7. Plasma definition

Can a plasma be maintained at temperatures of $T_e = 100$ K (Hint: Calculate the density limit using the plasma parameter and explain your result).

Solution:

$$\Lambda = n\lambda_D^3 = n^{-1/2}\left(\frac{\epsilon_0 k_B T_e}{e^2}\right)^{3/2} \gg 1$$

Taking the square

$$\begin{aligned} n^{-1}\left(\frac{\epsilon_0 k_B T_e}{e^2}\right)^3 &= n^{-1}\left(\frac{8.85\cdot 10^{-12}\cdot 1.38\cdot 10^{-23}\cdot 10^2}{1.6^2\cdot 10^{-38}}\right)^3\left[m^{-3}\right] \\ &= n^{-1}\left(4.8\cdot 10^5\right)^3\left[m^{-3}\right] = n^{-1}\cdot 1.1\cdot 10^{17}\left[m^{-3}\right] \end{aligned}$$

or taking number density as particle per cm^{-3}

$$\sqrt{n} \ll 3.3\cdot 10^5$$

which implies that n could be as small as 10^7or even 10^8 cm^{-3}. While this is true there is the additional aspect of how long such a plasma could be maintained and with a comparatively small value of the plasma parameter of for instance 10^2 the collision frequency is still quite large and would lead to relatively fast recombination leading to a loss of this plasma within a small small time frame.

8. Moments of a distribution function

A Maxwellian velocity distribution function is given by

$$f(\mathbf{v}) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + (v_z - v_{z0})^2\right]\right)$$

Compute the average (bulk) velocity of particles described by the distribution function. The kinetic energy can be split into a thermal portion and a part which is caused by the bulk motion of particles. Compute the thermal and the bulk kinetic energy? Determine the value of the bulk velocity for which the bulk kinetic energy equals the thermal energy.

Solution:

Average Velocity:

$$\mathbf{u} = \frac{1}{n}\int_v d^3v \mathbf{v} f(\mathbf{v})$$

Since the distribution is symmetric for the v_x and v_y components the corresponding integrals are 0. For the z component we obtain by substituting $w_z = v_z - v_{z0}$

$$\begin{aligned} u_z &= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v d^3v v_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + (v_z - v_{z0})^2\right]\right) \\ &= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \left(w_z + v_{z0}\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\ &= v_{z0}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \end{aligned}$$

Here

$$\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) = 1$$

by definition because multiplyng the above equation with n is the definition of the plasma density such that

$$u_z = v_{z0}$$

Another way to obtain this result is to realize that the given distribution function is a Maxwell distribution transformed into the frame moving with v_{z0}. Since the average velocity of a Maxwell distribtion is 0 the average velocity of the transformed distribution must be v_{z0}.

Total kinetic energy:

$$E_{kin,tot} = \int_v d^3v \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) f(\mathbf{v})$$

With the transformation $w_z = v_z - v_{z0}$ this becomes:

$$E_{kin,tot} = \frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \left(v_x^2 + v_y^2 + (w_z + v_{z0})^2\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right)$$

$$
\begin{aligned}
&= \frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \left(v_x^2 + v_y^2 + w_z^2\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\
&\quad +2v_{z0}\frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z w_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\
&\quad +\frac{m}{2} n v_{z0}^2 \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right)
\end{aligned}
$$

Here the first term is the kinetic energy in restframe of the Maxwellian or the thermal kinetic energy, the second term is 0 because of anti-symmetry in w_z (note the integral has the bounds $-\infty \leq w_z \leq \infty$) and the last term represents the bulk kinetic energy. Because of the normalization of the integral the bulk kinetic energy is simply

$$
E_{bulk} = \frac{m}{2} n v_{z0}^2 = \frac{1}{2}\rho u_z^2
$$

The thermal kinetic energy is

$$
\begin{aligned}
E_{th} &= \frac{m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z \left(v_x^2 + v_y^2 + w_z^2\right) \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right) \\
&= \frac{3m}{2} n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_v dv_x dv_y dw_z v_x^2 \exp\left(-\frac{m}{2k_B T}\left[v_x^2 + v_y^2 + w_z^2\right]\right)
\end{aligned}
$$

because the v_x^2 term, the v_y^2 term, and the w_z^2 term have identical contributions to the energy. Substituting $v_x = \sqrt{k_B T/m}\,\widetilde{v}_x$, and the same for v_y and w_z yields

$$
\begin{aligned}
E_{th} &= \frac{3m}{2} n \left(\frac{1}{2\pi}\right)^{3/2} \frac{k_B T}{m} \int_v d\widetilde{v}_x d\widetilde{v}_y d\widetilde{w}_z \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\left[\widetilde{v}_x^2 + \widetilde{v}_y^2 + \widetilde{w}_z^2\right]\right) \\
&= \frac{3}{2} n k_B T \left(\frac{1}{2\pi}\right)^{3/2} \int_{-\infty}^{\infty} d\widetilde{v}_x \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\widetilde{v}_x^2\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\widetilde{v}_y d\widetilde{w}_z \exp\left(-\frac{1}{2}\left[\widetilde{v}_y^2 + \widetilde{w}_z^2\right]\right) \\
&= \frac{3}{2} n k_B T \left(\frac{1}{2\pi}\right)^{3/2} \int_{-\infty}^{\infty} d\widetilde{v}_x \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\widetilde{v}_x^2\right) \int_0^{\infty} dv_r 2\pi v_r \exp\left(-\frac{1}{2}v_r^2\right)
\end{aligned}
$$

With

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\widetilde{v}_y d\widetilde{w}_z \exp\left(-\frac{1}{2}\left[\widetilde{v}_y^2 + \widetilde{w}_z^2\right]\right) &= \left[\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2\right)\right]^2 \\
&= \int_0^{\infty} dv_r 2\pi v_r \exp\left(-\frac{1}{2}v_r^2\right) \\
&= 2\pi\left[-\exp\left(-\frac{1}{2}v_r^2\right)\right]_0^{\infty} = 2\pi
\end{aligned}
\tag{1}
$$

The thermal energy becomes

$$E_{th} = \frac{3}{2} n k_B T \left(\frac{1}{2\pi}\right)^{1/2} \int\limits_{-\infty}^{\infty} d\widetilde{v}_x \widetilde{v}_x^2 \exp\left(-\frac{1}{2}\widetilde{v}_x^2\right)$$

We can use integration by parts and the expression in (1) or alternatively we can use the integral expressions

$$\begin{aligned} K_0 &= \int_{-\infty}^{+\infty} \exp\left(-a^2 u^2\right) du = \frac{\sqrt{\pi}}{a} \\ K_2 &= \int_{-\infty}^{+\infty} u^2 \exp\left(-a^2 u^2\right) du = \frac{\sqrt{\pi}}{2a^3} \end{aligned}$$

to obtain the thermal energy density as

$$E_{th} = \frac{3}{2} n k_B T$$

Bulk kinetic energy equal to the thermal energy:

$$\begin{aligned} \frac{1}{2} m n u_z^2 &= \frac{3}{2} n k_B T \\ &or \\ u_z &= \left(\frac{3 k_B T}{m}\right) \end{aligned}$$

9. MHD equations

Assume a scalar pressure, $\mathbf{L} = 0$, $Q^E = 0$, and $Q^\rho = 0$. Consider a function $h(\rho\, p) = \rho^a p^b$ and determine a and b such that the resulting equation for h assumes a total derivative, i.e., $\partial h/\partial t + \mathbf{u} \cdot \nabla h = 0$. For $\gamma = 5/3$ this becomes the equation for an entropy function because entropy is conserved for adiabatic changes. (Hint: Use the continuity and the pressure equation to eliminate the time derivatives of ρ and p).

Solution:

The total derivative of $h(\rho\, p) = \rho^a p^b$ is

$$\begin{aligned} \frac{dh}{dt} &= \frac{\partial\left(\rho^a p^b\right)}{\partial t} + \mathbf{u} \cdot \nabla \left(\rho^a p^b\right) \\ &= b\rho^a p^{b-1}\left[\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p\right] + a\rho^{a-1} p^b \left[\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho\right] \end{aligned}$$

Using the continuity

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

and the pressure equations

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u}$$

we obtain

$$\begin{aligned} \frac{dh}{dt} &= -\gamma b \rho^a p^b \nabla \cdot \mathbf{u} - a\rho^a p^b \nabla \cdot \mathbf{u} \\ &= -\left[\gamma b + a\right] \rho^a p^b \nabla \cdot \mathbf{u} = 0 \end{aligned}$$

such that $a/b = -\gamma$. Choosing $b = 1$ yields $a = -\gamma$ and

$$\frac{d\left(p/\rho^\gamma\right)}{dt} = 0$$

10. Normalization

a) Following the example presented in class, determine typical values for the electric field v_0, E_0, and the pressure p_0 from the MHD equations in terms of L_0, ρ_0, B_0.

b) Using the normalization procedure, derive the coefficients of the inertial term and of the Hall term in generalized Ohms's law. Show that these coefficients are $(c/\omega_{pe})^2/L_0^2$ and $c/(\omega_{pi}L_0)$ respectively.

c) What are the values of these coefficients for the plasma parameters from problem 4 and $L_0 = 1\,R_E$?

Solution:

a) From class we know the normalization for $j_0 = \frac{B_0}{\mu_0 L_0}$. We re-write the momentum equation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{2}$$

for all variables in terms of $\rho = \rho_0 \widehat{\rho}$, $\mathbf{u} = v_0 \widehat{\mathbf{u}}$, $\mathbf{B} = B_0 \widehat{\mathbf{B}}$, etc where the $\widehat{}$ indicates normalised values. The momentum equation becomes

$$\frac{\rho_0 v_0}{t_0}\frac{\partial \widehat{\rho}\widehat{\mathbf{u}}}{\partial \widehat{t}} + \frac{\rho_0 v_0^2}{L_0}\widehat{\nabla} \cdot (\widehat{\rho}\widehat{\mathbf{u}}\widehat{\mathbf{u}}) = -\frac{p_0}{L_0}\widehat{\nabla}\widehat{p} + \frac{B_0^2}{\mu_0 L_0}\widehat{\mathbf{j}} \times \widehat{\mathbf{B}}$$

Division by $\frac{B_0^2}{\mu_0 L_0}$ yields

$$\frac{\rho_0 v_0 \mu_0 L_0}{t_0 B_0^2}\frac{\partial \widehat{\rho}\widehat{\mathbf{u}}}{\partial \widehat{t}} + \frac{\rho_0 v_0^2 \mu_0}{B_0^2}\widehat{\nabla} \cdot (\widehat{\rho}\widehat{\mathbf{u}}\widehat{\mathbf{u}}) = -\frac{p_0 \mu_0}{B_0^2}\widehat{\nabla}\widehat{p} + \widehat{\mathbf{j}} \times \widehat{\mathbf{B}}$$

Now we set the coefficients to unity. The second term yields $v_0^2 = B_0^2/(\mu_0 \rho_0)$ which is the typical Alfven speed. The first term give just the identity $v_0 = L_0/t_0$ and the pressure term yields $p_0 = B_0^2/\mu_0$. Doing the same for Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{m_e}{e^2 n}\left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u})\right] - \frac{M}{e\rho}\nabla p_e + \frac{m_i}{e\rho}\mathbf{j} \times \mathbf{B} + \eta \mathbf{j} \tag{3}$$

yields $E_0 = v_0 B_0$

b) The coefficient of the inertial term in generalized Ohms's law becomes

$$\begin{aligned}
\frac{m_e}{e^2 n_0}\frac{1}{v_0 B_0}\frac{j_0}{t_0} &= \frac{m_e}{e^2 n_0}\frac{1}{v_0 B_0}\frac{B_0}{\mu_0 L_0}\frac{v_0}{L_0} = \frac{m_e}{\mu_0 e^2 n_0}\frac{1}{L_0^2} \\
&= \frac{c^2 \epsilon_0 m_e}{e^2 n_0}\frac{1}{L_0^2} = \frac{c^2}{\omega_{pe}^2}\frac{1}{L_0^2} \\
&= \left(\frac{\lambda_e}{L_0}\right)^2
\end{aligned}$$

and for the Hall term

$$\begin{aligned}
\frac{1}{en_0}\frac{1}{v_0 B_0} j_0 B_0 &= \frac{1}{en_0}\frac{B_0}{\mu_0 v_0 L_0} = \frac{(\mu_0 m_i n_0)^{1/2}}{en_0 \mu_0 L_0} \\
&= \frac{m_i^{1/2}}{(\mu_0 e^2 n_0)^{1/2} L_0} = \left(c^2 \frac{m_i \epsilon_0}{e^2 n_0}\right)^{1/2} \frac{1}{L_0} \\
&= c\left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{m_e \epsilon_0}{e^2 n_0}\right)^{1/2} \frac{1}{L_0} = \frac{c}{\omega_{pi}}\frac{1}{L_0} \\
&= \frac{\lambda_i}{L_0}
\end{aligned}$$

where we used the definitions

$$\begin{aligned}
\text{eletron plasma frequency :} \quad & \omega_{pe} = \left(\frac{n_e e^2}{\epsilon_0 m_e}\right)^{1/2} \\
\text{ion plasma frequency :} \quad & \omega_{pi} = \left(\frac{n_e e^2}{\epsilon_0 m_i}\right)^{1/2} \\
\text{electron inertial length :} \quad & \lambda_e = c/\omega_{pe} \\
\text{ion inertial length :} \quad & \lambda_i = c/\omega_{pi}
\end{aligned}$$

and the condition: $\epsilon_0 \mu_0 c^2 = 1$.

c) Numerical values for the coefficients using $n_0 = 1$ cm^{-3} and $L_0 = 6.4 \cdot 10^6$ m:

$$\begin{aligned}
\omega_{pe} &= 5.64 \cdot 10^4 \cdot \tilde{n}^{1/2} s^{-1} = 5.64 \cdot 10^4\, s^{-1} \\
\omega_{pi} &= \left(\frac{m_e}{m_i}\right)^{1/2} \omega_{pe} = \frac{\omega_{pe}}{\sqrt{1836}} = 1.32 \cdot 10^3\, s^{-1} \\
\lambda_e &= c/\omega_{pe} = \frac{3 \cdot 10^8}{5.64 \cdot 10^4} m = 5.32\, km \\
\lambda_i &= c/\omega_{pi} = \frac{3 \cdot 10^8}{1.32 \cdot 10^3} m = 227\, km \\
\left(\frac{\lambda_e}{L_0}\right)^2 &= \left(\frac{5.32 \cdot 10^3}{6.4 \cdot 10^6}\right)^2 = 6.9 \cdot 10^{-7} \\
\frac{\lambda_i}{L_0} &= \frac{2.27 \cdot 10^5}{6.4 \cdot 10^6} = 3.5 \cdot 10^{-2}
\end{aligned}$$

The results demonstrate that the electron inertial terms play a role only for structures on the few km scale and are minute on the 1 R_E scale. Ione inertial effects become important on a few hundred km scale but are rather small 1 R_E scale structure.