Chapter 2

Basic Plasma Properties

2.1 First Principles

2.1.1 Maxwell's Equations

In general magnetic and electric fields are determined by Maxwell's equations, corresponding boundary conditions and the source (charges and currents) distributions.

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_c \tag{2.1}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \tag{2.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.4}$$

where $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields.

$c = 3 \cdot 10^8 ms^{-1}$, $\epsilon_0 = 8.85 \cdot 10^{-12} Fm^{-1}$, and $\mu_0 = 4\pi \cdot 10^{-7} Hm^{-1}$. Sometimes it is convenient to express the electromagnetic fields in terms of an electric potential Φ and a vector potential $\mathbf{A}$ such that

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\nabla\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t) \end{aligned}$$

which requires to solve the electromagnetic field equations for the potentials for instance in the form

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{1}{\epsilon_0}\rho_c(\mathbf{r}, t) \tag{2.5}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}(\mathbf{r}, t) \tag{2.6}$$

where Φ and $\mathbf{A}$ satisfy the Lorentz gauge $\partial\Phi/\partial t + c^2 \nabla \cdot \mathbf{A} = 0$.

2.1.2 Lorentz Equations of Motion

In electromagnetic fields the motion of charged particles is determined by the fields through the equations of motion

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i \tag{2.7}$$

$$\frac{d\mathbf{v}_i}{dt} = \frac{q_i}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\mathbf{r}_i,t}. \tag{2.8}$$

If the forces are external, the corresponding equations of motion are complete and the particle motion is often called test particle motion. However, if the other particles contribute to the force on a particular particle the forces have to be evaluated. The electromagnetic forces in a plasma depends on the charge and current densities which are determined by the collective particle interaction:

$$\rho_{cs} = q_s n_s = q_s \int_{-\infty}^{\infty} d^3v f_s(\mathbf{x}, \mathbf{v}, t)$$

$$\mathbf{j}_s = q_s n_s \mathbf{u}_s = q_s \int_{-\infty}^{\infty} d^3v \mathbf{v} f_s(\mathbf{x}, \mathbf{v}, t)$$

with $\rho_c = \sum_s \rho_s$ and $\mathbf{j} = \sum_s \mathbf{j}_s$.

In a plasma the number of particles in a physical system is usually rather large. In addition the overwhelming majority of problems deal with the collective particle behavior rather than the individual one. Discrete particle dynamics can be important in some areas of plasma physics for sufficiently small ('microscopic') length or time scales. The collective behavior is usually well described in a fluid approximation.

2.1.3 Plasma Properties and Parameters

A plasma is a gas of charged particles, which consists of free positive and negative charge carriers. To allow free motion of the particles the typical potential energy must be much smaller then the typical kinetic energy

$$\langle e\phi \rangle \ll \left\langle \frac{m}{2} v^2 \right\rangle = k_B T$$

- More than 99% of all know matter in the universe is in the plasma state.
- Magnetosphere provides an ideal laboratory to examine complex processes in a natural plasma by in situ observations.

Debye Shielding: Coulomb potential of a charge q:

$$\phi = q/(4\pi\epsilon_0 r). \tag{2.9}$$

Test charge shielded by other charges:

$$\phi_D = \frac{q}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right) \tag{2.10}$$

with the Debye length defined by

$$\lambda_D^{-2} = \frac{n_0 e^2}{\epsilon_0 k_B} \left(\frac{1}{T_e} + \frac{1}{T_i}\right) \tag{2.11}$$

or $\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_e e^2}\right)^{1/2}$ for temperatures $T_e \simeq T_i$, number density n_e, Boltzmann constant $k_B = 1.38 \cdot 10^{-23} J K^{-1}$ and electron charge $e = 1.6 \cdot 10^{-19} C$. This potential is sometimes called the Yukawa potential.

Exercise: Show that the Yukawa potential satisfies the Poisson equation

$\nabla^2 \Phi = -\frac{1}{\epsilon_0} q_t \delta(\mathbf{r}) + \frac{1}{\lambda_D^2} \Phi(\mathbf{r})$

Exercise: Derive the Poisson equation for the case of a test charge $q = e$ in a plasma (Hint: First demonstrate that the density of species s is $n_s = n_0 \exp(-q\Phi/k_B T_s)$; see section 2.2; then expand the exponential under the assumption $q\Phi \ll k_B T_s$).

Exercise: Compute the net charge of the shielding cloud.

Quasi-neutrality for any physical length $L \gg \lambda_D$ otherwise binary interaction should be considered.

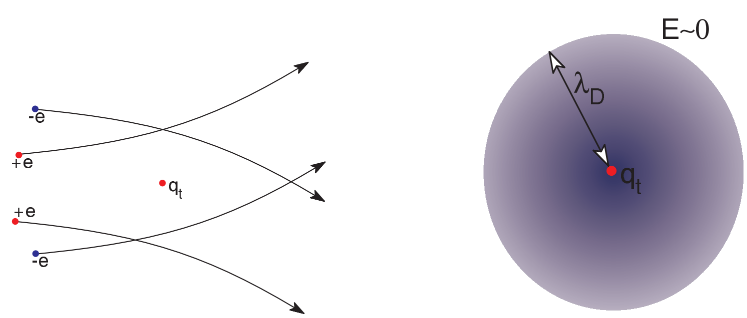

Figure 2.1: Sketch of the particle trajectories and of the Debye shielding charge distribution in the vicinity of a test charge.

Plasma Parameter: The number of of particles in a Debye sphere is $N = \frac{4\pi}{3} n \lambda_D^3$. The condition that the potential energy is much smaller than the kinetic energy yields

$$\Lambda = n\lambda_D^3 \gg 1 \tag{2.12}$$

with Λ being the so-called plasma parameter.

Exercise: Calculate the electron thermal speed, Debye length, and the plasma parameter for
(a) the tail magnetosphere with $T_e = 10^7$ K, $n_0 = 10^6$ m^{-3}
(b) the ionosphere with $T_e = 10^3$ K, $n_0 = 10^{12}$ m^{-3}

Exercise: Can a plasma be maintained at temperatures of $T_e = 100$ K (Hint: Calculate the density limit using the plasma parameter and explain your result).

Exercise: $\Lambda \propto n_0^{-1/2} T^{3/2}$ While the dependence on temperature seems intuitively clear the density dependence appears odd because lower densities mean less particles and less shielding. Why does the plasma parameter improve (increase) with decreasing density?

Plasma Frequency: Typical oscillations in a plasma are electron plasma oscillations with the frequency

$$\omega_{ps} = \left(\frac{n_e q_s^2}{m_s \epsilon_0} \right)^{1/2} \tag{2.13}$$

Collisions: In partially ionized plasma the collision time between electrons and neutrals must be much larger then the inverse electron plasma frequency

$$\omega_{pe} \tau_n \gg 1.$$

The collision frequency in a plasma is given by

$$\nu_c = \sqrt{\frac{\pi}{2}} \frac{1}{32\pi} \frac{\omega_{pe}}{\Lambda} \ln [12\pi\Lambda]$$

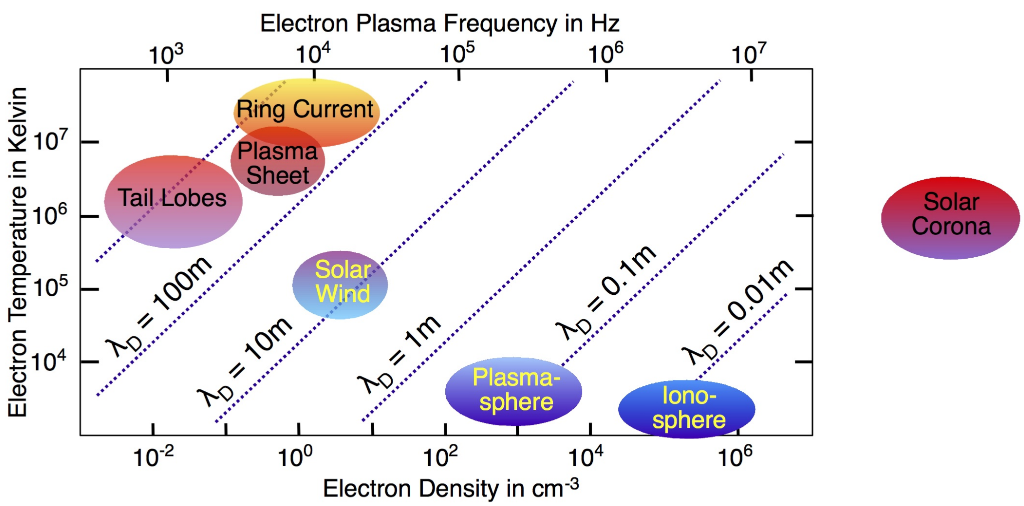

Figure 2.2: Magnetospheric plasma parameters.

2.2 Kinetic Equations

To describe a plasma one can solve the coupled system of *Maxwell's equations* and the particle *equations of motion*. However, there are more efficient methods to solve the plasma dynamics using the above approximations.

The first step toward a fluid description is the introduction of a phase fluid with a phase space consisting of the set of ordinary coordinates and velocities $(\mathbf{r}, \mathbf{u})$. The entire ensemble of n particles has $3n$ spatial and $3n$ velocity degrees of freedom. By integrating the corresponding *Liouville equation* (see statistical mechanics) over $3(n-1)$ spatial coordinate and $3(n-1)$ coordinates one obtains the *Boltzmann equation* for the so-called single particle distribution function $f(\mathbf{r}, \mathbf{u}, t)$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f \quad = \quad \left.\frac{\partial f}{\partial t}\right|_{coll} \tag{2.14}$$

In the case of thermal equilibrium f assumes locally a Maxwell distribution in velocity space and the collision term on the rhs of 2.14 vanishes. Equation 2.14is clearly a fluid (advection) equation though in a 6 dimensional space. The lhs can be interpreted as the total derivative of $f(\mathbf{r}, \mathbf{u}, t)$ along a trajectory given by the 6 dimensional velocity $\mathbf{v}^{(6)} = (\mathbf{v}, \frac{\mathbf{F}}{m})$ where $\mathbf{F}$ is the Lorentz force.

In the magnetosphere collisions can be neglected almost everywhere except for the ionosphere (and on rather long time scale in the plasma sphere). In the absence of collisions the total derivative along the path determined by $\mathbf{v}^{(6)}$ is

$$\frac{df}{dt} = 0$$

in analogy of the advection equation $\partial f/\partial t + \mathbf{v} \cdot \nabla f = df/dt$. This implies that the value of f is conserved along this trajectory in the six-dimensional phase space. This particularly means that any maximum of the distribution remains exactly the same. If in addition the force is conservative, the 6 dimensional flow is incompressible, that means that the phase space volume of a contour with any constant value of f is conserved during the evolution as illustrated in Figure 2.3.

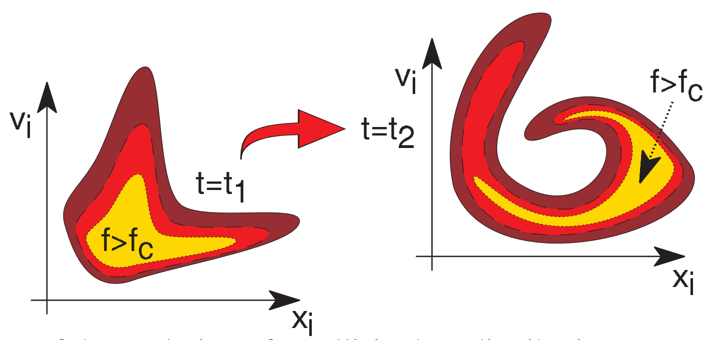

Figure 2.3: Illustration of the evolution of a collisionless distribution conserving the area inside contours of constant f.

The *Vlasov equations* consist of the *collisionless Boltzmann equation* 2.14 complemented by *Maxwell's equations*. The collision term on the rhs can consider many different physical or chemical processes. Chemical reactions, ionization or recombination, friction, diffusion, and energy exchange collisions are contained in the collision term. Details depend on the corresponding processes.

The description with the single particle distribution function 2.14 and the actual equations of motion 2.7 and 2.8 are in many respects equivalent. In mathematical terms the Lorentz equations of motion represent the characteristics of the collisionless Boltzmann equation. Each description has its particular advantages and disadvantages.

In the presence of constants of motion (steady state $\partial/\partial t = 0$, or ignorable coordinate directions, e.g., $\partial/\partial y = 0$) any function of the constants of motion $F_s(H_s, P_{sy})$ for species s with the Hamiltonian

$H_s = m_s v^2/2 + q_s\Phi$ and the generalized momentum $P_{sy} = m_s v_y + q_s A_y$ solves the collisionless Boltzmann equation. However, it still remains to solve Maxwell's equation which will be demonstrated in section 7. The solution to plasma waves can be found by linearizing the Vlasov equations.

2.3 Derivation of the Fluid Plasma Equations

Fluid equations are probably the most widely used equations for the description of inhomogeneous plasmas. While the phase fluid which is governed by the Boltzmann equation represents a first example, many applications do not require the precise velocity distribution at any point in space. Ordinary fluid equations for gases and plasmas can be obtained from the Boltzmann equation or can be derived using properties like the conservation of mass, momentum, and energy of the fluid.

2.3.1 Definitions

The equations of ordinary fluids and gases as well as those for magnetofluids (plasmas) can be obtained from equation 2.14 in a systematic manner. Defining the 0th, 1st, and 2nd moment of the integral over the distribution function f as mass density ρ, fluid bulk velocity $\mathbf{u}$, and pressure tensor $\underline{\Pi}$

$$n(\mathbf{x},t) = \int_\infty^\infty d^3v f(\mathbf{x},\mathbf{v},t) \tag{2.15}$$

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{n}\int_\infty^\infty d^3v \mathbf{v} f(\mathbf{x},\mathbf{v},t) \tag{2.16}$$

$$\underline{\Pi}(\mathbf{x},t) = m\int_\infty^\infty d^3v (\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) f(\mathbf{x},\mathbf{v},t). \tag{2.17}$$

Note that all quantities should carry an index for the particle species which is omitted here for convenience. With these definitions one also obtains number density $\rho(\mathbf{x},t) = mn$, charge density $\rho_c(\mathbf{x},t) = qn$, momentum density $\mathbf{p}(\mathbf{x},t) = \rho\mathbf{u}$, current density $\mathbf{j}(\mathbf{x},t) = q\mathbf{u}$, and scalar pressure (the isotropic portion of the pressure) $p(\mathbf{x},t) = \frac{1}{3}Tr(\underline{\Pi})$ where the individual particle mass m and charge q are used. The fluid equations are then determined by the moments of the Boltzmann equation, i.e.,

$$\int_\infty^\infty d^3v\, \mathbf{v^i} (Boltzmann\, Equ.)$$

To account for the collision term in (2.14) we define

$$Q^\rho(\mathbf{x},t) = m\int_\infty^\infty d^3v \left.\frac{\partial f}{\partial t}\right|_c \tag{2.18}$$

$$\mathbf{Q}^p(\mathbf{x},t) = m\int_\infty^\infty d^3v (\mathbf{v}-\mathbf{u}) \left.\frac{\partial f}{\partial t}\right|_c \tag{2.19}$$

$$Q^E(\mathbf{x},t) = \frac{1}{2}m\int_\infty^\infty d^3v (\mathbf{v}-\mathbf{u})^2 \left.\frac{\partial f}{\partial t}\right|_c \tag{2.20}$$

The precise form of these terms depends on the particular properties of the systems and will not be specified at this point.

2.3.2 Fluid Moments

To provide an example for the evaluation of the moments of the Boltzmann equations let us evaluate the 0th moment of the integral.

$$\int_{\infty}^{\infty} d^3v \left(\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f \right) = \int_{\infty}^{\infty} d^3v \left. \frac{\partial f}{\partial t} \right|_{coll}$$

The first term of the equation becomes

$$\int_{\infty}^{\infty} d^3v \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int_{\infty}^{\infty} d^3v f(\mathbf{x}, \mathbf{v}, t) = \frac{\partial n(\mathbf{x}, t)}{\partial t}.$$

The second term is

$$\int_{\infty}^{\infty} d^3v \mathbf{v} \cdot \nabla f = \int_{\infty}^{\infty} d^3v \nabla \cdot (\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)) = \nabla \cdot \int_{\infty}^{\infty} d^3v \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) = \nabla \cdot (\mathbf{u}(\mathbf{x}, t) n(\mathbf{x}, t))$$

and the third term is

$$\int_{\infty}^{\infty} d^3v \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = \sum_i \int_{\infty}^{\infty} d^3v \frac{F_i}{m} \frac{\partial f}{\partial v_i} = \sum_i \int_{\infty}^{\infty} d^2v \left[\frac{F_i}{m} f \right]_{v_i=-\infty}^{v_i=\infty} - \sum_i \int_{\infty}^{\infty} d^3v \frac{f}{m} \frac{\partial F_i}{\partial v_i}$$

The terms on the rhs. in the above equation are 0 because the $f = 0$ for $v_i = \left\{ \begin{array}{c} +\infty \\ -\infty \end{array} \right.$ for each component v_i and in because $\partial F_i / \partial v_i = 0$ (see homework for the Lorentz force). Therefore

$$\int_{\infty}^{\infty} d^3v \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = 0$$

The collision term of the Boltzmann equation reduces to Q^ρ (see equation 2.18) such that in summary the 0th moment of the Boltzmann equation reduces to

$$\frac{\partial n(\mathbf{x}, t)}{\partial t} + \nabla \cdot (n\mathbf{u}) = \frac{1}{m} Q^\rho. \tag{2.21}$$

This is the usual continuity equation for the particle number density with a source term on the right side. Multiplying (2.21) with the particle mass yields the continuity equation for mass density

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = Q^\rho \tag{2.22}$$

The source term describes production or annihilation of mass for instance through chemical reactions or ionization or recombination. It is noted that (2.22) is for one species only. In the case of several neutral constituents or ion species a corresponding continuity equation is obtained for each species. The total production rate of mass has to be zero.

Similar to the 0th moment the 1st moment of the Boltzmann equation [$m\int_{\infty}^{\infty} d^3v\,\mathbf{v}\,(Boltzmann\,equation)$] and 2nd moment [$\frac{3}{2}m\int_{\infty}^{\infty} d^3v\,v^2\,(Boltzmann\,equation)$] yield the equations for the fluid momentum (or velocity) and energy

$$\frac{\partial \rho\mathbf{u}}{\partial t} = -\nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \nabla\cdot\underline{\Pi} + qn(\mathbf{E}+\mathbf{u}\times\mathbf{B}) + \mathbf{u}Q^{\rho} + \mathbf{Q}^{p} \tag{2.23}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{\gamma-1}p + \frac{1}{2}\rho u^2\right) &= -\nabla\cdot\left(\frac{1}{2}\rho u^2\mathbf{u} + \frac{1}{\gamma-1}p\mathbf{u} + \mathbf{u}\cdot\underline{\Pi} + \mathbf{L}\right) \\ &\quad + qn\mathbf{u}\cdot\mathbf{E} + \frac{1}{2}u^2Q^{\rho} + \mathbf{u}\cdot\mathbf{Q}^{p} + Q^{\mathbf{E}}\end{aligned} \tag{2.24}$$

with the heat flux $\mathbf{L}(\mathbf{x},t) = \frac{1}{2}m\int_{\infty}^{\infty} d^3v(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})^2 f(\mathbf{x},\mathbf{v},t)$ and γ is the ratio of specific heats, i.e., $\gamma = 5/3$ if a gas has 3 degrees of freedom for motion. Note that

- As before the "fluid" Lorentz force $\mathbf{E}+\mathbf{u}\times\mathbf{B}$ requires to solve the corresponding field equations (2.1) - (2.4) .
- The Lorentz force (or any velocity dependent force now depends on the bulk velocity and velocity is a dependent variable in the fluid equations rather than an independent variable as in the Boltzmann equation.
- The source terms for mass Q^{ρ}, momentum $\mathbf{Q}^{p}$, and energy Q^{E} depend on system properties and need to be specified through these or through a systematic collision operator and the corresponding velocity integrals. The terms reflect mass generation and annihilation Q^{ρ}, momentum exchange through friction $\mathbf{Q}^{p}$, and energy exchange collisions Q^{E}. The friction term $\mathbf{Q}^{p}_{s}$ for species s can be expressed in terms of an effective collision frequency $\mathbf{Q}^{p}_{s} = \sum_{t\neq s}\nu_{st}m_s n(\mathbf{v}_t - \mathbf{v}_s)$
- The pressure tensor is often split into a scalar pressure and a viscous tensor $\underline{\Pi} = p\underline{1} + \underline{\mathbf{w}}$, with $p = \frac{1}{3}Tr(\Pi)$ and the viscous tensor $\underline{\mathbf{w}}$.
- Often a kinematic viscosity $\sigma_{ik} = a\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + b\frac{\partial u_l}{\partial x_l}\delta_{ik}$ is used on the right of the momentum equation yielding a term $\nabla\cdot\underline{\sigma} = \eta\Delta\mathbf{u} + \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla\cdot\mathbf{u})$.
- Elimination of $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right)$ in the energy equation (with the aid of (2.22) and (2.23)) yields

$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p + \nabla\cdot p\mathbf{u}\right) = -(\underline{\Pi}\cdot\nabla)\cdot\mathbf{u} - \nabla\cdot\mathbf{L} + Q^{E} \tag{2.25}$$

The fluid equations without the source terms imply the conservation of the corresponding property (mass, momentum, and energy). Consider the mass in a given volume defined by

$$M_V = \int_V \rho d^3x$$

The change of mass in the volume is

$$\frac{dM_V}{dt} = \int_V \frac{\partial\rho}{\partial t}d^3x = -\int_V \nabla\cdot\mathbf{v}\rho d^3x = -\oint_{S_V}\rho\mathbf{v}\cdot d\mathbf{s}$$

where S_V is the surface of V. In other words the mass in the volume V changes only if there is a nonzero density flux (velocity) across the surface of the volume. Similarly momentum and energy are conserved. However, for the energy one has to include momentum and energy which is contained in the fields as well.

Exercise: Determine the integral of the 1st order moment for the first two terms in the Boltzmann equation.

Exercise: Derive the 1st order moment force term for a gravitational force and the Lorentz force (velocity dependent).

Exercise: Do the same for the energy equation (i.e., multiply the Boltzmann equation (2.14) with $\frac{1}{2}mv^2$ and integrate).

2.3.3 Typical Fluid Approximations

Equations (2.22) - (2.24) establish the typical set of fluid equations which are used in many simulations of fluids and gases like weather simulations, air flow around aircraft or cars, water flow in pipes or round boats, and many other research and technical applications. Using the set of equations (2.22), (2.23), and (2.25) we can derive most equations commonly used in fluid simulations:

- For a known velocity profile $\mathbf{u}$ and no sources $Q^\rho = 0$ it is sufficient to model the continuity equation for instance to derive the evolution of density of a gas or the concentration of dust, aerosols, etc. in any medium like air water etc.:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

- With the total derivative along the fluid path defined as $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ the advection equation is

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u}$$

- A system is called incompressible if the density of a fluid parcel along the convection path does not change, i.e., $d\rho/dt = 0$ which is equivalent of the condition $\nabla \cdot \mathbf{u} = 0$ for the convection.

- For no sources Q^ρ, $\mathbf{Q}^p = 0$, no viscosity $\underline{\sigma} = 0$ (scalar pressure) and gravitational acceleration for the force term one obtains Euler's equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \; (+\, \mathbf{g})$$

- With a kinematic viscosity included the momentum equation is known as the Navier-Stokes equation:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta \Delta \mathbf{u} + \left(\zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{u}) \; (+\, \rho \mathbf{g} + ..)$$

- Neglecting pressure and external force terms and assuming a simplified viscosity one obtains Burger's equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} = 0$$

- Diffusion and heat conduction: A diffusion equation can be obtained from the continuity equation and a redefinition of the bulk velocity in the presence of several particle species. However, the more straightforward equation for diffusion is obtained for heat conduction (i.e., diffusion of temperature). Equation (2.25) can be re-written in different forms. Defining $\frac{p}{\gamma-1} = \rho\varepsilon$ yields an equation for the internal energy ε of a gas. More commonly used is the ideal gas law $p = nkT$ to re-write the energy equation into an equation for temperature. Assuming scalar pressure, constant density, and a heat flux driven by a temperature gradient $\mathbf{L} = -\kappa \nabla T$ on obtains

$$\frac{\partial T}{\partial t} = -\mu \Delta T$$

- Steady state equation are generated from the above sets by assuming $\partial/\partial t = 0$. For steady condition the velocity is often determined from a potential which can be scalar if the flow is assumed irrotational ($\mathbf{u} = \nabla \Phi$) or incompressible flow is modeled sometimes by a vector potential ($\mathbf{u} = \nabla \times \mathbf{V}$).

Exercise: Using the continuity equation and the stated assumptions derive Euler's equation.

Exercise: Derive the equation for heat conduction with the stated assumptions.

Exercise: Derive the heat conduction equation for nonzero velocity $\mathbf{u}$.

Exercise: Derive the continuity equation and momentum equation for irrotational flow.

Exercise: Assume a scalar pressure, $\mathbf{L} = 0$, and $Q^E = 0$ in the pressure equation (2.25). Consider a function $g = p^a \rho^b$ and determine a and b such that the resulting equation for g assumes a conservative form, i.e., $\partial g/\partial t + \nabla \cdot g\mathbf{u} = 0$.

Exercise: Assume a scalar pressure, $\mathbf{L} = 0$, and $Q^E = 0$ in the pressure equation (2.25). Consider a function $h = p^a \rho^b$ and determine a and b such that the resulting equation for h assumes a total derivative, i.e., $\partial h/\partial t + \mathbf{u} \cdot \nabla h = 0$. For $\gamma = 5/3$ this becomes the equation for an entropy function because entropy is conserved for adiabatic changes.

2.4 Plasma (Two-) Fluid Equations

In the absence of ionization and energy exchange collisions, considering a simple two component (ion and electron) plasma, and assuming a quasi-neutral plasma (implying $n_e = n_i$) one obtains the so-called two fluid equations. With quasi-neutrality the continuity equation is

$$\frac{\partial n(\mathbf{x}, t)}{\partial t} + \nabla \cdot (n \mathbf{u}_s) = 0. \tag{2.26}$$

Momentum and energy equations are unchanged except for neglecting the collision terms.

$$
\begin{aligned}
\frac{\partial \rho_s \mathbf{u}_s}{\partial t} &= -\nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s) - \nabla \cdot \underline{\Pi}_s + q_s n (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \mathbf{Q}_s^p && (2.27)\\
\frac{1}{\gamma - 1} \frac{\partial p_s}{\partial t} &= -\frac{1}{\gamma - 1} \nabla \cdot p_s \mathbf{u}_s - (\underline{\Pi}_s \cdot \nabla) \cdot \mathbf{u}_s - \nabla \cdot \mathbf{L}_s + Q_s^E && (2.28)
\end{aligned}
$$

where $\mathbf{Q}_s^p$ describes friction between the two components and effectively describes an electric resistance. Effectively one can express the friction terms as $\mathbf{Q}_s^p = \sum_{t \neq s} \nu_{st} m_s n (\mathbf{v}_t - \mathbf{v}_s)$ where ν_{st} is the collision frequency for particles of species s to collide with particles of species t. Momentum and energy conservation (for a closed system) further imply

$$
\begin{aligned}
\sum_s \mathbf{Q}_s^p &= 0 \\
\sum_s Q_s^E &= -\sum_s \mathbf{u}_s \cdot \mathbf{Q}_s^p
\end{aligned}
$$

Specifically for a neutral plasma consisting of electrons and one ion species these relations imply

$$
\begin{aligned}
\frac{\nu_{ei}}{\nu_{ie}} &= \frac{m_i}{m_e} \\
Q_e^E + Q_i^E &= (\mathbf{u}_i - \mathbf{u}_e) \, \mathbf{Q}_e^p = \frac{1}{en} \mathbf{j} \cdot \mathbf{Q}_e^p
\end{aligned}
$$

Often it is also assumed that the pressure is scalar or gyrotropic, i.e., is different parallel and perpendicular to the magnetic field. Also the heat conduction term is often neglected. Further use of these equations will be made in sections 3 and 4.

2.5 Single Fluid or MHD Equations

While considerably much simpler the two fluid equation contain still considerable complexity which is not needed for many plasma systems. Thus it is desirable to formulate a more appropriate set of equations which is applicable for large scale systems. This set of equations are the so-called MHD equations. With $q_i = -q_e = e$ the total current density is

$$\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e)$$

We can also define the total mass density as $\rho = n(m_i + m_e)$, effective mass $M = m_i + m_e$, and bulk velocity or total mass density flux as

$$\rho \mathbf{u} = n(m_i \mathbf{u}_i + m_e \mathbf{u}_e)$$

Exercise: Show that these definitions yield

$$
\begin{aligned}
\nabla \cdot j &= 0 \\
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} &= 0 && (2.29)
\end{aligned}
$$

With the above definitions we can now uniquely express $\mathbf{u}_i$ and $\mathbf{u}_e$ in terms of $\mathbf{u}$ and $\mathbf{j}$ the goal being to derive equations which substitute the two-fluid equations for momentum (2.27) and energy (2.28) density. It is also assumed that the pressure is scalar for both the electron and the ion components even though this is not necessary. By taking the sum of the momentum equations and substituting $\mathbf{u}_i$ and $\mathbf{u}_e$ one obtains:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \frac{m_e m_i}{e^2} \nabla \cdot (\frac{1}{\rho}\mathbf{j}\mathbf{j}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{2.30}$$

with $p = p_i + p_e$. Note that the total momentum has to be conserved such that $\mathbf{Q}_i^p + \mathbf{Q}_e^p = 0$. A second equation is required for uniqueness (there are two momentum equations for the two fluids). This is obtained by multiplying the ion equation with q_i/m_i and the electron equation with q_e/m_e and the sum of the modified equations:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{m_e}{e^2 n}\left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u})\right] - \frac{1}{en}\nabla p_e + \frac{1}{en}\mathbf{j} \times \mathbf{B} + \eta \mathbf{j} \tag{2.31}$$

with the resistivity $\eta = m_e \nu_c / ne^2$ where ν_c is the collision frequency between electrons and ions (or neutrals). This equation is usually termed generalized Ohm's law. In the above equation the first term on the rhs is often called the inertia term because it represent the electron inertia in this equation. The second term is the electron pressure force and the third term is the Hall term.

Note that thus far there has been no approximation in our derivation (except for the pressure isotropy which is not really required) such that the above form of Ohm's law is fully equivalent to the two-fluid equations.

Finally one can take the sum of the electron and ion pressure equations and keep the electron pressure equation unmodified to obtain

$$\begin{aligned} \frac{1}{\gamma - 1}\left(\frac{\partial}{\partial t} p + \nabla \cdot p\mathbf{u}\right) &= -p\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{L} + \eta \mathbf{j}^2 + ... \\ \frac{1}{\gamma - 1}\left(\frac{\partial}{\partial t} p_e + \nabla \cdot p_e\mathbf{u}_e\right) &= -p_e\nabla \cdot \mathbf{u}_e - \nabla \cdot \mathbf{L}_e + \eta \mathbf{j}^2 \end{aligned} \tag{2.32}$$

Note that the ohmic heating term is present both in the sum of the pressure equations as well as in the electron equation. Here it is assumed that the frictional heating occurs almost entirely for the electrons because of their small mass compared to ions. For the MHD equations we will now neglect the terms associated with the electron pressure, and the first, second, and third term on the rhs of generalized Ohm's law. Summarizing the equations and complementing them with the non-relativistic Maxwell equations one obtains the following set of equations termed the resistive MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \tag{2.33}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{2.34}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \tag{2.35}$$

$$\frac{1}{\gamma - 1}\left(\frac{\partial}{\partial t} p + \nabla \cdot p\mathbf{u}\right) = -p\nabla \cdot \mathbf{u} + \eta \mathbf{j}^2 \tag{2.36}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{2.37}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2.38}$$

In Maxwell's equations the displacement current has been neglected by assuming that there are no electromagnetic waves propagating at the speed of light. Divergence $\mathbf{B}$ is satisfied through the initial condition (show this) and quasi-neutrality satisfies Coulomb's equation. The continuity and momentum equations are exact (except for the pressure isotropy) and the main assumptions (simplifications) went into Ohm's law and to some degree into the total pressure equation (and the elimination of the electron pressure equation). These assumptions can be better understood by examining Ohm's law (2.31) through a normalization or dimensional analysis. The ideal MHD equations are give by the above set with $\eta = 0$.

The normalization is applied by measuring all quantities in typical units, e.g., the magnetic induction $\mathbf{B}$ in units of a typical magnetic field B_0 such that $\mathbf{B} = B_0\hat{\mathbf{B}}$ where $\hat{\mathbf{B}}$ is now of order unity, density in units of a typical density $\rho_0 = m_0 n_0$, Length in units of a typical length scales L_0, etc. Thus velocities should be measured in units of the Alfvén speed $u_0 = B_0/\sqrt{\mu_0\rho_0}$ and time in units of Alfvén travel time $t_0 = L_0/u_0$. Now we can examine the coefficients of the different terms in Ohm's law. Applying this scaling yields coefficients $c^2/\omega_{pe}^2 L_0^2$ for the first term on the rhs and $c/\omega_{pi}L_0$. The terms $c/\omega_{pe} = (\epsilon_0 m_e c^2/n_0 e^2)^{1/2}$ and $c/\omega_{pi} = (m_i/m_e)^{1/2} c/\omega_{pe}$ are called electron and ion inertia scales (or electron and ion skin depth because of the extinction length of waves in a medium). Thus it is justified to neglect these terms if the gradients in the system are on a much larger length scale then these inertia scales. Using this normalization general Ohms law becomes

$$\hat{\mathbf{E}} + \hat{\mathbf{u}} \times \hat{\mathbf{B}} = \left(\frac{c}{\omega_{pe}L_0}\right)^2 \left[\frac{\partial\hat{\mathbf{j}}}{\partial\hat{t}} + \nabla\cdot\left(\hat{\mathbf{u}}\hat{\mathbf{j}} + \hat{\mathbf{j}}\hat{\mathbf{u}}\right)\right] - \frac{c}{2\omega_{pi}L_0}\hat{\nabla}\hat{p}_e + \frac{c}{\omega_{pi}L_0}\hat{\mathbf{j}} \times \hat{\mathbf{B}} + \tilde{\eta}\hat{\mathbf{j}} \quad (2.39)$$

$$with \qquad \tilde{\eta} = \frac{\eta t_0}{\mu_0 L_0^2} = \frac{\eta}{\mu_0 u_0 L_0} = \frac{1}{R} \quad (2.40)$$

where a dimensional analysis of magnetic diffusion (section 7.2) yields R as the so-called Lundquist of magnetic Reynoldsnumber. Note that the factor of 2 in the electron pressure gradient term arises from a normalization of thermal pressure with $p_0 = B_0^2/(2\mu_0)$. In this normalization the plasma $\beta = 2\mu_0 p/B^2$ (ratio of thermal to magnetic pressure or energy density) becomes simply $\hat{\beta} = \hat{p}/\hat{B}^2$. Typical magnetospheric values of c/ω_{pi} are a few 100 km and of c/ω_{pe} about 10 km demonstrating that these coefficients are small if $L_0 \sim 1R_E$ or larger. This justifies to neglect the first 3 terms on the right of general Ohm's law for many applications. Particularly the first term involving the derivative of the current (electron inertial term) is small for almost all applications unless structure evolves on the tiny scale of about 10 km. The resistive term is 0 based on Coulomb collisions. However, resistivity can be caused by plasma turbulence in strong electric current or strong density or pressure gradients. In such cases resistivity can be important at thin boundaries (see also section 7).

Note, that equation (2.40) demonstrates that the effective resistivity $\tilde{\eta}$ (or inverse Lundquist number) in systems with fixed collision frequency and fixed typical velocity is proportional to the inverse length scale L_0.

As a second application of the normalization, consider a plasma interacting with a neutral background. In this case the normalized momentum equation is

$$\frac{\partial\hat{\rho}\hat{\mathbf{u}}}{\partial\hat{t}} + \hat{\nabla}\cdot(\hat{\rho}\hat{\mathbf{u}}\hat{\mathbf{u}}) = -\frac{1}{2}\hat{\nabla}\hat{p} + \hat{\mathbf{j}} \times \hat{\mathbf{B}} - t_0\nu_{in}\hat{\rho}\left(\hat{\mathbf{u}} - \hat{\mathbf{u}}_n\right)$$

Thus the effective plasma (ion) neutral collision frequency is measured in units of $1/t_0$ or $\tilde{\nu}_{in} = t_0\nu_{in} = L_0\nu_{in}/u_0 = L_0\nu_{in}\sqrt{\mu_0\rho_0}/B_0$. Considering a fixed magnetic field and density (and therefore density), we can identify the importance of the length scale. For $L_0 \sim u_0/\nu_{in}$, the coefficient for the friction term is of order unity. For the case $L_0 \gg u_0/\nu_{in}$ the momentum equation is strongly dominated by the plasma-neutral friction. This is for instance used in the ionosphere where Ohm's law is expressed as $\mathbf{j} = \underline{\underline{\sigma}} \cdot \mathbf{E}$ with the conductivity $\underline{\underline{\sigma}}$, and is derived from the electron and ion momentum equations by neglecting the inertial and pressure gradient terms. Note, however, that this is justified only on sufficiently large scales, and that small scale and high frequency phenomena require to consider the complete form of Ohm's law (2.31).

Exercise: Assume a plasma density of 1 cm^{-3}, temperature equivalent to 1 keV, and a magnetic field of 20 nT which are typical for the near Earth magnetotail. Determine electron and ion inertia scales. Assume that quasi-neutrality is violated in a sphere with the radius of the electron inertia length by 1 % (e.g. 1% of the ion charge is not compensated by electrons. If outside were a vacuum what is the electric field outside the sphere? What velocity perpendicular to the magnetic field is required by Ohm's law to generate an electric field magnitude equal to that on the surface of the sphere?

Exercise: For the plasma in the prior exercise, determine the temperature in degrees Kelvin. Determine the energy density in kW hours/m^3 and kW hours /R_e^3 (1 $R_E = 6370$ km). For the sake of simplicity assume that the plasma sheet is represented by a cylinder with 10 R_E radius and 100 R_E length. How long could a power plant with an output of 1000 MW operate on the energy stored in the plasma?

2.6 Properties of the Two-Fluid and MHD equations:

The MHD equations are a very commonly used plasma approximation. They conserve mass, momentum, and energy. As mentioned above they are valid on scales larger than the ion inertia scale. It is important to note that the ideal MHD equation do not have any intrinsic physical length scale. This implies for instance a self-similarity in the sense that the dynamics on small physical length scales is exactly the same as for large scale systems with the only difference that the larger system evolve slower. This can be illustrated by normalizing the equations to a particular length L_0 which implies that the typical time scale is $\tau_0 = L_0/u_A$ (with $u_A = B_0/\sqrt{\mu_0\rho_0}$). For a system which is identical except that is is 10 times larger the length scale is $10L_0$ and the time scale is $10\tau_0$. Thus a simple re-normalization yields exactly the same dynamics.

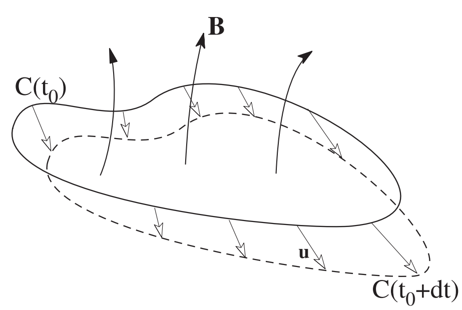

Figure 2.4: Illustration of the frozen-in condition.

Ohm's law in the ideal MHD equations $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ implies that the magnetic flux is frozen into the plasma motion. This can be seen from the following arguments. The magnetic flux through the surface C is the surface integral

$$\Phi_C = \int_C \mathbf{B} \cdot d\mathbf{s}$$

with $d\mathbf{s}_C$ being the surface element of the contour C. The contour elements move with the fluid velocity $\mathbf{u}$. The change of the magnetic flux from time t_0 do $t_0 + dt$ is

$$\begin{aligned}
\Phi_C(t_0+dt) - \Phi_C(t_0) &= \int_{C(t_0)}^{C(t_0+dt)} \mathbf{B} \cdot d\mathbf{s} + dt \int_{C(t_0)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \\
&= dt \oint_{\partial C(t_0)} \mathbf{B} \cdot (\mathbf{u}_C \times d\mathbf{l}) - dt \int_{C(t_0)} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} \\
&= dt \oint_{\partial C(t_0)} (\mathbf{B} \times \mathbf{u}_C) \cdot d\mathbf{l} - dt \oint_{\partial C(t_0)} (-\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\
&= dt \oint_{\partial C(t_0)} [(\mathbf{u} - \mathbf{u}_C) \times \mathbf{B}] \cdot d\mathbf{l}
\end{aligned}$$

where the first term on the rhs represents the contribution from the change of the shape of C and the second term the contribution from the change of $\mathbf{B}$. It follows that

$$\frac{d\Phi_C}{dt} = 0$$

if the surface is moving with the fluid $\mathbf{u}_C = \mathbf{u}$. In other words the amount of magnetic flux through any given cross sectional area of the MHD fluid does not change in time if this area which moves with the fluid. The frozen-in condition can also be understood in the following way. Two fluid elements are always connected by a magnetic field line if they were connected at one time by a field line (a line defined by the direction of the magnetic field at any moment in time). In other words a field line can be identified by fluid plasma elements. This property is sometimes called line conservation. This requires that ideal Ohm's law applies, i.e., electrical resistivity is zero.

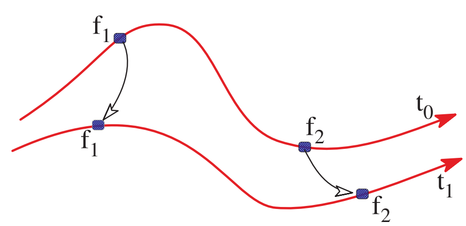

Figure 2.5: Illustration of line conservation

A more complete form of Ohm's law should be considered if gradients on smaller scales exist in a plasma. Since the ion inertia scale is by a factor of $\sqrt{m_i/m_e}$ larger than electron inertia effects the first terms to consider are the Hall term and the electron pressure term (note that the electron pressure is typically an order of magnitude smaller than the ion pressure such that contributions from this term are small. It is interesting to note that

$$\mathbf{u}_e = \mathbf{u} - \frac{m_i}{e\rho}\mathbf{j} \tag{2.41}$$

Comparing this with generalized Ohm's law we can re-write this neglecting the electron inertia scale as

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = -\frac{M}{e\rho}\nabla p_e \tag{2.42}$$

In other words the addition of the Hall term transforms Ohm's law from $\mathbf{E}+\mathbf{u}\times\mathbf{B}=0$ in MHD into $\mathbf{E}+\mathbf{u}_e\times\mathbf{B}=0$. With this form it is clear that the frozen-in condition for the magnetic flux applies now to the electron fluid (note that one can also include a scalar electron pressure term in Ohm's law if the density does not vary strongly).

There will be various applications using the fluid and the kinetic equations. Typical applications consider waves, discontinuities and shocks, instabilities, steady state solutions, and equilibrium solutions. Particularly for the last topic it is important to note the following terminology.

Steady state assumes time stationary solutions with nonzero velocity, $\partial/\partial t = 0$ and $\mathbf{u} \neq 0$.

Equilibrium solutions assume $\partial/\partial t = 0$ and $\mathbf{u} = 0$. Note that for kinetic systems the velocity in phase space is always nonzero for physical systems. Also the electron velocity is nonzero in current regions.

Electrostatic solutions assume $\partial \mathbf{B}/\partial t = 0$. This implies $\nabla \times \mathbf{E} = 0$ or $\mathbf{E} = -\nabla\phi$. In this case Ohm's law must be replaced by the Poisson equation ($\nabla \cdot \mathbf{E} = 0$).

2.7 Tables of Plasma Parameters

The first table gives some fundamental constants:

Property	Symbol	SI	cgs
Speed of light	c	3×10^{8} m s^{-1}	3×10^{10} m s^{-1}
Boltzmann constant	k	1.38×10^{-23} J K^{-1}	1.38×10^{-16} erg K^{-1}
Electron mass	m_e	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_i	1836 m_e	1836 m_e
Elementary charge	e	1.6×10^{-19}C	4.8×10^{-10} statcoulomb
Dielectric constant	ϵ_0	8.85×10^{-12} F m^{-1}	-
Permeability of free space	μ_0	$4\pi\times10^{-7}$ H m^{-1}	-

Other relations:

$$\begin{aligned} \epsilon_0\mu_0 &= \frac{1}{c^2} \\ 1eV &= 1.16\times10^{4}K \end{aligned}$$

The following table presents several basic plasma frequencies, length scales and velocities:

Property	Symbol	SI	cgs
Plasma frequency	ω_{pe}	$\left(\frac{n_e e^2}{\epsilon_0 m_e}\right)^{1/2}$	$\left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2}$
Electron gyro frequency	ω_{ge}	$\frac{eB}{m_e}$	$\frac{eB}{m_e c}$
Coulomb collision frequency	ν_{ei}	$\frac{\omega_{pe}}{4\pi n\lambda_D^3}\ln\Lambda$	$\frac{\omega_{pe}}{4\pi n\lambda_D^3}\ln\Lambda$
Debye length	λ_D	$\left(\frac{\epsilon_0 kT}{ne^2}\right)^{1/2}$	$\left(\frac{kT}{4\pi ne^2}\right)^{1/2}$
Skin depth (electron inertia)	λ_e	$\frac{c}{\omega_{pe}}$	$\frac{c}{\omega_{pe}}$
Electron Gyroradius	r_{ge}	$\frac{v_{the}}{\omega_{ge}}$	$\frac{v_{the}}{\omega_{ge}}$
Electron Thermal velocity	v_{the}	$\left(\frac{kT}{m_e}\right)^{1/2}$	$\left(\frac{kT}{m_e}\right)^{1/2}$
Alfvén speed	v_A	$\frac{B}{(\mu_0 n m_i)^{1/2}}$	$\frac{B}{(4\pi n m_i)^{1/2}}$
Number of particles in a Debye sphere	$N_D = \frac{1}{g}$	$\frac{4\pi}{3}n\lambda_D^3$	$\frac{4\pi}{3}n\lambda_D^3$

The corresponding ion properties are

$$\begin{aligned}
\omega_{pi} &= \sqrt{m_e/m_i}\omega_{pe} \\
\omega_{gi} &= (m_e/m_i)\omega_{ge} \\
\nu_{ie} &= (m_e/m_i)\nu_{ei} \\
r_{gi} &= \sqrt{m_i/m_e} r_{ge} \\
v_{thi} &= \sqrt{m_i/m_e} v_{the}
\end{aligned}$$

The following list provides numerical values for the various plasma parameters in a convenient form. Because it is more common in the field of magnetospheric physics everything is measured in cgs units in this table, i.e., n in cm^{-3}, B in Gauss, T in eV, and $\ln\Lambda \approx 20$ (ions are protons). Note that 1 T = 10^4 Gauss and 1 nT =10^{-5} Gauss.

$$\begin{aligned}
\omega_{pe} &= 5.64\times 10^4\, n^{1/2}\;\; [rad/sec] \\
\omega_{ge} &= 1.76\times 10^7\, B\;\; [rad/sec] \\
\omega_{gi} &= 9.58\times 10^3\, B\;\; [rad/sec] \\
\nu_{ei} &= 1.1\times 10^{-5}\, \ln(\Lambda) n T_e^{-3/2}\;\; \left[sec^{-1}\right] \\
\lambda_{De} &= 7.43\times 10^2\, n^{-1/2} T_e^{1/2}\;\; [cm] \\
\lambda_e &= 5.31\times 10^5\, n^{-1/2}\;\; [cm] \\
\lambda_i &= 2.28\times 10^7\, n^{-1/2}\;\; [cm] \\
r_{ge} &= 2.38\times 10^0\, T_e^{1/2} B^{-1}\;\; [cm] \\
r_{gi} &= 1.02\times 10^2\, T_i^{1/2} B^{-1}\;\; [cm] \\
v_{the} &= 4.19\times 10^7\, T_e^{1/2}\;\; [cm/sec] \\
v_{thi} &= 9.79\times 10^5\, T_i^{1/2}\;\; [cm/sec] \\
v_A &= 2.18\times 10^{11}\, B n^{-1/2}\;\; [cm/sec] \\
N_D &= 1.72\times 10^9 T_e^{3/2} n^{-1/2}
\end{aligned}$$

The last table is an overview of typical plasma properties in the magnetosheath/mantle, the outer magnetosphere/tail, and the inner magnetosphere(acceleration) regions.

	Mantle/Sheath	Outer M'Sphere/Tail	Inner M'Sphere/Accel. R
n [cm^{-3}]	10 [1-100]	1 [0.01-5]	10^2 [0.1-10^4]
B [nT]	20 [5-100]	40 [10-100]	10^4 [10^3 -10^5]
T_e [eV]	50 [10-10^3]	500 $[10^2 - 10^3]$	10^3 [1-10^4]
T_i [eV]	100 [10-10^3]	2$\times 10^3$ $[10^3 - 10^4]$	10^3 [1-10^4]
ω_{pe}	1.8$\times 10^5$	5.7$\times 10^4$	5.6$\times 10^5$
ω_{ge}	3.5$\times 10^3$	7.0$\times 10^3$	1.8$\times 10^6$
ω_{gi}	1.9$\times 10^0$	3.6$\times 10^0$	9.6$\times 10^2$
ν_{ei}	6.2$\times 10^{-6}$	2.0$\times 10^{-8}$	7.0$\times 10^{-7}$
λ_{De} [m]	1.7$\times 10^1$	1.7$\times 10^2$	2.3$\times 10^1$
λ_e [m]	1.7$\times 10^3$	5.3$\times 10^3$	5.3$\times 10^2$
λ_i [m]	7.3$\times 10^4$	2.3$\times 10^5$	2.3$\times 10^4$
r_{ge} [m]	8.4$\times 10^2$	1.3$\times 10^3$	7.5$\times 10^0$
r_{gi} [m]	5.0$\times 10^4$	1.1$\times 10^5$	3.2$\times 10^2$
v_{the} [m/sec]	3.0$\times 10^6$	9.4$\times 10^6$	1.3$\times 10^7$
v_{thi} [m/sec]	9.8$\times 10^4$	4.4$\times 10^5$	3.1$\times 10^5$
v_A [m/sec]	1.4$\times 10^5$	8.7$\times 10^5$	2.18$\times 10^7$
N_D	1.9$\times 10^{11}$	1.9$\times 10^{13}$	5.4$\times 10^{12}$