11. MHD Equations: (a) Consider a two component (electrons and ions) charge neutral ($\rho_c = 0$) plasma where the total bulk velocity is defined by $\rho\mathbf{u} = n\left(m_i\mathbf{u}_i + m_e\mathbf{u}_e\right)$ with $\rho = n\left(m_i + m_e\right)$ and compute $\mathbf{u}_i$ and $\mathbf{u}_e$ as a function of $\mathbf{u}$ and the current density $\mathbf{j}$ only.

(b) Use the two fluid momentum equations for this plasma (eq. 2.27 in the manuscript) for isotropic pressure with $p = p_i + p_e$ to derive the single fluid momentum equation.

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \frac{m_e m_i}{e^2}\nabla\cdot(\frac{1}{\rho}\mathbf{j}\mathbf{j}) = -\nabla p + \mathbf{j}\times\mathbf{B}$$

(c) Demonstrate that this equation conserves momentum, i.e., that it can be brought into the form $\partial\rho\mathbf{u}/\partial t = -\nabla\cdot\underline{\underline{\mathbf{T}}}$. (Hint: Use Ampere's law and appropriate vector identities to modify the $\mathbf{j}\times\mathbf{B}$ term).

12. Dipole field in cylindrical coordinates: (a) Demonstrate that the transformation between spherical coordinate (using latitude λ) and cylindrical coordinates (z, R, φ) is given by

$$\begin{aligned}\mathbf{e}_r &= \mathbf{e}_z\sin\lambda + \mathbf{e}_R\cos\lambda\\ \mathbf{e}_\lambda &= \mathbf{e}_z\cos\lambda - \mathbf{e}_R\sin\lambda\end{aligned}$$

(b) Using this transformation, compute the Earth's dipole field in cylindrical coordinates. Show that this field can also be derived from the ϕ component of the vectorpotential $A_\varphi\left(z,R\right) = -\kappa R/\left(z^2+R^2\right)^{3/2}$ with $\kappa = \mu_0 M_E/\left(4\pi\right)$.

(c) Demonstrate that $f\left(z,R\right) = RA_\varphi\left(z,R\right)$ satisfies $\mathbf{B}\cdot\nabla f = 0$, i.e., such that $f = \mathrm{const}$ represents magnetic field lines in cylindrical coordinates. Derive the field line equation and demonstrate that it is identical to the equation we have derived in class for spherical coordinates.

13. Dipole magnetic field: (a) Assume the magnetic field of the Earth to be dipolar. Two magnetic field lines are radially separated by 1000 km in the magnetic equator at a distance of 5 Earth radii (5 R_E). What is the separation at the Earth's surface?

(b) Consider the superposition of a constant IMF $\mathbf{B}_{IMF} = -B_0\mathbf{e}_z$ to the dipole field of the Earth. Compute the radial distance of the X-line that separates open and closed field for $B_0 = 3\cdot10^{-9}T$ and $B_0 = 3\cdot10^{-8}T$. Determine the latitude on the Earth's surface of the boundary between closed and open magnetic field (polar cap) for the two IMF values. (Assume as illustrated in class that the IMF and the Earth's dipole field can be linearly superimposed. Hint: Use the field line equation to map the magnetic field lines connected to the X-line to the Earth's surface.)

Please turn in the solutions to the homework on Friday, 2/21/2014