\mathbf{e}_\phi = -\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi
\end{aligned}
$$

Gradient Drift velocity:

$$
\mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}\left(\mathbf{B}\times\nabla B\right) = -\frac{mv_\perp^2}{2q}\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi
$$

For a 1 MeV particle

$$
\begin{aligned}
\mathbf{v}_{\nabla 1MeV} &= -10^6\frac{3\cdot 6^2}{3.11\cdot 10^{-5}\cdot 6.4\cdot 10^6}\mathbf{e}_\phi\mathrm{m\,s^{-1}} \\
&= -10^6\cdot 0.543\mathbf{e}_\phi\mathrm{m\,s^{-1}} = 543\,\mathrm{km\,s^{-1}}
\end{aligned}
$$

Since the drift velocity is proportional to particle energy we obtain $\mathbf{v}_{\nabla 0.5MeV} = 272\,\mathrm{km\,s^{-1}}$. The time to reach an observer at a distance d for the 1 MeV particles is $t_{1MeV} = d/v_{1MeV}$ and for the 0.5 MeV particles $t_{1MeV} = d/v_{0.5MeV}$. Therefore the time difference is expected to be

$$
\Delta t = d\left(\frac{1}{v_{0.5MeV}} - \frac{1}{v_{1MeV}}\right) \qquad \Longrightarrow \qquad d = \Delta t\frac{v_{1MeV} - v_{0.5MeV}}{v_{1MeV}v_{0.5MeV}} = \frac{\Delta t}{v_{1MeV}} = 3.26{\cdot}10^4 km = 5.1R_E
$$

c) For a 100keV, $\mathbf{v}_\nabla = 54.3\,\mathrm{km\,s^{-1}}$ for a 10 keV, and $\mathbf{v}_\nabla = 5.43\,\mathrm{km\,s^{-1}}$ for a MeV particle. The time difference for a distribution with a fraction f of the 1 MeV energy is

$$
\Delta t = d\left(\frac{1}{v_x} - \frac{1}{v_{1MeV}}\right) = d\frac{v_{1MeV} - fv_{1MeV}}{fv_{1MeV}^2} = \frac{d}{v_{1MeV}}\frac{1-f}{f}
$$

Using $t_0 = d/v_{1MeV} = 1$ minute we obtain $\Delta t_{100keV} = 1\cdot 0.9/0.1\, t_0 = 9$ minutes and $\Delta t_{10keV} = 1\cdot 0.99/0.01\, t_0 = 99$ minutes.

3. Normalization

a) Following the example presented in class, determine typical values for the electric field E_0 and the pressure p_0 from the MHD momentum equation in terms of L_0, ρ_0, B_0.

b) Using the normalization procedure, derive the coefficients of the inertial term and of the Hall term in generalized Ohm's law. Show that these coefficients are $(c/\omega_{pe})^2/L_0^2$ and $c/(\omega_{pi}L_0)$ respectively.

Solution:

a) From class we know the normalization for $j_0 = \frac{B_0}{\mu_0 L_0}$. We re-write the momentum equation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{3}$$

for all variables in terms of $\rho = \rho_0 \widehat{\rho}$, $\mathbf{u} = v_0 \widehat{\mathbf{u}}$, $\mathbf{B} = B_0 \widehat{\mathbf{B}}$, etc where the $\widehat{}$ indicates normalised values. The momentum equation becomes

$$\frac{\rho_0 v_0}{t_0}\frac{\partial \widehat{\rho}\widehat{\mathbf{u}}}{\partial \widehat{t}} + \frac{\rho_0 v_0^2}{L_0}\widehat{\nabla} \cdot (\widehat{\rho}\widehat{\mathbf{u}}\widehat{\mathbf{u}}) = -\frac{p_0}{L_0}\widehat{\nabla}\widehat{p} + \frac{B_0^2}{\mu_0 L_0}\widehat{\mathbf{j}} \times \widehat{\mathbf{B}}$$

Division by $\frac{B_0^2}{\mu_0 L_0}$ yields

$$\frac{\rho_0 v_0 \mu_0 L_0}{t_0 B_0^2}\frac{\partial \widehat{\rho}\widehat{\mathbf{u}}}{\partial \widehat{t}} + \frac{\rho_0 v_0^2 \mu_0}{B_0^2}\widehat{\nabla} \cdot (\widehat{\rho}\widehat{\mathbf{u}}\widehat{\mathbf{u}}) = -\frac{p_0 \mu_0}{B_0^2}\widehat{\nabla}\widehat{p} + \widehat{\mathbf{j}} \times \widehat{\mathbf{B}}$$

Now we set the coefficients to unity. The second term yields $v_0^2 = B_0^2/(\mu_0 \rho_0)$ which is the typical Alfven speed. The first term give just the identity $v_0 = L_0/t_0$ and the pressure term yields $p_0 = B_0^2/\mu_0$. Doing the sam for Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = -\frac{M}{e\rho}\nabla p_e + \frac{m_i}{e\rho}\mathbf{j} \times \mathbf{B} + \eta \mathbf{j} \tag{4}$$

yields $E_0 = v_0 B_0$

b) The coefficient of the Hall term in generalized Ohms's law becomes

$$\begin{aligned} \frac{m_i}{e\rho_0}\frac{1}{v_0 B_0} j_0 B_0 = \frac{1}{e n_0}\frac{B_0}{\mu_0 v_0 L_0} &= \frac{(\mu_0 m n_0)^{1/2}}{e n_0 \mu_0 L_0} \\ &= \left(c^2 \frac{m \epsilon_0}{e^2 n_0}\right)\frac{1}{L_0} \\ &= \frac{c}{\omega_{pi} L_0} \end{aligned}$$