1. Coordinate Systems

a) At what times at winter solstice, summer solstice, spring equinox, and fall equinox are geocentric equatorial inertial coordinates (GEI) and geographic coordinates (GEO) approximately equal?

b) For which dates are geocentric solar ecliptic (GSE) and geocentric solar magnetospheric coordinates approximately the same? For which dates is the difference largest?

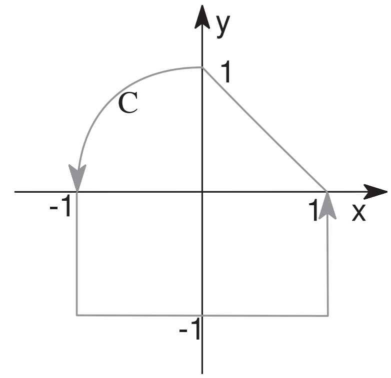
Solution:

(a) GEI is an inertial coordinate system which for all practical purposes is invariant for the astronomical background. GEO coordinates have the z axis (rotation axis) in common with GEI. At Spring equinox x_{GEI} points toward the sun, therefore the Greenwich meridian x_{GEO} has to be at noon (compare Figure 1.5) for the systems to agree. At Summer solstice x_{GEI} points toward dawn, i.e., the coordinates agree at 6 am UT. Correspondingly at Fall equinox the coordinate systems agree at midnight and at Winter solstice they agree at 6 pm UT.

(b) The answer is only approximate because there is a diurnal variation of the GSM coordinates since the rotation axis and magnetic dipole axis are offset. At the solstices the rotation axis is in the plane determined by x_{GSE} (toward the sun) and z_{GSE} (normal to ecliptic). Therefore the component of the rotation axis perpendicular to x_{GSE} is identical to the normal of the ecliptic. However, because of the diurnal variation of the magnetic north direction this occurs only twice a day during a time period around the solstices. The largest deviation between the coordinate systems occurs during the equinoxes. At that time the rotation axis is perpendicular to the sunward x direction and deviates by 23 degrees from the normal of the ecliptic. Magnetic north again has a diurnal variation of 11 degrees such that the maximum deviation between GSE and GSM is a rotation of about 34 degrees of the z and y axes at one time during the day at the equinoxes.

2. Conservative and Non-Conservative Forces

(a) A force field $\mathbf{K}(x, y)$ is determined by $\mathbf{K} = [xy + 1, y/(x^2 + y^2), 1]$. Calculate the closed line integral $\oint_C \mathbf{K} \cdot d\mathbf{s}$ along the contour C indicated in the figure on the right.
(b) Can $\mathbf{K}$ be derived from a potential and what does this imply? Summarize conditions or tests to demonstrate that a force (system) is conservative.



Solution:

(a) Calculate the closed line integral $\oint_C \mathbf{K} \cdot d\mathbf{s}$ along C:

We have to parameterize the line element $d\mathbf{s}$ for the contour:

(i) $(1, 0) - (0, 1)$: The line is $y = -x + 1$; with $x = s$ and $y = -s + 1$ the first path is

$$\begin{aligned}
I_a &= \int_1^0 (s(-s+1)+1)\,ds - \int_1^0 \frac{-s+1}{s^2+(-s+1)^2}ds \\
&= \int_0^1 \left(s^2 - s - 1\right) ds - \int_0^1 \frac{s-1}{2s^2-2s+1}ds \\
&= -\frac{7}{6} - \frac{1}{4}\int_0^1 \frac{4s-2-2}{2s^2-2s+1}ds \\
&= -\frac{7}{6} - \frac{1}{4}\ln\left(2s^2-2s+1\right)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{1}{2s^2-2s+1}ds \\
&= -\frac{7}{6} + \frac{1}{2}\arctan(2s-1)\Big|_0^1 \\
&= -\frac{7}{6} + \arctan(1) \\
&= -\frac{7}{6} + \frac{\pi}{4}
\end{aligned}$$

(ii) $(0, 1) - (-1, 0)$: Parameter φ with $x = \cos(\varphi)$, $y = \sin\varphi$, and $\varphi \in [\pi/2, \pi]$.

$$\begin{aligned}
I_b &= -\int_{\pi/2}^{\pi} (\cos\varphi\sin\varphi + 1)\sin\varphi d\varphi + \int_{\pi/2}^{\pi} \sin\varphi\cos\varphi d\varphi \\
&= -\int_{\pi/2}^{\pi} \left(\cos\varphi\sin^2\varphi + \sin\varphi\right) d\varphi + \int_{\pi/2}^{\pi} \sin\varphi\cos\varphi d\varphi \\
&= \left(-\frac{1}{3}\sin^3\varphi + \cos\varphi + \frac{1}{2}\sin^2\varphi)\right)_{\pi/2}^{\pi} \\
&= \frac{1}{3} - 1 - \frac{1}{2} \\
&= -\frac{7}{6}
\end{aligned}$$

(iii) $(-1,0)-(-1,-1)$: With $x=-1$ and $y=s$

$$I_c = \int_0^{-1} \frac{s}{s^2+1} ds = \frac{1}{2} \ln \left(s^2+1\right)_0^{-1} = \frac{1}{2} \ln 2$$

(iv) $(-1,-1)-(1,-1)$: With $x=s$ and $y=-1$

$$I_d = \int_{-1}^{1} (-s+1) ds = \left(-\frac{s^2}{2}+s\right)_{-1}^{1} = 2$$

(v) $(1,-1)-(1,0)$: With $x=1$ and $y=s$

$$I_e = \int_{-1}^{0} \frac{s}{s^2+1} ds = \frac{1}{2} \ln \left(s^2+1\right)_{-1}^{0} = -\frac{1}{2} \ln 2$$

Thus the sum over all terms is

$$\oint_C \mathbf{K} \cdot d\mathbf{s} = -\frac{1}{3} - \frac{\pi}{4}$$

(b) Can $\mathbf{K}$ be derived from a potential and what does this imply?

- No, $\mathbf{K}$ cannot be derived from a potential because the closed line integral is nonzero. This implies that the force is not conservative!

Conditions or tests to demonstrate that a force (system) is conservative.

- $\oint_C \mathbf{K} \cdot d\mathbf{s}$ over any closed contour must be zero.
- The force can derived from a potential $\mathbf{K} = -\nabla\phi$.
- $\nabla \times \mathbf{K} = 0$ for a conservative force.