Appendix A

Appendix

A.1 Bounce motion

Evaluation of the integral

$$\tau_b = 4\int_0^{\lambda_m} \frac{ds}{v_\parallel} = 4\int_0^{\lambda_m} \frac{ds}{d\lambda}\frac{d\lambda}{v_\parallel} \tag{A.1}$$

With $v_\parallel = v\cos\alpha = v\sqrt{1-\sin^2\alpha} = v\sqrt{1-(B/B_{eq})\sin^2\alpha_{eq}}$, the magnetic field

$$B(\lambda, L) = \frac{B_E}{L^3}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}$$

where $B_{eq} = B_E/L^3$, and the length of a field line (arc) element as a function of r or λ by $(ds)^2 = (dr)^2 + r^2\,(d\lambda)^2$such that

$$\begin{aligned}\frac{ds}{d\lambda} = \sqrt{\left(\frac{dr}{d\lambda}\right)^2 + r^2} &= \sqrt{(2r_{eq}\sin\lambda\cos\lambda)^2 + r_{eq}^2\cos^4\lambda} \\ &= r_{eq}\cos\lambda\sqrt{1+3\sin^2\lambda}\end{aligned}$$

we can derive the bounce integral as

$$\tau_b = 4\frac{r_{eq}}{v}\int_0^{\lambda_m}\cos\lambda\sqrt{1+3\sin^2\lambda}\left[1-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right]^{-1/2} d\lambda \tag{A.2}$$

A.2 Evaluation of the gradient curvature drift in a dipolar magnetosphere

We need to determine the integrand

$$\Delta\psi = 4\int_0^{\lambda_m} \frac{v_d}{r\cos\lambda}\frac{ds}{v_\parallel} \tag{A.3}$$

with

$$\mathbf{v}_d = \left(\frac{1}{2}v_\perp^2 + v_\parallel^2\right)\frac{\mathbf{B}\times(\nabla B)}{\omega_g B^2} \tag{A.4}$$

Relations used for the derivation include $r = r_{eq}\cos^2\lambda$ and $B_E = \mu_0 M_E/4\pi R_E^3$ and

$$\begin{aligned}
\nabla\Psi &= \frac{\partial\Psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Psi}{\partial\lambda}\mathbf{e}_\lambda + \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\varphi}\mathbf{e}_\varphi \\
B &= B_E\frac{R_E^3}{r^3}\left(1+3\sin^2\lambda\right)^{1/2} \\
B_r &= -2B_E R_E^3\frac{\sin\lambda}{r^3} \\
B_\lambda &= B_E R_E^3\frac{\cos\lambda}{r^3}
\end{aligned}$$

Evaluation of $\mathbf{B}\times\nabla B/B^3$:

$$\begin{aligned}
\nabla B &= B_E R_E^3\nabla\left[\frac{1}{r^3}\left(1+3\sin^2\lambda\right)^{1/2}\right] \\
&= -3B_E R_E^3\frac{1}{r^4}\left(1+3\sin^2\lambda\right)^{1/2}\mathbf{e}_r + 3B_E R_E^3\frac{\sin\lambda\cos\lambda}{r^4}\left(1+3\sin^2\lambda\right)^{-1/2}\mathbf{e}_\lambda \\
&= 3B_E R_E^3\frac{1}{r^4}\left(1+3\sin^2\lambda\right)^{-1/2}\left[-\left(1+3\sin^2\lambda\right)\mathbf{e}_r + \sin\lambda\cos\lambda\,\mathbf{e}_\lambda\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathbf{B}\times(\nabla B)}{B^3} &= \frac{3}{B^3}B_E^2R_E^6\frac{1}{r^7}\left(1+3\sin^2\lambda\right)^{-1/2}\left[-2\sin\lambda\cos\lambda\sin\lambda + \cos\lambda\left(1+3\sin^2\lambda\right)\right] \\
&= \frac{1}{B^3}3B_E^2R_E^6\frac{1}{r^7}\left(1+3\sin^2\lambda\right)^{-1/2}\cos\lambda\left[1+\sin^2\lambda\right] \\
&= \frac{3r^2}{B_E R_E^3}\cos\lambda\left[1+\sin^2\lambda\right]\left(1+3\sin^2\lambda\right)^{-2} \\
&= \frac{3L^2}{B_E R_E}\cos^5\lambda\left[1+\sin^2\lambda\right]\left(1+3\sin^2\lambda\right)^{-2}
\end{aligned}$$

Evaluation of $\frac{1}{2}v_\perp^2 + v_\parallel^2$

$$\begin{aligned}
\frac{1}{2}v_\perp^2 + v_\parallel^2 &= v^2\left(1-\sin^2\alpha+\frac{1}{2}\sin^2\alpha\right) \\
&= \frac{1}{2}v^2\left(2-\sin^2\alpha\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}v^2\left(2-(B/B_{eq})\sin^2\alpha_{eq}\right) \\
&= \frac{1}{2}v^2\left(2-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right)
\end{aligned}
$$

The combination of these yields:

$$
\begin{aligned}
\mathbf{v}_d &= \left(\frac{1}{2}v_\perp^2+v_\parallel^2\right)\frac{\mathbf{B}\times(\nabla B)}{\omega_g B^2} \\
&= \frac{3}{2}mv^2\frac{L^2}{qB_E R_E}\cos^5\lambda\frac{1+\sin^2\lambda}{\left(1+3\sin^2\lambda\right)^2}\left(2-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right) \qquad \text{(A.5)}
\end{aligned}
$$

From appendix 1

$$
\frac{ds}{v_\parallel}=\frac{r_{eq}}{v}\cos\lambda\sqrt{1+3\sin^2\lambda}\left[1-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right]^{-1/2}d\lambda
$$

and $r\cos\lambda = r_{eq}\cos^3\lambda$ and $B_{eq}=B_E/L^3$

Such that

$$
\begin{aligned}
\Delta\psi &= 4\int_0^{\lambda_m}\frac{v_d}{r\cos\lambda}\frac{ds}{v_\parallel} \\
&= \frac{4}{v}\frac{3}{2}mv^2\frac{1}{qB_{eq}r_{eq}}\int_0^{\lambda_m}\frac{\cos^3\lambda\left(1+\sin^2\lambda\right)}{\left(1+3\sin^2\lambda\right)^{3/2}}\frac{\left(2-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right)}{\left[1-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right]^{1/2}}d\lambda \qquad \text{(A.6)}
\end{aligned}
$$

To determine the average angular drift velocity

$$
\langle\omega_d\rangle=\frac{\Delta\psi}{2\pi\tau_b}
$$

with $\tau_b = 4\frac{r_{eq}}{v}I_1$ and the definitions $W=mv^2/2$ and

$$
I_1 = \int_0^{\lambda_m}\frac{\cos\lambda\sqrt{1+3\sin^2\lambda}}{\left[1-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right]^{1/2}}d\lambda \qquad \text{(A.7)}
$$

$$
I_2 = \int_0^{\lambda_m}\frac{\cos^3\lambda\left(1+\sin^2\lambda\right)}{\left(1+3\sin^2\lambda\right)^{3/2}}\frac{\left(2-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right)}{\left[1-\sin^2\alpha_{eq}\frac{\sqrt{1+3\sin^2\lambda}}{\cos^6\lambda}\right]^{1/2}}d\lambda \qquad \text{(A.8)}
$$

the angular drift velocity becomes

$$
\langle\omega_d\rangle=\frac{3W}{2\pi qB_{eq}r_{eq}^2}\frac{I_2}{I_1} \qquad \text{(A.9)}
$$

A.3 Definition of the radius of curvature

Assume: $x(t)$, $y(t)$, and $z(t)$

the line element along the curve $r(t) = (x(t), y(t), z(t))$ is

$$ds = dt\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} \tag{A.10}$$

Arc length:

$$s = \int_{t_0}^{t} ds$$

Tangent:

$$\mathbf{e}_t = \frac{d\mathbf{r}}{ds} \tag{A.11}$$

The unit vector $\mathbf{e}_n$ defined by

$$\mathbf{e}_n = \frac{d^2\mathbf{r}}{ds^2}\frac{1}{r_c} \tag{A.12}$$

is the principal normal with the radius of curvature defined by

$$r_c = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} \tag{A.13}$$

Exercise: Use the field line equation for the dipole field to compute the radius of curvature from $x = r(\theta)\sin\theta$ and $x = r(\theta)\cos\theta$ as a function of θ.

A.4 Kinetic Waves in a Magnetized Plasma

Derivation of the general dispersion relation

Ampere's law and the induction equation

$$\begin{aligned}\nabla \times \mathbf{B} &= \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{j} \\ \nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t}\end{aligned}$$

yield

$$\nabla^2\mathbf{E} - \nabla\left(\nabla \cdot \mathbf{E}\right) - \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial \mathbf{j}}{\partial t}$$

Using a plane wave ansatz $\delta\mathbf{E}(\omega,\mathbf{k}) = \delta\mathbf{E}_0(\omega,\mathbf{k})\exp(i\mathbf{k}\mathbf{x} - i\omega t)$ yields

$$\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\underline{\underline{1}} - \mathbf{k}\mathbf{k}\right]\delta\mathbf{E}_0(\omega,\mathbf{k}) = i\omega\mu_0 j_0(\omega,\mathbf{k})$$

With Ohm's law $j(\omega,\mathbf{k}) = \sigma(\omega,\mathbf{k})\delta\mathbf{E}_0(\omega,\mathbf{k})$ the general dispersion relation given by

$$D(\omega,\mathbf{k}) = \det\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\underline{\underline{1}} - \mathbf{k}\mathbf{k} - i\omega\mu_0\sigma(\omega,\mathbf{k})\right] = 0$$

With the definition of the dielectric tensor

$$\underline{\underline{\epsilon}}(\omega,\mathbf{k}) = \underline{\underline{1}} + \frac{i}{\omega\epsilon_0}\sigma(\omega,\mathbf{k})$$

the general dispersion relation can be re-written a

$$\det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{k}\mathbf{k}}{k^2} - \underline{\underline{1}}\right) + \underline{\underline{\epsilon}}(\omega,\mathbf{k})\right] = 0$$

Magnetized Collisionless Plasma Waves

Here we assume that the theory behind kinetic waves in an unmagnetized plasma is known. In a uniform medium the linearized Vlasov equations are

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{q}{m}\mathbf{v}\times\mathbf{B}_0\cdot\frac{\partial}{\partial\mathbf{v}}\right)\delta f(\mathbf{v},\mathbf{x},t) &= -\frac{q}{m}\left(\delta\mathbf{E} + \mathbf{v}\times\delta\mathbf{B}\right)\cdot\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}} \\
\delta\mathbf{j} &= \sum_s q_s\int d^3v\mathbf{v}\delta f_s \\
\delta\rho_e &= \sum_s q_s\int d^3v\delta f_s
\end{aligned}$$

where the rest frame electric field is 0, $\mathbf{B}_0 = B_0\mathbf{e}_z$, and we have omitted the species index s in the collisionless Boltzmann for convenience. The lhs of the linearized Boltzmann equation represents the total time derivative along the six-dimensional path $(\mathbf{x}(t),\mathbf{v}(t))$ such that

$$\frac{d\delta f\left[\mathbf{v}(t),\mathbf{x}(t),t\right]}{dt} = -\frac{q}{m}\left(\delta\mathbf{E}\left[\mathbf{x}(t),t\right] + \mathbf{v}\times\delta\mathbf{B}\left[\mathbf{x}(t),t\right]\right)\cdot\frac{\partial f_0\left[\mathbf{v}(t)\right]}{\partial\mathbf{v}(t)}$$

which can formally be integrated in time

$$\delta f\left[\mathbf{v}(t),\mathbf{x}(t),t\right] = -\frac{q}{m}\int_{-\infty}^{t}dt'\left\{\delta\mathbf{E}\left[\mathbf{x}(t),t\right] + \mathbf{v}\times\delta\mathbf{B}\left[\mathbf{x}(t),t\right]\right\}\cdot\frac{\partial f_0\left[\mathbf{v}(t)\right]}{\partial\mathbf{v}(t)}$$

This integral requires the knowledge of $(\mathbf{x}(t),\mathbf{v}(t))$ for all particles for $t' < t$. For linear perturbations the particle trajectories are usually assumed to be determined by the equilibrium electric and magnetic fields such that

$$\begin{aligned}
v_x(t'-t) &= v_\perp\cos\left[\omega_g\left(t'-t\right)+\psi\right] & x(t'-t) - x &= v_\perp/\omega_g\left\{\sin\left[\omega_g\left(t'-t\right)+\psi\right] - \sin\left[\psi\right]\right\} \\
v_y(t'-t) &= v_\perp\sin\left[\omega_g\left(t'-t\right)+\psi\right] & y(t'-t) - y &= -v_\perp/\omega_g\left\{\cos\left[\omega_g\left(t'-t\right)+\psi\right] - \cos\left[\psi\right]\right\} \\
v_z(t'-t) &= v_\parallel & z(t'-t)z &= -v_\parallel(t'-t)
\end{aligned}$$

With the transformation $\tau = t' - t$ the integration becomes

$$\delta f(\mathbf{v}) = -\frac{q}{m}\int_{-\infty}^{0} d\tau \left\{\delta\mathbf{E}(\tau) + \mathbf{v}\times\delta\mathbf{B}(\tau)\right\}\cdot\frac{\partial f_0\left[\mathbf{v}(\tau)\right]}{\partial\mathbf{v}(\tau)}$$

Introducing a plane wave ansatz $\exp\left[i\left(\mathbf{k}\cdot(\mathbf{x}(\tau)-\mathbf{x}) - \omega\tau\right)\right]$ and using the induction equation $\mathbf{k}\times\delta\mathbf{E} = \omega\delta\mathbf{B}$ the perturbed distribution function becomes

$$\delta f(\mathbf{v}) = -\frac{q\delta\mathbf{E}(\mathbf{k},\omega)}{m\omega}\cdot\int_{-\infty}^{0} d\tau \exp\left[i\left(\mathbf{k}\cdot(\mathbf{x}(\tau)-\mathbf{x}) - \omega\tau\right)\right]\Big\{\mathbf{v}\mathbf{k} + \underline{\underline{1}}\left(\omega - \mathbf{k}\cdot\mathbf{v}(\tau)\right)\Big\}\cdot\frac{\partial f_0\left[\mathbf{v}(\tau)\right]}{\partial\mathbf{v}(\tau)} \tag{A.14}$$

Using $\mathbf{k} = k_\perp\mathbf{e}_x + k_\parallel\mathbf{e}_z$ and the particle orbits the phase in the exponential can be written as

$$\begin{aligned}\varphi(\tau) &= \left(\mathbf{k}\cdot(\mathbf{x}(\tau)-\mathbf{x}) - \omega\tau\right)\\ &= \Big(k_\parallel v_\parallel - \omega\Big)\tau + \xi\left[\sin\left(\omega_g\tau+\psi\right) - \sin\left(\psi\right)\right]\end{aligned}$$

with $\xi = k_\perp v_\perp/\omega_g$. The operator $\nabla_\mathbf{v} f_0$ can be expressed as

$$\begin{aligned}\nabla_\mathbf{v} f_0 &= 2\left(\mathbf{v} - v_z\mathbf{e}_z\right)\frac{\partial f_0}{\partial v_\perp^2} + 2v_z\frac{\partial f_0}{\partial v_z^2}\mathbf{e}_z\\ &= \left[2v_\perp\frac{\partial f_0}{\partial v_\perp^2}\cos\left(\omega_g\tau+\psi\right), 2v_\perp\frac{\partial f_0}{\partial v_\perp^2}\sin\left(\omega_g\tau+\psi\right), 2v_z\frac{\partial f_0}{\partial v_z^2}\right]\end{aligned}$$

Since $v_\perp^2$and v_z are constants of motion we can integrate $\partial f_0/\partial v_\perp^2$ and $\partial f_0/\partial v_z^2$ can be removed from the integral. The sine and cosine functions in the exponent can be integrated with the aid of

$$\exp\left[ix\sin\phi\right] = \textstyle\sum_{l=-\infty}^{\infty} J_l(x)\exp\left[il\phi\right]$$

$$\begin{aligned}\exp\left[i\xi\sin\left(\omega_g\tau+\psi\right)\right] &= \sum_{l=-\infty}^{\infty} J_l(\xi)\exp\left[il\left(\omega_g\tau+\psi\right)\right]\\ \exp\left[-i\xi\sin\psi\right] &= \sum_{l'=-\infty}^{\infty} J_{l'}(\xi)\exp\left[-il'\psi\right]\\ \exp\left[i\xi\sin\left(\omega_g\tau+\psi\right) - i\xi\sin\psi\right] &= \sum_{l=-\infty}^{\infty} J_l J_{l'}\exp\left[il\left(\omega_g\tau+\psi\right)\right]\end{aligned}$$

Note also that equation (A.14) can be further simplified by introducing the derivative

$$\left(\omega - \mathbf{k}\cdot\mathbf{v}(\tau)\right)\exp\left[i\varphi\left(\tau\right)\right] = -i\frac{d}{d\tau}\exp\left[i\varphi\left(\tau\right)\right]$$

Now all integrals over time are of the form

$$\int_{-\infty}^{0} d\tau\left(v_x(\tau), v_y(\tau), 1\right)\exp\Big[i\Big(k_\parallel v_\parallel - \omega\Big)\tau + i\xi\left[\sin\left(\omega_g\tau+\psi\right) - \sin\left(\psi\right)\right]\Big]$$

With

$$
\begin{aligned}
\delta\mathbf{j} &= \sum_s q_s \int d^3v\, \delta f(\mathbf{v}) \\
&= \sum_s \frac{\epsilon_0 \omega_{ps}^2}{n_0 \omega} \int_0^\infty \int_{-\infty}^{\infty} \int_0^{2\pi} v_\perp dv_\perp dv_\parallel d\psi \delta\mathbf{E}(\mathbf{k},\omega) \cdot \\
& \int_{-\infty}^{0} d\tau \exp\left[i\left(\mathbf{k}\cdot(\mathbf{x}(\tau)-\mathbf{x}) - \omega\tau\right)\right] \left\{\mathbf{v}\mathbf{k} + \underline{\underline{1}}\left(\omega - \mathbf{k}\cdot\mathbf{v}(\tau)\right)\right\} \left[2\left(\mathbf{v} - v_z\mathbf{e}_z\right)\frac{\partial f_0}{\partial v_\perp^2} + 2v_z \frac{\partial f_0}{\partial v_z^2}\mathbf{e}_z\right]
\end{aligned}
$$

and noting that $\mathbf{j}(\omega,\mathbf{k}) = \underline{\underline{\sigma}}(\omega,\mathbf{k}) \cdot \delta\mathbf{E}_0(\omega,\mathbf{k})$ with $\underline{\underline{\epsilon}}(\omega,\mathbf{k}) = \underline{\underline{1}} + \frac{i}{\omega\epsilon_0}\underline{\underline{\sigma}}(\omega,\mathbf{k})$ we can derive the dielectric tensor as

$$
\begin{aligned}
\underline{\underline{\epsilon}}(\omega,\mathbf{k}) &= \left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right)\underline{\underline{1}} - \sum_s \sum_{l=-\infty}^{\infty} \frac{\omega_{ps}^2}{n_{0s}\omega^2} \\
& \int_0^\infty \int_{-\infty}^{\infty} v_\perp dv_\perp dv_\parallel \int_{-\infty}^{0} d\tau \exp\left(i\varphi\left(\tau\right)\right) \left\{i\mathbf{v}\left(\tau\right)\mathbf{k} - \underline{\underline{1}}\frac{d}{d\tau}\right\} \cdot \frac{\partial f_0\left[\mathbf{v}(\tau)\right]}{\partial \mathbf{v}(\tau)}
\end{aligned}
$$

Integrating over time using the above relations and properties yields the dielectric tensor for a magnetized plasma as

$$
\begin{aligned}
\underline{\underline{\epsilon}}(\omega,\mathbf{k}) &= \left(1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\right)\underline{\underline{1}} - \sum_s \sum_{l=-\infty}^{\infty} \frac{2\pi\omega_{ps}^2}{n_{0s}\omega^2} \\
& \int_0^\infty \int_{-\infty}^{\infty} v_\perp dv_\perp dv_\parallel \left(k_\parallel \frac{\partial f_0}{\partial v_\parallel} + \frac{l\omega_{gs}}{v_\perp}\frac{\partial f_0}{\partial v_\perp}\right) \frac{\mathbf{S}_{ls}\left(v_\parallel, v_\perp\right)}{k_\parallel v_\parallel - l\omega_{gs} - \omega}
\end{aligned}
$$

with the tensor $\mathbf{S}_{ls}$ of the form

$$
\mathbf{S}_{ls}\left(v_\parallel, v_\perp\right) = \begin{bmatrix} \frac{l^2\omega_{gs}^2}{k_\perp^2} J_l^2 & \frac{ilv_\perp\omega_{gs}}{k_\perp} J_l \dot{J}_l & \frac{lv_\parallel\omega_{gs}}{k_\perp} J_l^2 \\ -\frac{ilv_\perp\omega_{gs}}{k_\perp} J_l \dot{J}_l & v_\perp^2 \dot{J}_l^2 & -iv_\parallel v_\perp J_l \dot{J}_l \\ \frac{lv_\parallel\omega_{gs}}{k_\perp} J_l^2 & iv_\parallel v_\perp J_l \dot{J}_l & v_\parallel^2 J_l^2 \end{bmatrix}
$$

with the Bessel functions J_l, and $\dot{J}_l = dJ_l/d\xi$ with the argument $\xi_s = k_\perp v_\perp/\omega_{gs}$.

This is the most general expression for the dielectric tensor in a plasma with a uniform magnetic field.

Simplifications and special cases:

- The purely electrostatic dispersion relation is obtained by taking the dot product of $\underline{\underline{\epsilon}}(\omega\,\mathbf{k})$ with $\mathbf{k}$ from both sides.
- The condition $\omega - k_\parallel v_\parallel - l\omega_{gs} = 0$ defines particle resonances. In the case of an unmagnetized plasma they reduce to the Landau resonance $\omega = k_\parallel v_\parallel$.
- Expansion for cold plasma $\xi_s = k_\perp v_\perp/\omega_{gs}$-> 0.

Weak Damping and Weak Instability

We have in the previous parts assumed wave fields of the form

$$\delta A = \sum_k A_k \exp\left(i\mathbf{k}\cdot\mathbf{x} - i\omega t\right)$$

The amplitude of the waves will decrease in time if ω has an imaginary part which is negative $\omega = \omega_r + i\gamma$ with $\gamma < 0$. Weak damping assumes $|\gamma(\omega_r, \mathbf{k})| \ll \omega_r(\mathbf{k})$. In other words the damping proceeds slowly compared to the real wave period ω_r. In this case we can expand the dispersion relation

$$D\left(\omega, \gamma, \mathbf{k}\right) = D_r\left(\omega, \gamma, \mathbf{k}\right) + iD_i\left(\omega, \gamma, \mathbf{k}\right) = 0$$

around the real frequency $\omega = \omega_r$

$$D\left(\omega_r, \gamma, \mathbf{k}\right) = D_r\left(\omega_r, 0, \mathbf{k}\right) + i\gamma\left.\frac{\partial D\left(\omega_r, \gamma, \mathbf{k}\right)}{\partial \omega_r}\right|_{\gamma=0} + iD_i\left(\omega_r, \gamma, \mathbf{k}\right) = 0$$

with the solutions

$$\begin{aligned} D_r\left(\omega_r, 0, \mathbf{k}\right) &= 0 \\ \gamma(\omega_r, \mathbf{k}) &= \frac{D_i\left(\omega_r, \gamma, \mathbf{k}\right)}{\partial D\left(\omega_r, \gamma, \mathbf{k}\right)/\partial \omega_r|_{\gamma=0}} \end{aligned}$$

For instability one can follow the same line of arguments. Note that the exponential growth of the wave for $\gamma > 0$ implies that the amplitude of the wave assumes eventually that of the equilibrium plasma in which case the approach to linearize the equations is not anymore applicable. For instabilities with $\gamma(\omega_r, \mathbf{k}) \ll \omega_r(\mathbf{k})$ we can use the same expansion as in the case of weak damping and obtain the same solution for the real and imaginary parts of ω.

A.5 Collisionless Waves in an Anisotropic Plasma

With the definition of the dielectric tensor

$$\underline{\underline{\epsilon}}(\omega\,\mathbf{k}) = \underline{\underline{1}} + \frac{i}{\omega\epsilon_0}\sigma(\omega\,\mathbf{k})$$

the general dispersion relation is

$$\det\left[\frac{k^2c^2}{\omega^2}\left(\frac{\mathbf{kk}}{k^2} - \underline{\underline{1}}\right) + \underline{\underline{\epsilon}}(\omega\,\mathbf{k})\right] = 0$$

For an anisotropic (gyrotropic) distribution function the dielectric tensor is becomes

$$\underline{\underline{\epsilon}}(\omega\,\mathbf{k}) = \underline{\underline{1}} + \sum_s \begin{pmatrix} \epsilon_{s1} & \epsilon_{s2} & \epsilon_{s4} \\ -\epsilon_{s2} & \epsilon_{s1} - \epsilon_{s0} & \epsilon_{s5} \\ \epsilon_{s4} & \epsilon_{s5} & \epsilon_{s3} \end{pmatrix}$$

with

$$\begin{aligned}
\epsilon_{s0} &= \frac{2\omega_{ps}^2}{\omega k_\parallel v_{ths\parallel}} \sum_l \eta_s \Lambda_l'(\eta_s) \left[Z(\zeta_{s,l}) - \frac{k_\parallel v_{ths\parallel}}{2\omega} A_s Z'(\zeta_{s,l})\right] \\
\epsilon_{s1} &= \frac{\omega_{ps}^2}{\omega k_\parallel v_{ths\parallel}} \sum_l \frac{l^2 \Lambda_l(\eta_s)}{\eta_s} \left[Z(\zeta_{s,l}) - \frac{k_\parallel v_{ths\parallel}}{2\omega} A_s Z'(\zeta_{s,l})\right] \\
\epsilon_{s2} &= \frac{i\mathrm{sign}(q_s)\omega_{ps}^2}{\omega k_\parallel v_{ths\parallel}} \sum_l l \Lambda_l'(\eta_s) \left[Z(\zeta_{s,l}) - \frac{k_\parallel v_{ths\parallel}}{2\omega} A_s Z'(\zeta_{s,l})\right] \\
\epsilon_{s3} &= -\frac{\omega_{ps}^2}{k_\parallel^2 v_{ths\parallel}^2} \sum_l \left(1 - \frac{A_s}{A_s+1}\frac{l\omega_{gs}}{\omega}\right)\left(1 + \frac{l\omega_{gs}}{\omega}\right) \Lambda_l(\eta_s) Z'(\zeta_{s,l}) \\
\epsilon_{s4} &= \frac{k_\perp}{2k_\parallel}\frac{\omega_{ps}^2}{\omega\omega_{gs}} \sum_l \left(A_s + 1 - \frac{l\omega_{gs}}{\omega}A_s\right)\left(1 + \frac{l\omega_{gs}}{\omega}\right) \frac{l\Lambda_l(\eta_s)}{\eta_s} Z'(\zeta_{s,l}) \\
\epsilon_{s5} &= \frac{i\mathrm{sign}(q_s)}{k_\perp k_\parallel}\frac{\omega_{ps}^2}{2\omega\omega_{gs}} \sum_l \left(A_s + 1 - \frac{l\omega_{gs}}{\omega}A_s\right) \Lambda_l'(\eta_s) Z'(\zeta_{s,l})
\end{aligned}$$

where the sum is take from $l = -\infty$ to $l = \infty$ and

$$\begin{aligned}
A_s &= \frac{T_{s\perp}}{T_{s\parallel}} - 1 \\
\zeta_{s,l} &= \frac{\omega - l\omega_{gs}}{k_\parallel v_{ths\parallel}} \\
\Lambda_l(\eta_s) &= I(\eta_s)\exp(-\eta_s) \\
\eta_s &= \frac{k_\perp^2 v_{ths\perp}^2}{2\omega_{gs}^2} = \frac{k_\perp^2 T_{s\perp}}{m_s\omega_{gs}^2} = \frac{k_\perp^2 r_{gs}^2}{2}
\end{aligned}$$

with the modified Bessel function $I(\eta_s)$ and the plasma dispersion function

$$Z(\zeta) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{\exp\left(-z^2\right) dz}{z - \zeta}$$

A.6 Plasma Dispersion Function

$$Z(\zeta) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{\exp\left(-z^2\right) dz}{z - \zeta}$$

Differential equation:

$$\frac{dZ(\zeta)}{d\zeta} = -2\left[1 + \zeta Z(\zeta)\right]$$

Useful relations

$$\begin{aligned} Z(-\zeta) &= -Z(\zeta) + 2\pi^{1/2} i \exp\left(-\zeta^2\right) \quad \text{for Im}\zeta > 0 \\ \widetilde{Z}(\zeta) &= Z(\zeta) - 2\pi^{1/2} \exp\left(-\zeta^2\right) \quad \text{for Im}\zeta < 0 \end{aligned}$$

where $\widetilde{Z}(\zeta)$ is the analytic continuation of $Z(\zeta)$. The complex conjugate of the plasma dispersion function is

$$\left[Z(\zeta)\right]^* = Z(\zeta^*) - 2\pi^{1/2} \exp\left(-\zeta^2\right)$$

Expansion for $\zeta < 1$:

$$\begin{aligned} Z(\zeta) &= i\pi^{1/2} \exp\left(-\zeta^2\right) - 2\zeta\left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - ...\right) \\ &= \pi^{1/2} \sum_{n=0}^{\infty} \frac{i^{n+1}\zeta^n}{\Gamma(1 + n/2)} \end{aligned}$$

Expansion for $\zeta \gg 1$:

$$Z(\zeta) = -\frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + ...\right) + \sigma\pi^{1/2} i \exp\left(-\zeta^2\right)$$

with

$$\sigma = \begin{cases} 0, & \text{for Im}\zeta < 0 \\ 1, & \text{for Im}\zeta = 0 \\ 2, & \text{for Im}\zeta > 0 \end{cases}$$

The plasma dispersion function is related to the error function:

$$\begin{aligned} \text{erf}(z) &= \left(1 + \frac{2i}{\pi^{1/2}} \int_0^z e^{t^2} dt\right) e^{-z^2} \\ Z(\zeta) &= i\pi^{1/2} \text{erf}(\zeta) \end{aligned}$$

A.7 Bessel Functions and Modified Bessel Functions

Bessel's differential equation

$$x^2y'' + xy' + \left(x^2 - n^2\right) y = 0$$

Solutions are the Bessel functions of the first kind

$$J_n(x) = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!\Gamma\left(n+v+1\right)} \left(\frac{x}{2}\right)^{n+2v}$$

Modified Bessel functions of the first kind

$$I_n(x) = i^{-n} J_n(x) = \sum_{v=0}^{\infty} \frac{1}{v!\Gamma\left(n+v+1\right)} \left(\frac{x}{2}\right)^{n+2v}$$

are the solution of the differential equation

$$x^2y'' + xy' - \left(x^2 - n^2\right) y = 0$$

Recurrence relations

$$\begin{aligned} \frac{2n}{x} J_n(x) &= J_{n-1}(x) + J_{n+1}(x) \\ \frac{d}{dx}\left(x^n J_n(x)\right) &= x^n J_{n-1}(x) \\ \frac{d}{dx}\left(x^{-n} J_n(x)\right) &= -x^{-n} J_{n+1}(x) \end{aligned}$$

and for the modified Bessel functions

$$\frac{2n}{x} I_n(x) = I_{n-1}(x) - I_{n+1}(x)$$

Asymptotic expansion for $x \gg 1$

$$\begin{aligned} J_n(x) &= \left(\frac{2}{\pi x}\right)^{1/2} \left[\cos\left(x - \pi n/2 - \pi/4\right)\right] + O(1/x) \\ I_n(x) &= \frac{\exp x}{(2\pi x)^{1/2}} \left[1 + O(1/x)\right] \end{aligned}$$