Chapter 4

Single Particle Dynamics

The motion of charged particles in assumed electric and magnetic field can provide insight into many important physical properties of plasmas. While it does not provide the full plasma dynamics it can provide clues on the collective behavior.

4.1 Gyro Motion

Lorentz force equation:

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

In the absence of a magnetic field the equation reduces to

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \times \mathbf{B}$$

Assuming a constant and uniform magnetic field, the scalar product with $\mathbf{v}$ yields the conservation of energy

$$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = 0 \tag{4.1}$$

To solve the equations of motion let us consider a magnetic field of $\mathbf{B} = B\mathbf{e}_z$. Denoting the direction perpendicular to z with $\perp$ the components of the equations of motion are

$$\begin{aligned} \frac{dv_z}{dt} &= 0 \\ \frac{d\mathbf{v}_\perp}{dt} &= \frac{qB}{m}\mathbf{v}_\perp \times \mathbf{e}_z \end{aligned}$$

with the solution

$$\begin{aligned} v_x &= v_\perp \cos(\omega_g t + \phi) \\ v_y &= v_\perp \sin(\omega_g t + \phi) \\ v_z &= v_\parallel \end{aligned} \tag{4.2}$$

with $v_\perp^2 = v_x^2 + v_y^2$ and the trajectory

$$\begin{aligned} x - x_0 &= \frac{v_\perp}{\omega_g} \sin(\omega_g t + \phi) \\ y - y_0 &= -\frac{v_\perp}{\omega_g} \cos(\omega_g t + \phi) \\ z - z_0 &= v_\| t \end{aligned} \tag{4.3}$$

For $v_{z0} = 0$ the trajectories are circles in which the particles gyrate with a frequency of

$$\omega_g = qB/m. \tag{4.4}$$

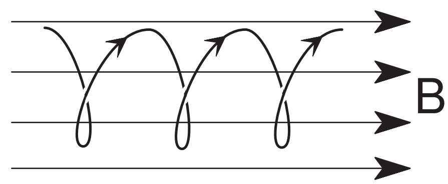

Figure 4.1: Helical ion trajectory in a uniform magnetic field.

With a finite velocity in z the trajectories are helical lines with the same gyro-frequency. The location of the center of the particle gyro motion (x_0, y_0, z) is called the guiding center of the particle motion. The gyro-radius for the particle motion is

$$r_g = \frac{v_\perp}{\omega_g} = \frac{m v_\perp}{|q| B}. \tag{4.5}$$

The angle of the particle trajectory with the magnetic field is called pitch angle

$$\alpha = \tanh^{-1} \frac{v_\perp}{v_\|} \tag{4.6}$$

4.2 Particle Drifts

The discussion of the particle motion in the prior section is highly idealized because it assumes a uniform magnetic field, no other forces, and time independence for the magnetic field. The motion changes with these effects included.

4.2.1 Electric Force Drift

Let us first consider the presence of a uniform and time independent electric field $\mathbf{E} = E_\| \mathbf{e}_z + E_x \mathbf{e}_x$. This requires to consider the full Lorentz force equations. The motion parallel to the magnetic field can be separated from the perpendicular motion and describes a simple acceleration along the magnetic field.

$$m dv_z/dt = qE_\| \tag{4.7}$$

The equations for the perpendicular motion are

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q\mathbf{E}_\perp}{m} + \frac{q}{m}\mathbf{v} \times \mathbf{B} \tag{4.8}$$

We can solve this equation either by explicit integration or finding a useful transformation in which the perpendicular electric field vanishes. Substituting $\mathbf{v}_\perp = \mathbf{u} + \mathbf{w}$ with a time constant component $\mathbf{w}$ the equation of motion becomes

$$\frac{d\mathbf{u}}{dt} = \frac{q\mathbf{E}_\perp}{m} + \frac{q}{m}\mathbf{v}_E \times \mathbf{B} + \frac{q}{m}\mathbf{u} \times \mathbf{B}$$

Choosing the velocity $\mathbf{w}$ such that $\frac{q\mathbf{E}_\perp}{m} + \frac{q}{m}\mathbf{v}_E \times \mathbf{B} = 0$ reduces the equation for $\mathbf{u}$ to the case with no electric force. Using the Lorentz transformation we obtain this velocity to be

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{4.9}$$

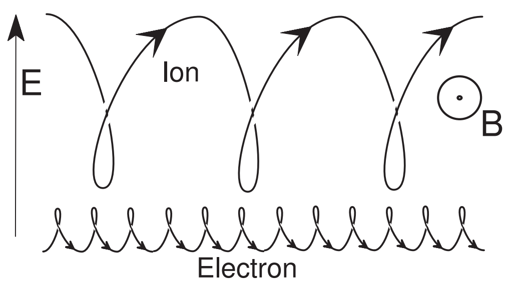

Figure 4.2: Illustration of the $\mathbf{E} \times \mathbf{B}$ drift.

Thus a particle which drifts across the magnetic field with this so-called $\mathbf{E} \times \mathbf{B}$ drift does not feel the perpendicular electric field. The drift is independent of the particle charge because it represents the transformation into the frame where the perpendicular electric field $\mathbf{E}'_\perp = 0$.

The above result can be generalized by substituting $q\mathbf{E}$ with a general constant force term $\mathbf{F}$. The resulting particle drift exposed a this constant force is

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \tag{4.10}$$

4.2.2 Polarization Drift

Taking the cross-product of the Lorentz equation with $\mathbf{B}/B^2$yields

$$\frac{m}{qB^2}\frac{d\mathbf{v}}{dt} \times \mathbf{B} = \mathbf{E} \times \frac{\mathbf{B}}{B^2} + \mathbf{v}_\perp$$

For uniform fields $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and averaging the above equation over a gyro period we obtain the drift velocity as

$$\mathbf{v}_D = \mathbf{w}_E + \frac{1}{\omega_g B}\frac{d\mathbf{E}_\perp}{dt} \tag{4.11}$$

The first term is the already known $\mathbf{E} \times \mathbf{B}$ drift and the second is caused by a slow change of the electric field and called the polarization drift.

$$\mathbf{v}_P = \frac{1}{\omega_g B} \frac{d\mathbf{E}_\perp}{dt} \tag{4.12}$$

Here slow means a change on a time scale much larger than the gyro period. Note that the time derivative in can be caused by a temporal change of the electric field or by a spatial variation of the electric field over the gyro scale. In that case $d\mathbf{E}_\perp/dt = \mathbf{v} \cdot \nabla\mathbf{E}$. The corresponding correction for the electric drift is second order and becomes

$$\mathbf{v}_E = \left(1 + \frac{1}{4} r_g^2 \nabla^2\right) \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{4.13}$$

4.2.3 Magnetic Gradient Drift

In an inhomogeneous magnetic field with a magnetic gradient the gyro radius is a function of the location in the magnetic field. The result is the so-called magnetic gradient drift. Expanding the magnetic field at the location of the guiding center

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{r} \cdot \nabla\mathbf{B}_0$$

Substituting this expansion in the Lorentz force equation yields

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0) + \frac{q}{m}\left(\mathbf{v} \times (\mathbf{r} \cdot \nabla)\,\mathbf{B}_0\right)$$

Splitting the velocity into the gyro and a drift motion term $\mathbf{v} = \mathbf{v}_g + \mathbf{v}_\nabla$ and assuming that the drift velocity is much smaller than the gyro velocity

$$\frac{d\mathbf{v}_\nabla}{dt} = \frac{q}{m}(\mathbf{v}_\nabla \times \mathbf{B}_0) + \frac{q}{m}\left(\mathbf{v}_g \times (\mathbf{r} \cdot \nabla)\,\mathbf{B}_0\right)$$

where we have subtracted the gyro motion of the Lorentz equation and neglected $\mathbf{v}_\nabla$ in the last term of the equation. Averaging over a gyro period gets rid of the time derivative on the left side. Taking the cross product with $\mathbf{B}/B^2$ yields

$$\mathbf{v}_\nabla = \frac{1}{B^2} \left\langle \left(\mathbf{v}_g \times (\mathbf{r} \cdot \nabla)\,\mathbf{B}_0\right) \times \mathbf{B}_0 \right\rangle$$

where $\langle\rangle$ denotes the average over the gyro period. Assuming $\mathbf{B}_0 = B_0(x)\mathbf{e}_z$ one can substitute $\mathbf{v}_g$ and $\mathbf{r}$ with the solution for the simple gyro motion

$$\mathbf{v}_\nabla = \frac{1}{B_0} \left\langle \mathbf{v}_g x \frac{dB_0}{dx} \right\rangle$$

one can substitute $\mathbf{v}_g$ and x with the solution for the simple gyro motion. Only the average along y is nonzero and in it general form the drift becomes

$$\mathbf{v}_\nabla = \frac{m v_\perp^2}{2qB^3} \left(\mathbf{B} \times \nabla B\right) \tag{4.14}$$

The gradient drift is perpendicular to the magnetic field and perpendicular to the gradient of the field.

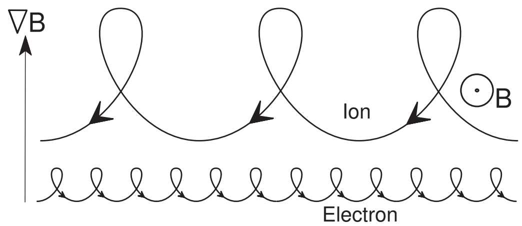

Figure 4.3: Illustration of the magnetic gradient drift.

4.2.4 Magnetic Curvature Drift

A particle which moves along a curved magnetic field line experiences a centrifugal force on its guiding center. This force is

$$\mathbf{F}_c = m v_\parallel^2 \frac{\mathbf{r}_c}{r_c^2}$$

where $\mathbf{r}_c$ denotes the radius of curvature vector. With a local coordinate system $\mathbf{e}_1 = \mathbf{B}/B$ and $\mathbf{e}_2 = -\mathbf{r}_c/r_c$ and noting that $\mathbf{e}_2/r_c = \partial \mathbf{e}_1/\partial s$ where s is the line element along the field line we can rewrite the curvature term as

$$\frac{\mathbf{r}_c}{r_c^2} = -\frac{\partial}{\partial s}\left(\frac{\mathbf{B}}{B}\right) = -\frac{1}{B}\frac{\partial \mathbf{B}}{\partial s} + \frac{\mathbf{B}}{B^2}\frac{\partial B}{\partial s} \tag{4.15}$$

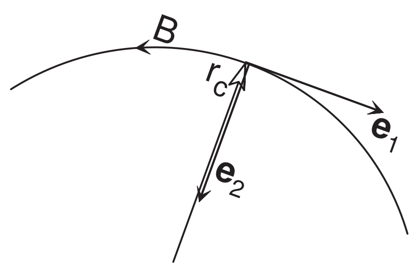

Figure 4.4: Illustration of the curvature radius.

We can neglect the last term because of the cross product with $\mathbf{B}$ in the drift equation. With $\partial/\partial s = \mathbf{B} \cdot \nabla$ the curvature force is

$$\mathbf{F}_c = -m v_\parallel^2 \frac{1}{B^2} (\mathbf{B} \cdot \nabla) \mathbf{B}$$

Note also that $(\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla B^2/2 = (\nabla \times \mathbf{B}) \times \mathbf{B} = \mu_0 \mathbf{j} \times \mathbf{B}$. In the absence of currents the curvature force becomes

$$\mathbf{F}_c = -m v_\parallel^2 \frac{1}{B} \nabla B$$

The resulting drift velocity is

$$\mathbf{v}_C = \frac{m v_\parallel^2}{q B^4} \mathbf{B} \times [(\mathbf{B} \cdot \nabla)\mathbf{B}] \tag{4.16}$$

or in the absence of electric currents

$$\mathbf{v}_C = \frac{m v_\parallel^2}{q B^3} \mathbf{B} \times (\nabla B) \tag{4.17}$$

4.2.5 Summary of Particle Drifts

$$\begin{aligned}
\text{Electric Field :} \quad & \mathbf{v}_E = \frac{1}{B^2}\mathbf{E}\times\mathbf{B} && (4.18)\\
\text{General Force :} \quad & \mathbf{v}_F = \frac{1}{qB^2}\mathbf{F}\times\mathbf{B} && (4.19)\\
\text{Polarisation :} \quad & \mathbf{v}_P = \frac{m}{qB^2}\frac{d\mathbf{E}_\perp}{dt} && (4.20)\\
\text{Curvature :} \quad & \mathbf{v}_C = \frac{mv_\parallel^2}{qB^4}\mathbf{B}\times\left[(\mathbf{B}\cdot\nabla)\,\mathbf{B}\right] && (4.21)\\
\text{Gradient :} \quad & \mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}\left(\mathbf{B}\times\nabla B\right) && (4.22)
\end{aligned}$$

Most drifts are associated with an electric current which is given by $\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e)$.

$$\begin{aligned}
Polarisation: \quad & \mathbf{j}_P = \frac{n\left(m_i+m_e\right)}{B^2}\frac{d\mathbf{E}_\perp}{dt} && (4.23)\\
Curvature: \quad & \mathbf{j}_C = \frac{n\left(m_i v_{i\parallel}^2 + m_e v_{e\parallel}^2\right)}{B^4}\mathbf{B}\times\left[(\mathbf{B}\cdot\nabla)\,\mathbf{B}\right] && (4.24)\\
Gradient: \quad & \mathbf{j}_\nabla = \frac{n\left(m_i v_{i\perp}^2 + m_e v_{e\perp}^2\right)}{2B^3}\left(\mathbf{B}\times\nabla B\right) && (4.25)
\end{aligned}$$

These drifts have been determined by model electric and magnetic fields. Thus they describe test particle motion if the electric and magnetic fields were in fact as assumed. However, it should be reminded that the currents due to the drifts alter the fields. If these changes are small compared to the background field it is justified to apply the drift model. The derived particle drifts do not contain any collective behavior. For this reason it is a nontrivial aspect to compare particle and fluid plasma drifts. We will return to this issue in a later chapter.

4.3 Adiabatic Invariants

For periodic motion with a period smaller than changes of the overall system the action integral

$$J_i = \oint p_i dq_i \tag{4.26}$$

is a constant of motion and an adiabatic invariant.

4.3.1 Magnetic Moment, First Adiabatic Invariant

In the absence of electric fields the parallel equation of motion is

$$m\frac{dv_\parallel}{dt} = -\mu\nabla_\parallel B = -\mu\frac{dB}{ds} \tag{4.27}$$

with the magnetic moment

$$\mu = \frac{mv_\perp^2}{2B} = \frac{W_\perp}{B}. \tag{4.28}$$

The parallel equation of motion also implies

$$\frac{dW_\parallel}{dt} = -\mu\frac{dB}{dt}$$

Conservation of energy

$$\frac{dW_\parallel}{dt} + \frac{dW_\perp}{dt} = 0$$

Substituting the magnetic moment

$$-\mu\frac{dB}{dt} + \mu\frac{dB}{dt} + B\frac{d\mu}{dt} = B\frac{d\mu}{dt} = 0$$

Which demonstrates that the magnetic moment is an invariant of the particle motion.

In the presence of an electric field and of a corresponding change $\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$ the magnetic moment is still conserved as long as the changes of the magnetic field are slow compared to the gyro motion. Note that the flux encircled by the gyro motion $\Phi_\mu = \pi r_g^2 B$ is

$$\Phi_\mu = \pi\frac{m^2v_\perp^2}{q^2B^2}B = \frac{2\pi m}{q^2}\mu \tag{4.29}$$

and therefore also conserved.

Magnetic Mirror

With the pitch angle α for the particle trajectory relative to the magnetic field orientation the magnetic moment becomes

$$\mu = \frac{mv^2\sin^2\alpha}{2B} \tag{4.30}$$

and in the absence of parallel electric fields (such that the energy is conserved) the pitch angle at two different location in the magnetic field must satisfy

$$\frac{\sin^2\alpha_2}{\sin^2\alpha_1} = \frac{B_2}{B_1} \tag{4.31}$$

This provides the information of the change of the pitch angle along the entire field line if it is known in one location. In particular we ca compute the condition for which the pitch angle becomes 90°, i.e., the condition for which a particle is reflected in the magnetic field. Assuming the field strength B_m at the point where a particle is mirrored we obtain anywhere on the field line

$$\sin\alpha = \sqrt{B/B_m} \tag{4.32}$$

In other words at a given location particle with a pitch angle smaller than α will be transmitted whereas particle with a larger pitch angle are mirrored.

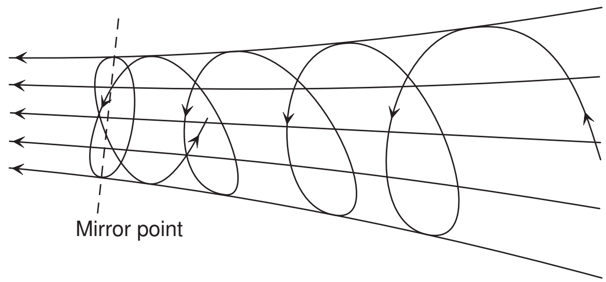

Figure 4.5: Illustration of magnetic mirror motion.

The is important for many laboratory and space plasmas. Particles are confined in a magnetic mirror by this mirror force. particle with a smaller pitch angle are lost from the magnetic field such that the distribution function miss this portion of phase space which is called the loss cone. The same principle governs the magnetosphere. Ions and electrons which penetrate into the lower ionosphere can undergo collisions with the neutrals and thereby are lost from the magnetospheric population.

Adiabatic Heating

Drift motion can bring particles into regions of the magnetosphere with a larger magnetic field strength. The conservation of the magnetic moment implies

$$W_{\perp 2} = \frac{B_2}{B_1} W_{\perp 1} \tag{4.33}$$

Thus particles can gain perpendicular energy (without a change of the parallel energy) by adiabatic motion. The process is similar to so-called betatron acceleration.

4.3.2 Second (Longitudinal) Adiabatic Invariant

The mirror motion implies a second quasi-periodic motion for a particle in a dipole field , i.e., the motion from one mirror point to the opposite and back with a bounce frequency of ω_b. For configurational changes on a time scale $\tau \gg 1/\omega_b$ the corresponding action integral

$$J = \oint m v_\parallel ds \tag{4.34}$$

is a longitudinal invariant of the particle motion.

In terms of an average parallel velocity $\left\langle v_\parallel \right\rangle$ the invariant is $J = 2ml \left\langle v_\parallel \right\rangle$ with l being the length of the entire field line between the mirror points. The square of this invariant implies for the parallel energy

$$\frac{\left\langle W_\parallel \right\rangle_2}{\left\langle W_\parallel \right\rangle_1} = \frac{l_1^2}{l_2^2} \tag{4.35}$$

Therefore as the length of the field line between mirror points changes, so does the parallel energy which is basic for so-called Fermi acceleration.

Exercise: Demonstrate that the momentum of an ideally reflecting ball which bounces between two walls which approach each other with a velocity u, satisfies $p_\parallel d = const$ where d is the distance between the walls and $p_\parallel$ is the momentum normal to the wall surface.

Thus the Adiabatic invariants provide insight into the change of anisotropy $A_W = \langle W_\perp \rangle / \left\langle W_\| \right\rangle$ for a distribution function.

$$\frac{A_{W2}}{A_{W1}} = \frac{B_2}{B_1} \frac{l_2^2}{l_1^2} \tag{4.36}$$

4.3.3 Third (Drift) Adiabatic Invariant

The third adiabatic invariant is the magnetic flux encircled by the (periodic) drift path of a particle.

$$\Phi = \oint v_d r d\psi \tag{4.37}$$

Similar to the other invariants it requires slow configurational changes $\tau \gg 1/\omega_d$ where $\omega_d is$ the frequency of the drift motion.

4.3.4 Violation of Adiabatic Invariants

Adiabatic invariants require that temporal changes of a configuration is slow compared to the period associated with the invariant. The relation

$$\omega_g \gg \omega_b \gg \omega_d \tag{4.38}$$

establishes a hierarchy of temporal scales. Configurational changes with $\tau \simeq 1/\omega_d$ destroy the third invariant while the first two are still conserved. Changes of order $\tau \simeq 1/\omega_b$ destroy the second and the third invariant and leave only the first invariant conserved. If changes occur on the gyro period scale all invariants are destroyed.

The same line of arguments can be made for spatial gradients because the particle motion is subject to the electromagnetic fields and spatial gradients can act similarly to temporal changes for the particle motion. There is also a hierarchy of spatial scales which limit the inhomogeneity of the field, for instance, gradients on the scale of the gyro radius will destroy the first adiabatic invariant.