3. Debye Screening

(a) Demonstrate that the plasma definition $\langle e\phi \rangle \ll \left\langle \frac{m}{2} v^2 \right\rangle = k_B T$ implies quasi-neutrality, i.e., $\Lambda = n\lambda_D^3 \gg 1$.

(b) The plasma parameter has the basic dependence $\Lambda \propto n_0^{-1/2} T^{3/2}$. While the dependence on temperature is intuitively clear, the density dependence appears odd because lower densities seems to imply fewer particles and less shielding. Why is intuition wrong and why does the plasma parameter improve (increase) with decreasing density?

Solution:

(a) With $\phi_{typ} \simeq e/(\epsilon_0 r_{typ})$ and $n \simeq 1/r_{typ}^3$

$$\Lambda = n \left(\frac{\epsilon_0 k_B T}{ne^2} \right)^{3/2} = n^{-1/2} \left(\frac{\epsilon_0 k_B T}{e^2} \right)^{3/2} \simeq r_{typ}^{3/2} \left(\frac{\epsilon_0 \frac{2}{3} \left\langle \frac{m}{2} v^2 \right\rangle}{e^2} \right)^{3/2} \simeq \left(\frac{\left\langle \frac{m}{2} v^2 \right\rangle}{\langle e\phi \rangle} \right)^{3/2} \gg 1$$

(b) Because of $n \simeq 1/r_{typ}^3$ decreasing density n implies a larger typical separation of particles. However, for a larger separation the potential energy becomes smaller ($\sim 1/r$), i.e., less important compared to the thermal energy, such that the plasma approximation improves for constant average kinetic energy.

4. Plasma properties

a) Assume a plasma density of 1 cm^{-3}, temperature equivalent to 1 keV, and a magnetic field of 20 nT which are typical for the near Earth magnetotail. Determine Debye radius and the plasma parameter Λ.

b) Compute the electron plasma frequency, the collision frequency for electrons, and the thermal velocity, and the mean free path of an electron based on these numbers. Can one neglect collisions for these electrons for the length scales of the magnetosphere of 100 R_E? (1 R_E=6400 km)

Solution: Note $1\,eV = 1.6 \cdot 10^{-19} J = k_B T_0 \;\hookrightarrow T_0 = 1.6 \cdot 10^{-19}/1.38 \cdot 10^{-23} K = 1.16 \cdot 10^4 K$

Measuring thermal energy in equivalents of $1eV$: $\varepsilon = \tilde{T} \cdot 1.6 \cdot 10^{-19} J$ (which yields an actual temperature of $T = \tilde{T} T_0$). Furthermore measuring density in particles/cm^3 through $n = \tilde{n} 10^6$

In this notation $\tilde{n} = 1$ and $\tilde{T} = 1000$. Below, λ_D and Λ are derived as general impression for input in units of eV and particles/cm^3. This was not needed for the homework.

(a) Debye length:

$$\begin{aligned} \lambda_D &= \left(\frac{\epsilon_0 k_B T}{n e^2}\right)^{1/2} = \left(\frac{\epsilon_0 e \tilde{T}}{n e^2}\right)^{1/2} \\ &= \left(\frac{8.85 \cdot 10^{-12} \tilde{T}}{10^6 \cdot 1.6 \cdot 10^{-19} \tilde{n}}\right)^{1/2} m = \left(\frac{8.85 \cdot 10}{1.6}\right)^{1/2} \left(\frac{\tilde{T}}{\tilde{n}}\right)^{1/2} m \\ &= 7.44 \left(\frac{\tilde{T}}{\tilde{n}}\right)^{1/2} m = 235 m \end{aligned}$$

Plasma parameter: Note that even though we used as input particles/cm^3 the Debye length has units of m. To get the plasma parameter we can just evaluate $\Lambda = n\lambda_D^3 = 1.3 \cdot 10^{13}$. To obtain a more general expression with n in particles/cm^3 as above we need λ_D in cm:

$$\begin{aligned} \Lambda &= n \left(\frac{\epsilon_0 k_B T}{n e^2}\right)^{3/2} = 10^6 \tilde{n} \lambda_D^3 \\ &= \tilde{n} \cdot 7.44^3 \cdot 10^{3 \cdot 2} \frac{\tilde{T}^{3/2}}{\tilde{n}^{3/2}} \\ &= 4.12 \cdot 10^8 \frac{\tilde{T}^{3/2}}{\tilde{n}^{1/2}} = 1.3 \cdot 10^{13} \end{aligned}$$

Sometimes the plasma parameter is defined as particles in a Debye sphere in which case one multiplies the result with a factor of $4\pi/3 \simeq 10$.

(b) Plasma frequency:

$$\begin{aligned} \omega_{pe} &= \left(\frac{n e^2}{m_e \epsilon_0}\right)^{1/2} = \left(\frac{10^6 \cdot 1.6^2 \cdot 10^{-2 \cdot 19}}{9.1 \times 10^{-31} \cdot 8.85 \cdot 10^{-12}}\right)^{1/2} \tilde{n}^{1/2} \\ &= \left(\frac{1.6^2 \cdot 10^3}{9.1 \cdot 8.85}\right)^{1/2} 10^4 \cdot \tilde{n}^{1/2} = 5.64 \cdot 10^4 \cdot \tilde{n}^{1/2} s^{-1} \end{aligned}$$

Collision frequency:

$$\begin{aligned}
\nu_{ei} &= \frac{\omega_{pe}}{4\pi\Lambda}\ln\Lambda = \frac{5.64\cdot 10^4}{4\pi\cdot 4.12\cdot 10^8}\frac{\tilde{n}}{\tilde{T}^{3/2}}\ln\left[4.12\cdot 10^8\frac{\tilde{T}^{3/2}}{\tilde{n}^{1/2}}\right]\ s^{-1}\\
&= 1.09\cdot 10^{-5}\frac{\tilde{n}}{\tilde{T}^{3/2}}\ln\left[4.12\cdot 10^8\frac{\tilde{T}^{3/2}}{\tilde{n}^{1/2}}\right]\ s^{-1}\\
&= 1.09\cdot 10^{-10}10^{1/2}\ln\left[1.3\cdot 10^{13}\right]\ s^{-1}\\
&= 3.44\cdot 10^{-10}\cdot 30.2\, s^{-1} = 1.04\cdot 10^{-8}\ s^{-1}
\end{aligned}$$

Alternatively (with a more accurate evaluation of the velocity dependence of the collision cross section)

$$\nu_c = \sqrt{\frac{\pi}{2}}\frac{1}{32\pi}\frac{\omega_{pe}}{\Lambda}\ln\left[12\pi\Lambda\right]$$

which gives an additional factor of $\sqrt{\pi/2}/8 = 0.157$ and a factor of 12π in the Coulomb logarithm $\ln\left[12\pi\Lambda\right] = \ln\left[4.9\cdot 10^{14}\right] = 33.8$ to yield

$$\nu_{ei} = 5.4\cdot 10^{-11}\cdot 33.8\, s^{-1} = 1.8\cdot 10^{-9}\ s^{-1}$$

Thermal velocity: There are different equally justified definitions but their use may depend on the particular application. Definitions: (a) $v_{th}^2 = \langle v^2\rangle = 3k_BT/m$ such that $\frac{1}{2}mv_{th}^2 = \frac{3}{2}k_BT =$ thermal energy; (b) by the average along only a single cartesian direction $v_{th}^2 = \langle v_i^2\rangle = k_BT/m$; (c) by the mean of the magnitude $v_{th} = \langle |v|\rangle = (8k_BT/\pi m)^{1/2}$ or; (d) by the halfwidth of a Maxwell distribution, i.e, $f \sim \exp\left(-v^2/v_{th}^2\right)$ which yields $v_{th}^2 = 2k_BT/m$. Since all contain the factor k_BT/m the result below uses definition (b) $v_{th} = \sqrt{k_BT/m}$.

$$v_{the} = \left(\frac{k_B}{m_e}\right)^{1/2} T_e^{1/2} = \left(\frac{1.38\cdot 10^{-23}}{9.1\cdot 10^{-31}}\right)^{1/2} T_e^{1/2} m\, s^{-1} = 3.9\cdot 10^3 T_e^{1/2} m\, s^{-1} = 1.3\cdot 10^7 m\, s^{-1}$$

with $T_e = 1.16\cdot 10^7 K$. Using temperature measured in eV $k_BT_e = e\tilde{T}_e$:

$$v_{the} = \left(\frac{e\tilde{T}_e}{m_e}\right)^{1/2} = \left(\frac{1.6\cdot 10^{-19}}{9.1\cdot 10^{-31}}\right)^{1/2} \tilde{T}_e^{1/2} m\, s^{-1} = 4.2\cdot 10^5 \tilde{T}_e^{1/2} m\, s^{-1} = 1.3\cdot 10^7 m\, s^{-1}$$

Mean free path:

$$\begin{aligned}
L_c &= \frac{v_{the}}{\nu_{ei}} = \frac{1.3\cdot 10^7}{1.04\cdot 10^{-8}} m = 1.25\cdot 10^{15} m = 1.95\cdot 10^8 R_E\\
&\quad \text{or for the more accurate collision frequency}\\
L_c &= \frac{1.3\cdot 10^7}{1.8\cdot 10^{-9}} m = 7.2\cdot 10^{15} m = 1.1\cdot 10^9 R_E
\end{aligned}$$

The mean free path is many order of magnitude larger than any dimension associated with the Earth's magnetosphere. Therefore the system is highly collisionless.

5. Plasma energy

(a) For the plasma in problem 4, determine the temperature in degrees Kelvin, and the thermal and magnetic energy density. How do these compare?

(b) Express the energy densities in kW hours/m^3 and kW hours /R_E^3. For the sake of simplicity assume that the plasma sheet is represented by a cylinder with 15 R_E radius and 100 R_E length. How long could a power plant with an output of 1000 MW operate on the thermal energy stored in the plasma?

Solution:

(a) With the note in Problem 4: A temperature of $1\,eV \hookrightarrow T_0 = 1.16 \cdot 10^4 K$

Thermal energy density:

$$\varepsilon_{th} = \frac{3}{2} n k_B T = 1.5 \cdot 10^6 \cdot 1.38 \cdot 10^{-23} \cdot 1.16 \cdot 10^7 m^{-3} J K^{-1} K = 2.4 \cdot 10^{-10} J/m^3$$

Magnetic energy density:

$$\varepsilon_B = B^2/2\mu_0 = \frac{2^2 \cdot 10^{-2 \cdot 8}}{2 \cdot 4 \cdot \pi \cdot 10^{-7}} m^{-3} J/m^3 = 1.6 \cdot 10^{-10} J/m^3$$

Thermal and magnetic energy density are comparable.

(b) Power plant part:

$$\begin{aligned} \varepsilon &= 2.4 \cdot 10^{-10} J/m^3 = \frac{1.6 \cdot 10^{-10}}{10^3 \cdot 3600} kW hours/m^3 \\ &= 6.7 \cdot 10^{-17} kW hours/m^3 \end{aligned}$$

- In kW hours /R_E^3: $\varepsilon = 6.7 \cdot 10^{-17} \cdot 6.4^3 \cdot 10^{18} = 1.7 \cdot 10^4$ kW hours /R_E^3.
- Energy in 15 R_E radius and 100 R_E length cylinder: $W = \pi \cdot 15^2 \cdot 10^2 \cdot 1.7 \cdot 10^4 = 1.2 \cdot 10^9$ kW hours
- Operation time for 1000 MW power plant $t = W/10^6 = 1200$ hours$\approx$ 50 days

6. Collisionless Boltzmann equation

(a) Show that any distribution function $f(\mathbf{x}, \mathbf{v}, t) = F(H, P_y)$ with $H = m\mathbf{v}^2/2 + q\phi(\mathbf{r})$ and $P_y = mv_y + qA_y$ solves the steady state ($\partial/\partial t = 0$) collisionless Boltzmann equation for $\partial/\partial y = 0$.

(b) Consider a distribution function of the form $F(H) = c_0 \exp(-H/k_B T)$. Show that the plasma density is given by $n(\mathbf{r}) = n_0 \exp(-q\phi/k_B T)$ and express c_0 in terms of n_0 and $k_B T$.

Solution:

(a) Collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0$$

Individual terms:

$$\frac{\partial f}{\partial t} = \frac{\partial F(H, P_y)}{\partial t} = 0$$

∇f terms:

$$\begin{aligned}
\mathbf{v} \cdot \nabla f &= \mathbf{v} \cdot \left(\frac{\partial F}{\partial H} \frac{\partial H}{\partial \phi} \nabla \phi + \frac{\partial F}{\partial P_y} \frac{\partial P_y}{\partial A_y} \nabla A_y \right) \\
&= -q \frac{\partial F}{\partial H} \mathbf{v} \cdot \mathbf{E} + q \frac{\partial F}{\partial P_y} \mathbf{v} \cdot \nabla A_y
\end{aligned}$$

and $\nabla_{\mathbf{v}} f$ terms:

$$\begin{aligned}
\frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f &= \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \left[\frac{\partial F}{\partial H} \nabla_{\mathbf{v}} H + \frac{\partial F}{\partial P_y} \nabla_{\mathbf{v}} P_y \right] \\
&= \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \left[\frac{m}{2} \frac{\partial F}{\partial H} \nabla_{\mathbf{v}} v^2 + m \frac{\partial F}{\partial P_y} \nabla_{\mathbf{v}} v_y \right] \\
&= q \frac{\partial F}{\partial H} \mathbf{E} \cdot \mathbf{v} + q \frac{\partial F}{\partial P_y} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{e}_y
\end{aligned}$$

Note $(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{e}_y = v_z B_x - v_x B_z$ with $B_x = \partial_y A_z - \partial_z A_y = -\partial_z A_y$ and $B_z = \partial_x A_y - \partial_y A_x = \partial_x A_y$. Further $E_y = -\partial_y \phi = 0$ because $\partial_y = 0$. Therefore

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{e}_y = -v_z \partial_z A_y - v_x \partial_x A_y = -\mathbf{v} \cdot \nabla A_y$$

such that the sum of the ∇f terms and the $\nabla_{\mathbf{v}} f$ terms is 0!

(b) With $F(H) = c_0 \exp(-H/k_B T)$ the density is given by

$$
\begin{aligned}
n(\mathbf{r}) &= \int_{-\infty}^{+\infty} d^3v f(\mathbf{r},\mathbf{v},t) = c_0 \int_{-\infty}^{+\infty} d^3v \exp\left(-H/k_BT\right) \\
&= c_0 \exp\left(-\frac{q\phi}{k_BT}\right) \int_{-\infty}^{+\infty} d^3v \exp\left(-\frac{m\mathbf{v}^2}{2k_BT}\right) \\
&= c_0 \exp\left(-\frac{q\phi}{k_BT}\right) \int_{-\infty}^{+\infty} dv_x \exp\left(-\frac{v_x^2}{v_{th}^2}\right) \int_{-\infty}^{+\infty} dv_y \exp\left(-\frac{v_y^2}{v_{th}^2}\right) \int_{-\infty}^{+\infty} dv_z \exp\left(-\frac{v_z^2}{v_{th}^2}\right) \\
&= c_0 \exp\left(-\frac{q\phi}{k_BT}\right) \left[\int_{-\infty}^{+\infty} dv \exp\left(-\frac{v^2}{v_{th}^2}\right)\right]^3 \\
&= v_{th}^3 c_0 \exp\left(-\frac{q\phi}{k_BT}\right) \left[\int_{-\infty}^{+\infty} d\tilde{v} \exp\left(-\tilde{v}^2\right)\right]^3 \\
&= \pi^{3/2} v_{th}^3 c_0 \exp\left(-\frac{q\phi}{k_BT}\right) = \left(\frac{2\pi k_BT}{m}\right)^{3/2} c_0 \exp\left(-\frac{q\phi}{k_BT}\right) \\
with \quad & v_{th}^2 = 2k_BT/m \qquad and \qquad \tilde{v} = v/v_{th}
\end{aligned}
$$

where we used the indefinite integral

$$
I_0 = \int_{-\infty}^{+\infty} d\tilde{v} \exp\left(-\tilde{v}^2\right) = \sqrt{\pi}
$$

in comparison with $n(\mathbf{r}) = n_0 \exp\left(-q\phi/k_BT\right)$ we find

$$
c_0 = \left(\frac{m}{2\pi k_BT}\right)^{3/2} n_0
$$