For the exam choose 3 from the 4 problems. All problems have equal weight for this exam. The exam is open book but closed homework.

1. Dipole magnetic field

a) Assume the magnetic field of the Earth to be dipolar. Consider a flux tube with a small quadratic cross section in the equatorial plane where the two inner corners are at the same radial distance r_0 and the two outer corners are at the distance $r_0(1+\varepsilon)$ with $\varepsilon \ll 1$. What is the latitudinal and azimuthal separation of the corner field lines at the Earth's surface. For the sake of numbers assume $r_0 = 4R_E$ and $\varepsilon = 10^{-3}$.

b) Is the cross section of the flux tube still a square and what is the aspect ratio at the Earth's surface? Does the aspect ratio depend on the L value (radial distance)?

c) Show explicitly that the magnetic flux through the crossection in the equatorial plane is the same as on the Earth's surface, i.e., that $\mathbf{B} \cdot \Delta\mathbf{A} = \mathrm{const}$ for your result (for the small value of ε you can assume $\mathbf{B}$ approximately constant for each crossection).

2. Anisotropic distribution function

An anisotropic (bi-Maxwellian) distribution function for a two-dimensional system ($\partial/\partial y = 0$) can be written as $F(H,\mu) = G(A)\exp(-\beta(A)H - \alpha(A)\mu B_0)$ with the Hamiltonian $H = (m/2)\left(v_\perp^2 + v_\parallel^2\right) + q\phi$, the adiabatic moment $\mu = (m/2)v_\perp^2/B$, the magnetic field strength B, and the y component of the vector potential A, where $G(A)$, $\beta(A)$, and $\alpha(A)$ are arbitrary functions.

a) Relate α and β to the temperatures $T_\perp$ and $T_\parallel$ for a bi-Maxwellian distribution.

b) Calculate the number density n by integrating F over velocity space to show

$$n(A,B) = \sqrt{\frac{2\pi k_B T_\parallel}{m}}\frac{2\pi k_B T_\perp}{m} G(A)\exp(-\beta q\phi)$$

c) Show that the parallel and perpendicular pressures are $p_\parallel = n(A,B)k_B T_\parallel$ and $p_\perp = n(A,B)k_B T_\perp$

d) Determine the pressure anisotropy $D = \frac{p_\parallel - p_\perp}{p_\parallel}$ as a function of B, B_0, β, and α.

3. Whistler waves

In the electron magnetohydrodynamic (EMHD) limit ($\mathbf{u} \equiv 0$) Ohm's law can be written as

$$\mathbf{E} = \frac{1}{ne}\mathbf{j} \times \mathbf{B} + \eta\mathbf{j}$$

Assume a uniform plasma with a magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$ and perturbations δB_x and $\delta B_y \propto \exp(ik_z z - i\omega t)$.

a) Use the continuity equation to show that $\delta n = 0$ in the EMHD ($\mathbf{u} \equiv 0$) approximation.

b) Use the the above form of Ohm's law with together with the induction equation and Ampere's law to derive equations for linear waves (in δB_x and δB_y). Show that the dispersion relation is $\omega = \pm k_z^2 B_0/(\mu_0 ne) - ik_z^2\eta/\mu_0 = \pm k_z V_A (k_z c/\omega_{pi}) - ik_z^2\eta/\mu_0$. These waves are the so-called whistler waves.

c) Show that the group and phase velocities are $\propto V_A(k_z c/\omega_{pi})$ for this wave for $\eta = 0$? What is the wave length when the phase velocity is V_A. Discuss the group and phase velocities in the limit of k_z to infinity.

4. Shocks

a) What are the conditions (in terms of the upstream velocity in the rest frame of the discontinuity) for the formation of a slow shock, a fast shock, and a rotational discontinuity?

b) Summarize the typical properties of slow shocks, fast shocks, and rotational discontinuities. What are switch-off and switch-on shocks and how do they relate to slow and fast shocks?

c) Assume a fixed upstream tangential field of B_0. Can the sheet current of a slow shock and a fast shock be greater than that of a rotational discontinuity? (The sheet current is $\left(B_{yu} - B_{yd}\right)/\mu_0$ for x pointing upstream and y along the upstream tangential field.). Are the directions of the slow shock current and the fast shock current the same as the direction of the current for the rotational discontinuity?

d) Why is the Earth's bow shock almost always a fast shock? What is the physical reason for the strong heating of plasma behind the bow shock?

e) What are the asymptotic values for the downstream density, magnetic field, and pressure if the upstream velocity goes to infinity? In this limit the downstream velocity also goes to infinity. How can this be if the downstream velocity has to satisfy the condition that it needs to be slower than the fast mode speed?