1. Dipole magnetic field

a) Assume the magnetic field of the Earth to be dipolar. Consider a flux tube with a small quadratic cross section in the equatorial plane where the two inner corners are at the same radial distance r_0 and the two outer corners are at the distance $r_0(1+\varepsilon)$ with $\varepsilon \ll 1$. What is the latitudinal and azimuthal separation of the corner field lines at the Earth's surface. For the sake of numbers assume $r_0 = 4R_E$ and $\varepsilon = 10^{-3}$.

b) Is the cross section of the flux tube still a square and what is the aspect ratio at the Earth's surface? Does the aspect ratio depend on the L value (radial distance)?

c) Show explicitly that the magnetic flux through the crossection in the equatorial plane is the same as on the Earth's surface, i.e., that $\mathbf{B} \cdot \Delta\mathbf{A} = \mathrm{const}$ for your result (for the small value of ε you can assume $\mathbf{B}$ approximately constant for each crossection).

Solution:

a) Flux tube mapping to Earth:

Fieldline equation: $r = R_{eq}\cos^2\lambda$

Latitude of outer field line at the Earth's surface: $\lambda_E = \arccos\sqrt{R_E/r_0} = \arccos\sqrt{1/4} = 60°$

Latitude of inner field line at the Earth's surface: $\lambda_{E1} = \arccos\sqrt{R_E/r_0(1+\varepsilon)} = \arccos\sqrt{1/4.04} = 60.16°$

Latitudinal difference: $\Delta\lambda = 0.0164° = 0.286 \cdot 10^{-3} rad$ or 1.83km

Taylor expansion to lowest order in $\varepsilon = r_1/r_0$:

$$\begin{aligned}\lambda_{E1} &= \arccos\sqrt{\frac{R_E}{r_0}} + \varepsilon\frac{d}{d\varepsilon}\left(\arccos\sqrt{\frac{R_E}{r_0(1+\varepsilon)}}\right) = \arccos\sqrt{\frac{1}{L}} + \varepsilon\frac{d}{d\varepsilon}\left(\arccos\sqrt{\frac{1}{L(1+\varepsilon)}}\right)\\ &= \arccos\sqrt{\frac{1}{L}} + \varepsilon\left(-\frac{\sqrt{1/L}}{\sqrt{1-1/L}}\left(-\frac{1}{2}(1+\varepsilon)^{-3/2}\right)\right) = \arccos\sqrt{\frac{1}{L}} + \varepsilon\frac{1}{2\sqrt{L-1}}\end{aligned}$$

such that the **latitudinal separation** is $\Delta\lambda = \frac{\varepsilon}{2\sqrt{L-1}} = \frac{10^{-3}}{2\sqrt{4-1}} = 0.289 \cdot 10^{-3} rad$ or

Latitudinal distance: $x_1 = \Delta\lambda R_E = \frac{\varepsilon}{2\sqrt{L-1}}R_E = 1.85$km.

Azimuth (Longitude): Since dipole field lines are in planes with $\phi = const$ the **azimuthal separation** (in rad) at the Earth's surface is

$$\Delta\phi = \frac{\varepsilon r_0}{r_0} = \varepsilon = 10^{-3} rad$$

The physical **distance in longitude** depends on latitude as well:

$$y_1 = \varepsilon R_E \cos\lambda_E = \varepsilon R_E\sqrt{1/L} = 3.2\,km$$

b) Aspect ratio:

Thus the aspect ratio on the Earth's surface is

$$\delta = \frac{y_1}{x_1} = \frac{\varepsilon R_E}{L^{1/2}}\frac{2\sqrt{L-1}}{\varepsilon R_E} = \frac{2\sqrt{L-1}}{L^{1/2}} = 2\sqrt{1-1/L}$$

The aspect ratio depends on the L shell or on latitude of the foot points on the Earth. For high latitudes $L \gg 1$ the aspect ratio approaches 2.

c) Crossections and magnetic flux

Crossection in the equatorial plane: $A_{eq} = \varepsilon^2 r_0^2 = \varepsilon^2 L^2 R_E^2$

Crossection on the Earth's surface:

$$\begin{aligned} A_E &= y_1 x_1 = \varepsilon R_E L^{-1/2} \frac{\varepsilon}{2\sqrt{L-1}} R_E \\ &= \frac{\varepsilon^2 R_E^2}{2L^{1/2}\sqrt{L-1}} \end{aligned}$$

Magnetic field in the equatorial plane: $B_{eq} = B_E / L^3$ (normal to the equator) such that the flux is

$$\Phi_{eq} = B_{eq} A_{eq} = \frac{\varepsilon^2 R_E^2}{L} B_E$$

On the Earth's surface we could use the magnitude of B at λ_E and then determine the radial component through the orientation of $\mathbf{B}$ on the surface. It is easier to directly use the radial component of $\mathbf{B}$ for the dipole field

$$\begin{aligned} B_r\left(r = 1R_E\right) &= -2B_E R_E^3 \frac{\sin\lambda_E}{r^3} = -2B_E \sin\lambda_E \\ &= -2B_E\sqrt{1-\cos^2\lambda_E} = -2B_E\sqrt{1-1/L} \end{aligned}$$

Magnetic flux on the Earth's surface:

$$\begin{aligned} \Phi_E &= B_{r,E} A_E = 2B_E\sqrt{1-1/L}\frac{\varepsilon^2 R_E^2}{2L^{1/2}\sqrt{L-1}} \\ &= \frac{\varepsilon^2 R_E^2}{L} B_E \end{aligned}$$

Note that using the field line equation with $r = R_E$ implies $1 = L\cos^2\lambda_E$ such that all quantities could be expressed as functions of λ_E instead of L. Note also that, instead of computing $\Delta\lambda$ through mapping of $r_0\left(1+\varepsilon\right)$, we could have computed $\Delta\lambda$ through λ_E and $\Delta\phi$ using magnetic flux conservation.

2. Anisotropic distribution function

An anisotropic (bi-Maxwellian) distribution function for a two-dimensional system ($\partial/\partial y = 0$) can be written as $F(H,\mu) = G(A)\exp\left(-\beta(A)H - \alpha(A)\mu B_0\right)$ with the Hamiltonian $H = (m/2)\left(v_\perp^2 + v_\parallel^2\right) + q\phi$, the adiabatic moment $\mu = (m/2)v_\perp^2/B$, the magnetic field strength B, and the y component of the vector potential A, where $G(A)$, $\beta(A)$, and $\alpha(A)$ are arbitrary functions.

a) Relate α and β to the temperatures $T_\perp$ and $T_\parallel$ for a bi-Maxwellian distribution.

b) Calculate the number density n by integrating F over velocity space to show

$$n(A,B) = \sqrt{\frac{2\pi k_B T_\parallel}{m}}\frac{2\pi k_B T_\perp}{m}G(A)\exp\left(-\beta q\phi\right)$$

c) Show that the parallel and perpendicular pressures are $p_\parallel = n(A,B)k_B T_\parallel$ and $p_\perp = n(A,B)k_B T_\perp$

d) Determine the pressure anisotropy $D = \frac{p_\parallel - p_\perp}{p_\parallel}$ as a function of B, B_0, β, and α.

Solution:

(a) Distribution function:

$$\begin{aligned}
F(H,\mu) &= G(A)\exp\left(-\beta(A)H - \alpha(A)\mu B_0\right) \\
&= G(A)\exp\left(-\beta\frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right) - \beta q\phi - \alpha\frac{B_0 v_\perp^2}{B}\right) \\
&= G(A)\exp\left(-\beta\frac{m}{2}v_\parallel^2 - \beta q\phi - \left(\alpha\frac{B_0}{B} + \beta\right)\frac{m}{2}v_\perp^2\right)
\end{aligned}$$

Such that $kT_\parallel = 1/\beta$ and $kT_\perp = 1/\lambda$ with $\lambda = \alpha\frac{B_0}{B} + \beta$

(b) Calculate the number density n by integrating F over velocity space.

$$\begin{aligned}
n(A) &= G(A)\exp\left(-\beta q\phi\right)\int\int\int d^3v\exp\left(-\beta\frac{m}{2}v_\parallel^2 - \lambda\frac{m}{2}v_\perp^2\right) \\
&= 2\pi G(A)\exp\left(-\beta q\phi\right)\int_{-\infty}^{\infty}dv_\parallel\exp\left(-\beta\frac{m}{2}v_\parallel^2\right)\int_0^\infty dv_\perp v_\perp\exp\left(-\lambda\frac{m}{2}v_\perp^2\right) \\
&= 2\pi G(A)\exp\left(-\beta q\phi\right)\sqrt{\frac{2\pi}{\beta m}}\left(-\frac{1}{\lambda m}\exp\left(-\lambda\frac{m}{2}v_\perp^2\right)\right)_0^\infty \\
&= \sqrt{\frac{2\pi}{\beta m}}\frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)
\end{aligned}$$

(c) Show that the parallel and perpendicular pressures are $p_\parallel = n(A,B)k_B T_\parallel$ and $p_\perp = n(A,B)k_B T_\perp$

$$\begin{aligned}
p_\parallel &= mG(A)\exp\left(-\beta q\phi\right)\int\int\int d^3v\, v_\parallel^2\exp\left(-\beta\frac{m}{2}v_\parallel^2 - \lambda\frac{m}{2}v_\perp^2\right) \\
&= m\frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)\int_{-\infty}^{\infty}dv_\parallel v_\parallel^2\exp\left(-\beta\frac{m}{2}v_\parallel^2\right) \\
&= m\frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)\frac{\sqrt{\pi}}{2}\left(\frac{2}{\beta m}\right)^{3/2} \\
&= \frac{2\pi}{\lambda m}G(A)\exp\left(-\beta q\phi\right)\sqrt{\frac{2\pi}{\beta m}}kT_\parallel \\
&= n(A)kT_\parallel
\end{aligned}$$

$$\begin{aligned}
p_\perp &= mG(A)\exp(-\beta q\phi)\int\int\int d^3v v_\perp^2 \exp\left(-\beta\frac{m}{2}v_\parallel^2 - \lambda\frac{m}{2}v_\perp^2\right) \\
&= 2\pi m\sqrt{\frac{2\pi}{\beta m}}G(A)\exp(-\beta q\phi)\int_0^\infty dv_\perp \frac{1}{2}v_\perp^3 \exp\left(-\lambda\frac{m}{2}v_\perp^2\right) \\
&= \pi m\sqrt{\frac{2\pi}{\beta m}}G(A)\exp(-\beta q\phi)\left[\left(-\frac{v_\perp^2}{\lambda m} - \frac{2}{\lambda^2 m^2}\right)\exp\left(-\lambda\frac{m}{2}v_\perp^2\right)\right]_0^\infty \\
&= \pi\sqrt{\frac{2\pi}{\beta m}}G(A)\exp(-\beta q\phi)\frac{2}{\lambda m}kT_\perp = n(A)kT_\perp
\end{aligned}$$

(d) Determine the pressure anisotropy $D = \frac{p_\parallel - p_\perp}{p_\parallel}$ as a function of B, B_0, β, and η.

$$\begin{aligned}
D &= \frac{T_\parallel - T_\perp}{T_\parallel} \\
&= \frac{1/\beta - 1/\lambda}{1/\beta} = 1 - \frac{\beta}{\lambda} \\
&= 1 - \frac{1}{\frac{\alpha B_0}{\beta B} + 1}
\end{aligned}$$

Assuming B and $B_0 > 0$ this particularly implies:

$$\frac{T_\perp}{T_\parallel} = \frac{1}{\frac{\alpha B_0}{\beta B} + 1} = \begin{cases} < 1 & if \quad \alpha/\beta > 0 \\ > 1 & if \quad \alpha/\beta < 0 \end{cases}$$

Note also that this requires $\frac{\alpha B_0}{\beta B} > -1$ otherwise $T_\perp$ or $T_\parallel$ is negative.

3. Whistler waves

In the electron magnetohydrodynamic (EMHD) limit ($\mathbf{u} \equiv 0$) Ohm's law can be written as

$$\mathbf{E} = \frac{1}{ne}\mathbf{j} \times \mathbf{B} + \eta \mathbf{j}$$

Assume a uniform plasma with a magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_z$ and perturbations δB_x and $\delta B_y \propto \exp(ik_z z - i\omega t)$.

a) Use the continuity equation to show that $\delta n = 0$ in the EMHD ($\mathbf{u} \equiv 0$) approximation.

b) Use the the above form of Ohm's law with together with the induction equation and Ampere's law to derive equations for linear waves (in δB_x and δB_y). Show that the dispersion relation is $\omega = \pm k_z^2 B_0 / (\mu_0 ne) - ik_z^2 \eta/\mu_0 = \pm k_z V_A (k_z c/\omega_{pi}) - ik_z^2 \eta / \mu_0$. These waves are the so-called whistler waves.

c) Show that the group and phase velocities are $\propto V_A (k_z c/\omega_{pi})$ for this wave for $\eta = 0$? What is the wave length when the phase velocity is V_A. Discuss the group and phase velocities in the limit of k_z to infinity.

Solution:

(a) Show that $\delta n = 0$ in the EMHD ($\mathbf{u} \equiv 0$) approximation.

Continuity equation:

$$\frac{\partial \delta n}{\partial t} = -\nabla \cdot n \delta \mathbf{u} = 0$$

(b) Derivation of the dispersion relation:

The linearized induction equation becomes

$$\frac{\partial \delta \mathbf{B}}{\partial t} = -\nabla \times \delta \mathbf{E} = -\nabla \times \left(\frac{1}{ne} \delta \mathbf{j} \times \mathbf{B}_0 + \eta \delta \mathbf{j} \right)$$

or

$$\begin{aligned} -i\omega \delta B_x &= ik_z \left(-\frac{B_0}{ne} \delta j_x + \eta \delta j_y \right) \\ -i\omega \delta B_y &= -ik_z \left(\frac{B_0}{ne} \delta j_y + \eta \delta j_x \right) \end{aligned}$$

With Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ the current density is given by

$$\begin{aligned} \delta j_x &= -\partial_z \delta B_y / \mu_0 = -ik_z \delta B_y / \mu_0 \\ \delta j_y &= \partial_z \delta B_x / \mu_0 = ik_z \delta B_x / \mu_0 \end{aligned}$$

Inserting $\delta \mathbf{j}$ yields

$$\begin{aligned} \omega \delta B_x &= -\frac{k_z^2}{\mu_0} \left(i \frac{B_0}{ne} \delta B_y + i\eta \delta B_x \right) = -ik_z^2 \frac{B_0}{\mu_0 ne} \delta B_y - i\eta \frac{k_z^2}{\mu_0} \delta B_x \\ \omega \delta B_y &= \frac{k_z^2}{\mu_0} \left(i \frac{B_0}{ne} \delta B_x - i\eta \delta B_y \right) = ik_z^2 \frac{B_0}{\mu_0 ne} \delta B_x - i\eta \frac{k_z^2}{\mu_0} \delta B_x \end{aligned}$$

Re-arranging:

$$\begin{aligned} \left(\omega + i \frac{\eta k_z^2}{\mu_0} \right) \delta B_x &= -ik_z^2 \frac{B_0}{\mu_0 ne} \delta B_y \\ \left(\omega + i \frac{\eta k_z^2}{\mu_0} \right) \delta B_y &= ik_z^2 \frac{B_0}{\mu_0 ne} \delta B_x \end{aligned}$$

Multiplication of the two equations yields

$$\left(\omega + i\frac{\eta k_z^2}{\mu_0}\right)^2 = \left(k_z^2 \frac{B_0}{\mu_0 ne}\right)^2$$

With

$$\frac{B_0}{\mu_0 ne} = \frac{B_0}{(m_p\mu_0 n)^{1/2}}\left(\frac{m_p}{\mu_0 ne^2}\right)^{1/2} = V_A c\left(\frac{\epsilon_0 m_p}{ne^2}\right)^{1/2} = V_A \frac{c}{\omega_{pi}}$$

the dispersion relation is

$$\omega = \pm k_z^2 V_A \frac{c}{\omega_{pi}} - ik_z^2 \frac{\eta}{\mu_0}$$

with the damping rate $\gamma = k_z^2\eta/\mu_0$. Here the resistive diffusion time for a wve length λ_z is $\tau_{diff,\lambda} = \mu_0\lambda_z^2/\eta$ such that the exponential damping rate is $\gamma = 4\pi^2/\tau_{diff,\lambda}$. Here $c/\omega_{pi} = \lambda_i$ the ion inertia length.

(c) Phase velocity (along z) and group velocity $d\omega/dk_z$:

$$\frac{\omega}{k_z} = \pm k_z V_A \lambda_i \qquad \frac{d\omega}{dk_z} = \pm 2k_z V_A \lambda_i$$

Discussion of the wave propagation:

For a phase velocity of $\omega/k_z = \pm V_A$ the $\mathbf{k}$ vector satisfies $k_z = 2\pi/\lambda_z = 1/\lambda_i$ or the wave length is $\lambda_z = 2\pi\lambda_i$. For any wavelength smaller than this the phase velocity is faster. In particular for $k_z \to \infty$ (or $\lambda_z \to 0$) both the group and the phase velocities approach infinity. Note, however, that for $\lambda_z \to 0$ also the damping becomes infinitely fast. The time for the wave to travel one wave length is $\tau_\lambda = \lambda_z/\left(\omega/k_z\right) = \lambda_z/\left(k_z V_A \lambda_i\right) = \lambda_z^2/\left(2\pi V_A \lambda_i\right)$. The amount of resistive damping during this time is

$$\tau_\lambda \gamma = \frac{\lambda_z^2}{2\pi V_A \lambda_i}\frac{4\pi^2\eta}{\mu_0\lambda_z^2} = \frac{2\pi\lambda_i}{V_A}\frac{\eta}{\mu_0\lambda_i^2} = \frac{\tau_{Ai}}{\tau_{diff,i}}$$

Here $\tau_{Ai} = 2\pi\lambda_i/V_A$ i.e., the travel time of an Aflven wave over the distance $2\pi\lambda_i$ and $\tau_{diff,i} = \mu_0\lambda_i^2/\eta$, i.e., the diffusion time for the ion inertia length. Note that $\tau_{Ai}/\tau_{diff,i} = 2\pi R$ with R being the Lundquist-number based on the ion inertia length.

4. Shocks

a) What are the conditions (in terms of the upstream velocity in the rest frame of the discontinuity) for the formation of a slow shock, a fast shock, and a rotational discontinuity?

b) Summarize the typical properties of slow shocks, fast shocks, and rotational discontinuities. What are switch-off and switch-on shocks and how do they relate to slow and fast shocks?

c) Assume a fixed upstream tangential field of B_0. Can the sheet current of a slow shock and a fast shock be greater than that of a rotational discontinuity? (The sheet current is $(B_u - B_d)/\mu_0$ for x pointing upstream and y along the upstream tangential field.). Are the directions of the slow shock current and the fast shock current the same as the direction of the current for the rotational discontinuity?

d) Why is the Earth's bow shock almost always a fast shock? What is the physical reason for the strong heating of plasma behind the bow shock?

e) What are the asymptotic values for the downstream density, magnetic field, and pressure if the upstream velocity goes to infinity? In this limit the downstream velocity also goes to infinity. How can this be if the downstream velocity has to satisfy the condition that it needs to be slower than the fast mode speed?

Solution:

(a) Upstream conditions:

Fast shock: Upstream velocity faster than the fast mode speed

Rotational discontinuity: Upstream velocity Alfven speed (and parallel to the magnetic field in the upstream region)

Slow shock: Upstream velocity smaller or equal to the Alfven speed but faster than slow mode speed.

(b) Properties:

Fast shock: Compression of density, pressure (required for all shocks), compression of magnetic field such that downstream field is bent away from shock normal.

Rotational discontinuity: Not a shock. Downstream plasma density and pressure equal to upstream density and pressure. Downstream tangential magnetic field is equal in magnitude and opposite in direction to upstream field. Normal magnetic field required.

Slow shock: Upstream velocity smaller or equal to the Alfven speed but faster than slow mode speed. Density and pressure increase; down stream magnetic field smaller than upstream field such that magnetic field is bent toward shock normal.

Switch-off shock: Slow shock for which downstream tangential magnetic field is 0.

Switch-on shock: Fast shock for which the upstream tangential magnetic field is 0.

(c) Sheet current:

Fast shock: The maximum downstream tangential magnetic field is $4B_0$. Thus the maximum sheet current is $I_{fs} = -3B_0/\mu_0$

Rotational discontinuity: The downstream tangential magnetic field is $-B_0$. Thus the sheet current is $I_{rd} = 2B_0/\mu_0$.

Slow shock: The minimum downstream magnetic field is 0, such that the maximum sheet current density is B_0/μ_0.

(d) Bow shock:

The solar wind speed relative to Earth is almost always faster than the fast mode speed. Therefore the bow shock is almost always a fast shock. The strong heating is due to the conversion of flow energy into magnetic and thermal energy. Since the amount of magnetic compression is limited for a strong shock, a large (often dominant amount of energy) is going into plasma heating.

(e) Asymptotic values:

The maximum plasma and magnetic field compression is a factor of 4 of the upstream values. The downstream pressure goes to infinity in the limit of infinite upstream plasma velocity. Although the downstream velocity goes to infinity in this limit the velocity remains sub-fast because the fast mode speed (for pressure also going to infinity) approaches infinity faster than the downstream flow velocity.