Chapter 9

Quiet Magnetotail

The basic elements of the magnetotail of the magnetosphere (Figure 9.1) are

- Mantle (current): Region of open field with high density magnetosheath plasma.
- Lobes: Low density, strong magnetic field region. Energy is stored in the lobe magnetic field.
- Plasma sheet: Region of higher density and higher thermal pressure close to the equatorial plane. The plasma $\beta \geq O(1)$.
- Current sheet or neutral sheet: region of the cross tail current generating the lobe field. Here the magnetic field is rather weak thus the name neutral sheet.
- Field aligned currents: Birkeland currents originate from Earthward boundary of the current sheet and close in the ionosphere.

The magnetotail is the region of the magnetosphere where energy is stored. The tail plays an important role in magnetospheric substorms. Here the process of storage and release of energy in the tail determines much of the magnetospheric dynamics and is important for the coupling to the ionosphere and all related space weather effects.

9.1 Magnetotail models

9.1.1 Magnetic field models

There are number of models which describe the magnetic field of the magnetosphere [*Tsyganenko*, 1990; *Tsyganenko and Stern*, 1996; *Tsyganenko*, 2000]. These models use a suitable set of base functions and then fit the coefficients of these functions from large databases of satellite observations. The models can easily be parameterized for various IMF conditions, solar wind pressure, dipole tilt etc.

It is important to understand that these models are not equilibrium models. They do not provide any plasma data and in general it is not possible fit an isotropic pressure distribution to generate equilibria just from the magnetic field model.

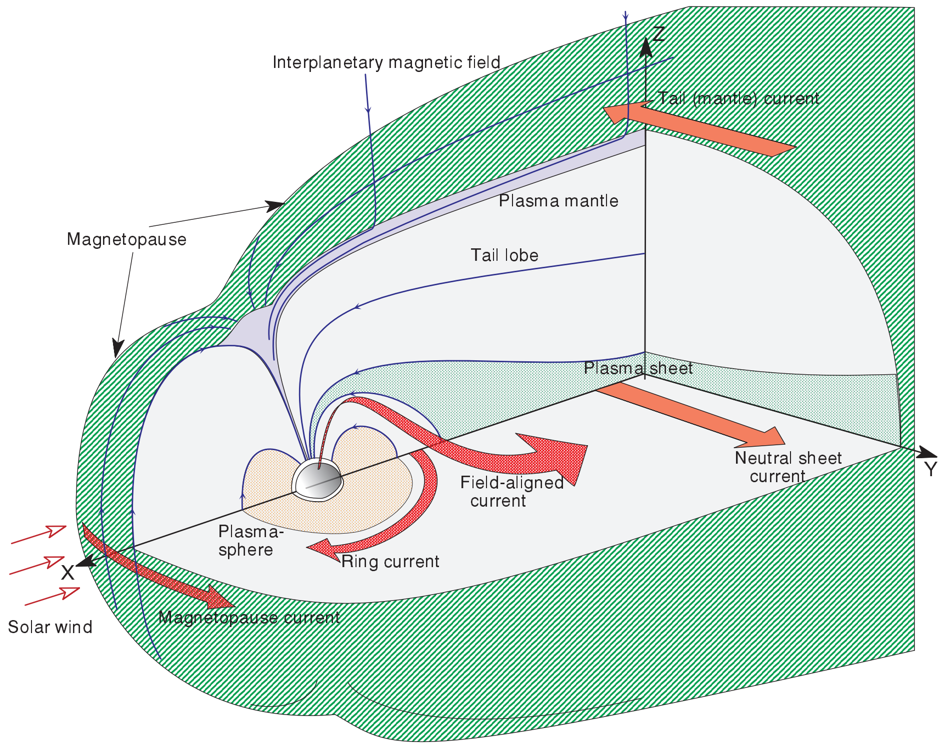

Figure 9.1: Sketch of the magnetosphere with various elements of the magnetotail and the magnetospheric current systems.

Magnetic field models have many different application. For instance they can provide a reference for spacecraft observations. These models are also used frequently to carry out test particle computations to study mechanisms which form distribution functions or to determine how particles can enter into certain regions of the magnetosphere. However, some caution is necessary particularly for this use. To compute particle trajectories both magnetic and electric fields are required. The latter are often assumed for instance as a constant in the cross tail direction these. A constant electric field, however, implies $\partial \mathbf{B}/\partial t = 0$ and therefore implies stationary convection in the magnetotail. It is highly questionable whether such a steady state convection exists. This point is discussed later in this section.

While the magnetic field models are not equilibrium models they can be used to obtain typical properties of the magnetic field such as the magnetic flux tube volume (Figure 9.2). This can for instance provide insight into magnetospheric convection.

9.1.2 Equilibrium Configuration

The magnetotail is surprisingly stable for long periods of time. Typically convection is small and the tail configuration is well described by equilibrium solutions. Analytic equilibria are available for the section of the magnetotail where the variation along the magnetotail (or in the cross-tail direction y) is small compared to the variation perpendicular to the current sheet (weakly two- or three-dimensional) (e.g., *Birn et al.*, 1975; *Schindler*, 1975; *Birn et al.*, 1977; *Schindler*, 1979a). Because of the variation in x the solutions are applicable only at sufficient distances ($\geq 10 R_E$) from the Earth. These equilibria can be constructed as fully kinetic solutions. In the MHD approximation they solve the static MHD equations.

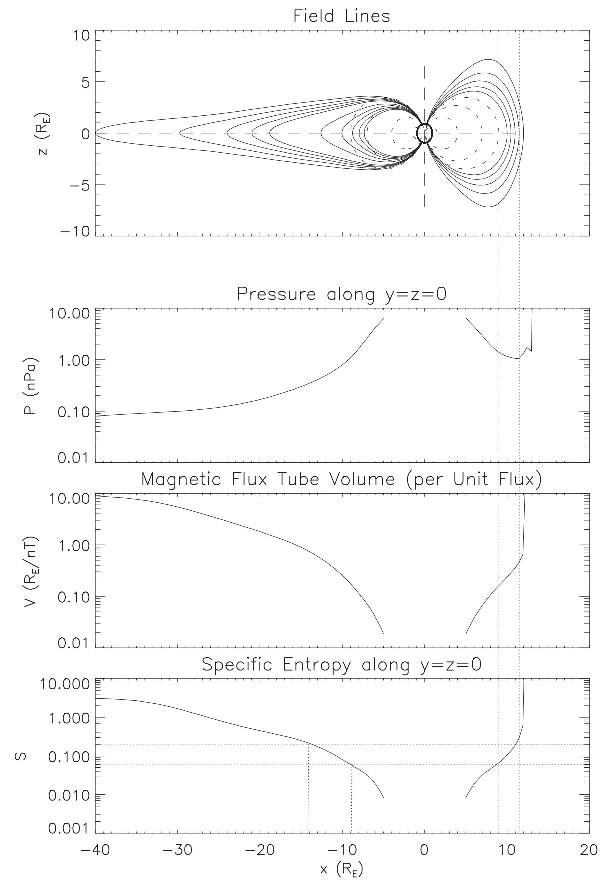

Figure 9.2: Illustration of the magnetic field and some associated properties such as pressure, magnetic flux tube volume, and entropy on magnetic field lines.

A simple example for this class of analytic solutions is the following solution which represents a two-dimensional modification of the classic Harris sheet configurations [*Harris*, 1962].

Kinetic equilibria can be constructed by assuming the distribution function as a function of the constants of motion. Specifically for the magnetotail the Harris solution can be derived by assuming

$$\frac{\partial}{\partial t} = 0 \quad \text{and} \quad \frac{\partial}{\partial y} = 0$$

Such that the constants of motion for the particle species s are

$$H_s = m_s v^2/2 + q_s \phi \quad \text{and} \quad P_{ys} = m_s v_y + q_s A_y$$

where $\phi$and A_y are the electric potential and the y component of the vector potential. Any function of the constants of motion $f_s(\mathbf{r}, \mathbf{v}) = F_s(H_s, P_{ys})$ solves the collisionless Boltzmann equation. To obtain/specify distribution functions that are in local thermodynamics equilibrium one can choose

$$F_s(H_s, P_{ys}) = c_s \exp(-\alpha_s H_s + \beta_s P_{ys})$$

where α_s and β_s need to be specified to obtain the required distribution functions. The exponent $W_s = -\alpha_s H_s + \beta_s P_{ys}$can be re-written as

$$\begin{aligned} W_s &= -\alpha_s\left[\frac{m_s}{2}\left(v_x^2+v_y^2+v_z^2\right)+q_s\phi-\frac{\beta_s}{\alpha_s}\left(m_s v_y+q_s A_y\right)\right] \\ &= -\alpha_s\left[\frac{m_s}{2}\left(v_x^2+\left(v_y-\frac{\beta_s}{\alpha_s}\right)^2+v_z^2\right)-\frac{m_s}{2}\frac{\beta_s^2}{\alpha_s^2}+q_s\phi-\frac{\beta_s}{\alpha_s}q_s A_y\right] \end{aligned}$$

This illustrates that $\alpha_s = 1/\left(k_B T_s\right)$ to obtain a Maxwell distribution (local thermodynamic equilibrium). Further, β_s/α_s should be interpreted as a constant velocity that shifts the distribution in the v_y direction. Note also that $\phi = \phi\left(\mathbf{r}\right)$ and $A_y = A_y\left(\mathbf{r}\right)$ which need to be determined through the solution of Maxwell's equations (Poisson equation and Ampere's law). Defining $w_s = \beta_s/\alpha_s$ and $\widehat{c}_s = c_s \exp\left(\alpha_s m_s w_s^2/2\right)$ (note that all terms in the exponential in $\widehat{c}_s$ are constants such that we can just re-define the normalization constant c_s. The complete distribution function for species s can now be written as

$$f_s\left(\mathbf{r},\mathbf{v}\right) = \widehat{c}_s \exp\left\{\frac{m_s}{2k_B T_s}\left(v_x^2+\left(v_y-w_s\right)^2+v_z^2\right)\right\}\exp\left\{-\frac{q_s}{k_B T_s}\left(\phi\left(\mathbf{r}\right)-w_s A_y\left(\mathbf{r}\right)\right)\right\}$$

Note that the first exponential depends only on velocity coordinates and the second depends only on spatial coordinates through ϕ and A_y. To obtain charge and current densities we need to integrate the distribution function over velocity space. The integral for the density also determines the normalization coefficient $\widehat{c}_s$. Let us abbreviate the second exponential function with

$$h\left(\phi\left(\mathbf{r}\right), A_y\left(\mathbf{r}\right)\right) = \exp\left\{-\frac{q_s}{k_B T_s}\left(\phi\left(\mathbf{r}\right)-w_s A_y\left(\mathbf{r}\right)\right)\right\}$$

Normalization: with the definition $n_s\left(\mathbf{r}\right) = n_{0s} h_s\left(\phi, A_y\right)$ and

$$\begin{aligned} n_s\left(\mathbf{r}\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_s\left(\mathbf{r},\mathbf{v}\right) d^3v \\ &= \widehat{c}_s h_s\left(\phi, A_y\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{\frac{m_s}{2k_B T_s}\left(v_x^2+\left(v_y-w_s\right)^2+v_z^2\right)\right\} d^3v \\ &= \widehat{c}_s\left(\frac{2\pi k_B T_s}{m_s}\right)^{3/2}\exp\left\{-\frac{q_s}{k_B T_s}\left(\phi\left(\mathbf{r}\right)-w_s A_y\left(\mathbf{r}\right)\right)\right\} \end{aligned}$$

the normalization constant is

$$\widehat{c}_s = n_{0s}\left(\frac{m_s}{2\pi k_B T_s}\right)^{3/2}$$

Charge density ρ_c and current density $\mathbf{j}_s = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{v} f_s\left(\mathbf{r},\mathbf{v}\right) d^3v$ for species s are now obtained as

$$\begin{aligned}\rho_{cs} &= q_s n_{0s} h_s\left(\phi, A_y\right)\\ \mathbf{j}_s &= q_s n_{0s} w_s h_s\left(\phi, A_y\right)\mathbf{e}_y\end{aligned}$$

Note that current density in the x and z direction is 0 because of the symmetry. the current density along y can be formally computed from the above integral or it can be argued that the bulk velocity must be $w_s$because the distribution function is a Maxwellian (symmetric) shifted by the the velocity $w_s$in the $v_y$direction.

Assuming a plasma of single charged ions and electrons yields for the total charge and the current density

$$\begin{aligned}\rho_c &= e\left\{n_{0i}\exp\left\{-\frac{e}{k_B T_i}\left(\phi - w_i A_y\right)\right\} - n_{0e}\exp\left\{\frac{e}{k_B T_e}\left(\phi - w_e A_y\right)\right\}\right\}\\ j_y &= e\left\{n_{0i} w_i\exp\left\{-\frac{e}{k_B T_i}\left(\phi - w_i A_y\right)\right\} - n_{0e} w_e\exp\left\{\frac{e}{k_B T_e}\left(\phi - w_e A_y\right)\right\}\right\}\end{aligned}$$

With these sources we need to solve the Poisson equation and Ampere's law. Since we seek a solution on scales much larger than the Debye length we can assume a neutral plasma. Here is customary to assume either quasi-neutrality $\rho_c = 0$ or exact neutrality $\phi = 0$. Since the assumption of quasi-neutrality leads to basically the same result but requires a more complicated treatment let us start directly with exact neutrality $\phi = 0$. The condition $\rho_c = 0$ yields

$$n_{0i}\exp\left\{\frac{e w_i}{k_B T_i} A_y\right\} - n_{0e}\exp\left\{-\frac{e w_e}{k_B T_e} A_y\right\} = 0$$

this equation is solved by

$$\begin{aligned}n_{0e} = n_{0i} &= n_0\\ w_e &= -\frac{T_e}{T_i} w_i\end{aligned}$$

Substitution in the equation for current density and using $\nabla \times \mathbf{B} = -\Delta\mathbf{A} = \mu_0 \mathbf{j}$ yields

$$-\Delta A_y = \lambda\exp\left\{A_y/\kappa\right\}$$

with $\lambda = \mu_0 e n_0 w_i\left(1 + \frac{T_e}{T_i}\right)$ and $\kappa = k_B T_i/\left(e w_i\right)$. The above equation is called the Grad-Shafranov equation. There are various analytic solutions to this equation. Note that the specific form of the current density is due to the choice of the distribution function. Note, however, that our choice only permits two-dimensional solutions. To derive the Harris sheet let us consider a one-dimensional solution of the Grad-Shafranov equation.

$$\frac{d^2 A_y}{dx^2} = -\lambda\exp\left(A_y/\kappa\right)$$

Multiplying this equation with dA_y/dx and re-arranging yields

$$\frac{d}{dx}\left(\frac{1}{2}\left(\frac{dA_y}{dx}\right)^2 + \frac{\lambda}{\kappa}\exp\left(A_y/\kappa\right)\right) = 0 \quad (9.1)$$

$$\text{or} \qquad \frac{1}{2}\left(\frac{dA_y}{dx}\right)^2 + \frac{\lambda}{\kappa}\exp\left(A_y/\kappa\right) = \text{const} \quad (9.2)$$

Note that $B_z = dA_y/dx$ such that the above expression can also be written as

$$\frac{B_z^2}{2\mu_0} + \frac{\lambda}{\mu_0\kappa}\exp\left(A_y/\kappa\right) = \text{const}$$

which is the equation for total pressure balance where $B_z^2/2\mu_0$ is the magnetic and $\frac{\lambda}{\mu_0\kappa}\exp\left(A_y/\kappa\right)$ the thermal plasma pressure. We can now integrate (9.2) to obtain the solution for A_y as

$$A_y = A_0 \ln\cosh\frac{x}{L}$$

Reducing the length scale L and $A_0$to typical plasma properties yields

$$\begin{aligned} L &= \lambda_i \frac{v_{thi}}{|w_i|}\sqrt{\frac{2T_i}{T_i+T_e}} \\ A_0 &= -2\frac{k_B T_i}{e w_i} \\ B_0 &= -\text{sgn}\left(w_i\right)\sqrt{2\mu_0\left(p_{i0}+p_{e0}\right)} \end{aligned}$$

with $p_{s0} = n_0 k_B T_s$ and $v_{thi} = \sqrt{k_B T_i/m_i}$. These relations show that the magnetic field amplitude B_0 scale with the square root of the thermal pressure. The width L of the Harris sheet is proportional to the ion inertial length and scales with the ratio of ion thermal velocity v_{thi} to drift velocity w_i. The magnetic field, number density, current density, and pressure are

$$\begin{aligned} B_z &= B_0\tanh\frac{x}{L} \\ n &= n_0\cosh^{-2}\frac{x}{L} \\ j_y &= -j_0\cosh^{-2}\frac{x}{L} \\ p &= p_0\cosh^{-2}\frac{x}{L} \end{aligned}$$

with $j_0 = B_0/\mu_0$ and $p_0 = B_0^2/2\mu_0$. The magnetic field, density, pressure, and current density distributions are shown in Figure 9.3. The Harris sheet equilibrium consists of a sheet of current that separates two regions of oppositely directed magnetic field of equal magnitude. The magnetic field pressure outside of the current sheet is balanced by the thermal pressure inside the current sheet.

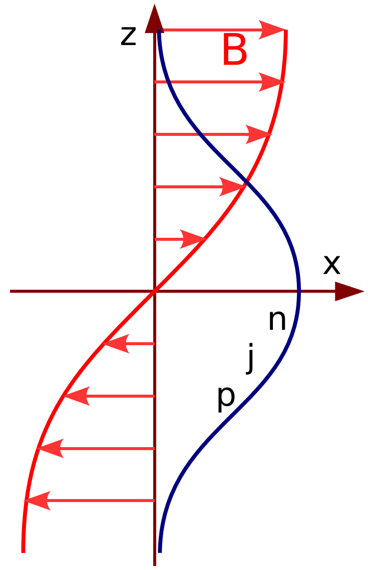

Figure 9.3: Illustration of the Harris magnetic field, density, pressure and current density configuration.

Asymptotically density, pressure, and current density go to zero at increasing distance from the current sheet. The configuration is clearly highly idealized and there are other analytic solutions of the Grad Shafranov equation. The importance of the Harris equilibrium is that it constitutes an exact analytic solution of the Vlasov equations. It has extensively been used to examine stability properties and linear instabilities of current sheets such as the magnetotail or the magnetopause currents.

However, it should be noted that equilibria are usually easier to obtain in a fluid or MHD plasma. To illustrate basic properties let us represent the magnetic field through

$$\mathbf{B}_\perp = \nabla \times A_z(x,y)\mathbf{e}_z = \nabla A_z(x,y) \times \mathbf{e}_z \tag{9.3}$$

From this form we can derive the current density in terms of the vector potential

$$\mathbf{j} = -\frac{1}{\mu_0}\Delta A_z \mathbf{e}_z + \frac{1}{\mu_0}\nabla B_z \times \mathbf{e}_z$$

such that the z component of the current density is

$$j_z = -\frac{1}{\mu_0}\Delta A_z$$

Substituting this representation of $\mathbf{B}$ and $\mathbf{j}$ into the static force balance equation in MHD demonstrates that the pressure must be also a function of A_z. Note that this is intuitively clear because pressure must be constant on magnetic field lines in an equilibrium. Including a nonzero B_z is also possible and it turns out that B_z is also a function of A_z. Summarizing all of these results and combining them with Ampere's law yields

$$
\begin{aligned}
\Delta A_z &= -\mu_0 \frac{d}{dA_z}\widetilde{p}(A_z) \\
\text{with} \qquad & \widetilde{p}(A_z) = p(A_z) + \frac{B_z(A_z)^2}{2\mu_0}
\end{aligned}
$$

This is also the Grad Shafranov equation, however now with a term on the right side that is a general function of A_z whereas this term was specified in our kinetic treatment through the assumption of local thermodynamic equilibrium.

The above equations can also be solved in two-dimensions. A common treatment particularly for the magnetotail is to assume that any variation in the direction down the tail is much smaller than the variations (gradients) perpendicular to the equatorial plane. One class of such solutions can be expressed through a modification of the Harris sheet solution:

$$
\begin{aligned}
A_y &= A_c \ln\cosh(z/l(\epsilon x)) + f(\epsilon x) \\
B_x &= -\partial A_y/\partial z = B_0(\epsilon x)\tanh(z/l(\epsilon x)) \\
B_z &= \partial A_y/\partial x
\end{aligned}
$$

Here $l(\epsilon x)$ is a general function which can be used to match either the pressure or magnetic field variation along the magnetotail. Figure 9.4 shows a configuration with a realistic pressure variation along the tail. The Earthward boundary is located at approximately 15 R_E. This class of analytic solutions is particularly useful for a number of studies such as the stability of particular configurations, pressure anisotropy in the tail, and as initial configurations for numerical simulations.

In addition to analytic equilibrium solutions there are numerical equilibrium solutions for the magnetotail. The resulting configurations are not subject to the constraints of analytic equilibria and can be extended much closer to the Earth. They provide insight into the mechanisms which cause field-aligned current and the structure at the earthward edge of the plasma sheet. There are two basic approaches one of which solves the MHD equilibrium equations directly using numerical iteration methods. The other approach starts from a magnetic field model with a good guess for the plasma and pressure distribution. A numerical relaxation method is then used to obtain an equilibrium configuration.

9.2 Convection in the Magnetotail

During periods of southward IMF one can attempt to approximate convection with a constant cross-tail component of the electric field. Considering a cut in the noon midnight meridian it is possible to evaluate the velocity perpendicular to the magnetic field using the $\mathbf{E} \times \mathbf{B}$ drift (Figure 3.10). For a constant electric field in the positive y direction (which should be equal to the dayside reconnection rate of$\approx 10^{-3}$ V/m) one obtains convection of lobe field lines (20 to 40 nT) toward the plasma sheet of a few 10 km/s. In the plasma sheet convection is directed toward the Earth with relatively large velocities of a few 100 km/s because of the weak magnetic field in the neutral sheet (few nT). The constant electric field implies $\partial \mathbf{B}/\partial t = 0$. Using long time averages (hours) the magnetic field may

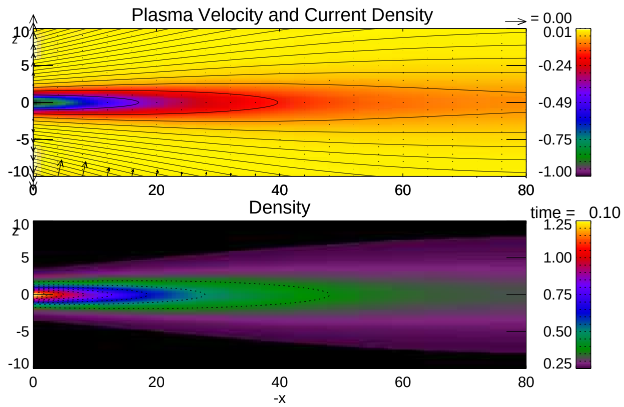

Figure 9.4: Magnetic field and current density (top) and plasma density of a two-dimensional tail equilibrium.

indeed not change much locally such that this pattern describes correctly the average convection over sufficiently long periods of time. However, there are important constraints to convection in the plasma sheet which need to be considered.

While stationary convection provides some insight into magnetospheric convection there are important constraints which prohibit stationary convection for most times. Using the MHD equations it is easy to show that the quantity $p\rho^{-\gamma}$ is conserved, i.e., $d\left(p\rho^{-\gamma}\right)/dt = 0$. Here $p\rho^{-\gamma}$ is a measure of entropy. By integrating over the length of a field line and defining the differential flux tube volume as

$$V = \int ds/B$$

one can find two additional conserved quantities which are the number of particles N and the specific entropy S on field lines.

$$\begin{aligned} N &= \int \frac{\rho dl}{B} \\ S &= pV^{\gamma} \end{aligned}$$

Strictly these conservation laws apply only in ideal MHD and if there is no loss of particles or kinetic energy into the ionosphere. However, for typical applications non MHD effects and/or loss in the ionosphere are negligible. The conservation laws imply that if a field line is convected from 40 R_E to 10 R_E the number of particles and the entropy remain constant. However, the field line volume is much larger for a typical field line originating at 40 R_E then it is at 10 R_E. Using the example of the Tsyganenko model (Figure 9.2) the flux tube volume changes by about two orders of magnitude which implies that the pressure has to increase by more the 10^3 to maintain a constant entropy during convection. Note that the local magnetic field does not change for steady convection. However, this would yield an entirely unrealistic pressure close at 10 R_E such that convection for this field model cannot be stationary [*Schindler*, 1979b; *Erickson and Wolf*, 1980; *Birn and Schindler*, 1983]. Note that a realistic pressure distribution, which yields approximately an equilibrium in the noon midnight meridian, yields an entropy that varies by two orders of magnitude (Figure 9.2) thereby excluding steady state convection. Both density and specific entropy distributions from satellite observations are not consistent with steady state convection from the mid-tail (30 to 40 R_E) to the near Earth region.

Several model describe a quasi-static (slow) evolution of the magnetotail (e.g., [*Schindler*, 1979b; *Schindler and Birn*, 1982]) and demonstrate that the electric field (or convection) is shielded from the plasma sheet.