11. MHD Equations: (a) Consider a two component (electrons and ions) charge neutral ($\rho_c = 0$) plasma where the total bulk velocity is defined by $\rho\mathbf{u} = n\left(m_i\mathbf{u}_i + m_e\mathbf{u}_e\right)$ with $\rho = n\left(m_i + m_e\right)$. Compute $\mathbf{u}_i$ and $\mathbf{u}_e$ as a function of $\mathbf{u}$ and the current density $\mathbf{j}$ only.

(b) Use the two fluid momentum equations for this plasma (eq. 2.27 in the manuscript) for isotropic pressure with $p = p_i + p_e$ to derive the single fluid momentum equation.

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \frac{m_e m_i}{e^2}\nabla\cdot(\frac{1}{\rho}\mathbf{j}\mathbf{j}) = -\nabla p + \mathbf{j}\times\mathbf{B} \tag{1}$$

(c) Demonstrate that this equation conserves momentum, i.e., that it can be brought into the form $\partial\rho\mathbf{u}/\partial t = -\nabla\cdot\underline{\underline{\mathbf{T}}}$. (Hint: Use Ampere's law and appropriate vector identities to modify the $\mathbf{j}\times\mathbf{B}$ term).

Solution:

(a) Using

$$\mathbf{j} = en\left(\mathbf{u}_i - \mathbf{u}_e\right) \tag{2}$$

$$\rho\mathbf{u} = n\left(m_i\mathbf{u}_i + m_e\mathbf{u}_e\right) \tag{3}$$

we can multiply equation (2) with m_e/e and take the sum to obtain $\mathbf{j}m_e/e + \rho\mathbf{u} = \left(m_i + m_e\right)n\mathbf{u}_i$ or

$$\mathbf{u}_i = \mathbf{u} + \frac{m_e}{enM}\mathbf{j}$$

We can now either use this result in equation (2) or (3) to obtain

$$\mathbf{u}_e = \mathbf{u} - \frac{m_i}{enM}\mathbf{j}$$

(b) Isotropic two fluid equations

$$\frac{\partial\rho_e\mathbf{u}_e}{\partial t} = -\nabla\cdot\left(\rho_e\mathbf{u}_e\mathbf{u}_e\right) - \nabla\cdot p_e - en(\mathbf{E} + \mathbf{u}_e\times\mathbf{B})$$

$$\frac{\partial\rho_i\mathbf{u}_i}{\partial t} = -\nabla\cdot\left(\rho_i\mathbf{u}_i\mathbf{u}_i\right) - \nabla\cdot p_i + en(\mathbf{E} + \mathbf{u}_i\times\mathbf{B})$$

Taking the sum of the two equations with $\rho\mathbf{u} = \rho_e\mathbf{u}_e + \rho_i\mathbf{u}_i$ yields for the time derivative term (with $\rho\mathbf{u} = n\left(m_i\mathbf{u}_i + m_e\mathbf{u}_e\right)$):

$$\frac{\partial\rho_e\mathbf{u}_e}{\partial t} + \frac{\partial\rho_i\mathbf{u}_i}{\partial t} = \frac{\partial\rho\mathbf{u}}{\partial t}$$

First term on the rhs (using the expressions for $\mathbf{u}_i$ and $\mathbf{u}_e$):

$$\rho_e\mathbf{u}_e\mathbf{u}_e + \rho_i\mathbf{u}_i\mathbf{u}_i = m_e n\left(\mathbf{u} - \frac{m_i}{enM}\mathbf{j}\right)\left(\mathbf{u} - \frac{m_i}{enM}\mathbf{j}\right) + m_i n\left(\mathbf{u} + \frac{m_e}{enM}\mathbf{j}\right)\left(\mathbf{u} + \frac{m_e}{enM}\mathbf{j}\right)$$

$$
\begin{aligned}
&= m_e n\left(\mathbf{uu}-\frac{m_i}{enM}\mathbf{uj}-\frac{m_i}{enM}\mathbf{ju}+\left(\frac{m_i}{enM}\right)^2\mathbf{jj}\right)\\
&\quad +m_i n\left(\mathbf{uu}+\frac{m_e}{enM}\mathbf{uj}+\frac{m_e}{enM}\mathbf{ju}+\left(\frac{m_e}{enM}\right)^2\mathbf{jj}\right)\\
&= Mn\mathbf{uu}+\frac{m_e m_i\left(m_e+m_i\right)}{e^2 nM^2}\mathbf{jj}=\rho\mathbf{uu}+\frac{m_e m_i}{e^2\rho}\mathbf{jj}
\end{aligned}
$$

For the pressure terms we can simply use the definition $p = p_i + p_e$.

Finally, for the last terms we obtain for the sum

$$
en(\mathbf{E}+\mathbf{u}_i\times\mathbf{B})-en(\mathbf{E}+\mathbf{u}_e\times\mathbf{B})=\mathbf{j}_i\times\mathbf{B}+\mathbf{j}_e\times\mathbf{B}=\mathbf{j}\times\mathbf{B}
$$

such that collecting all terms we recover the equation (1).

(c)

$$
\frac{\partial\rho\mathbf{u}}{\partial t}=-\nabla\cdot(\rho\mathbf{uu})-\nabla p+\mathbf{j}\times\mathbf{B}
$$

With $\nabla\times\mathbf{B}=\mu_0\mathbf{j}$ we obtain

$$
\frac{\partial\rho\mathbf{u}}{\partial t}=-\nabla\cdot(\rho\mathbf{uu})-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}
$$

Using the vector identities

$$
\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}\\
\nabla\cdot(\mathbf{AB}) &= (\nabla\cdot\mathbf{A})\mathbf{B}+(\mathbf{A}\cdot\nabla)\mathbf{B}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathbf{B}\times(\nabla\times\mathbf{B}) &= \frac{1}{2}\nabla(\mathbf{B}\cdot\mathbf{B})-(\mathbf{B}\cdot\nabla)\mathbf{B}\\
&= \frac{1}{2}\nabla B^2-\nabla\cdot(\mathbf{BB})+(\nabla\cdot\mathbf{B})\mathbf{B}\\
&= \frac{1}{2}\nabla B^2-\nabla\cdot(\mathbf{BB})
\end{aligned}
$$

because $\nabla\cdot\mathbf{B}=0$. Substitution in the momentum equation yields

$$
\begin{aligned}
\frac{\partial\rho\mathbf{u}}{\partial t} &= -\nabla\cdot(\rho\mathbf{uu})-\nabla p-\frac{1}{2\mu_0}\nabla B^2+\frac{1}{\mu_0}\nabla\cdot(\mathbf{BB})\\
&= -\nabla\cdot(\rho\mathbf{uu})-\nabla\cdot\left(p+\frac{1}{2\mu_0}B^2\right)\underline{\underline{\mathbf{1}}}+\frac{1}{\mu_0}\nabla\cdot(\mathbf{BB})
\end{aligned}
$$

or combining all terms

$$
\frac{\partial\rho\mathbf{u}}{\partial t}=-\nabla\cdot\left[\rho\mathbf{uu}+\left(p+\frac{1}{2\mu_0}B^2\right)\underline{\underline{\mathbf{1}}}-\frac{1}{\mu_0}\mathbf{BB}\right]
$$

which illustrates the concept of momentum conservation!

12. Dipole field in cylindrical coordinates: (a) Demonstrate that the transformation between spherical coordinate (using latitude λ) and cylindrical coordinates (z, R, φ) is given by

$$\begin{aligned}\mathbf{e}_r &= \mathbf{e}_z \sin\lambda + \mathbf{e}_R \cos\lambda \\ \mathbf{e}_\lambda &= \mathbf{e}_z \cos\lambda - \mathbf{e}_R \sin\lambda\end{aligned}$$

(b) Using this transformation, compute the Earth's dipole field in cylindrical coordinates. Show that this field can also be derived from the ϕ component of the vectorpotential $A_\varphi(z, R) = -\kappa R/\left(z^2 + R^2\right)^{3/2}$ with $\kappa = \mu_0 M_E/(4\pi)$.

(c) Demonstrate that $f(z, R) = RA_\varphi(z, R)$ satisfies $\mathbf{B}\cdot\nabla f = 0$, i.e., such that $f = \mathrm{const}$ represents magnetic field lines in cylindrical coordinates. Derive the field line equation and demonstrate that it is identical to the equation we have derived in class for spherical coordinates.

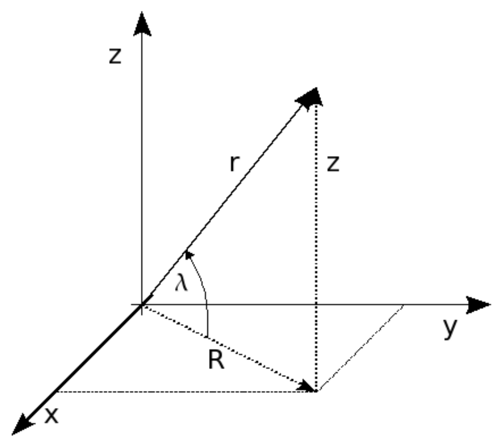
Solution:

(a) Cylindrical coordinates are determined by

$$\begin{aligned}z &= r\sin\lambda \\ R &= r\cos\lambda\end{aligned}$$

such that the inverse transformation is

$$\sin\lambda = z/r \qquad \cos\lambda = R/r$$
$$with \qquad r^2 = z^2 + R^2$$



The unit vector along the r direction has a z component of $\sin\lambda$ and an R component of $\cos\lambda$ such that it is given by $\mathbf{e}_z \sin\lambda + \mathbf{e}_R\cos\lambda$ (see Figure).

The unit vector along the λ direction has to have a negative value along the R direction and must be perpendicular to unit vector along the r direction. Therfore,

$$\begin{aligned}\mathbf{e}_r &= \mathbf{e}_z \sin\lambda + \mathbf{e}_R \cos\lambda \\ \mathbf{e}_\lambda &= \mathbf{e}_z \cos\lambda - \mathbf{e}_R \sin\lambda\end{aligned}$$

(b) In sperical coordinates the dipole magnetic field is given by

$$B_r = -2\kappa\frac{\sin\lambda}{r^3} \qquad B_\lambda = \kappa\frac{\cos\lambda}{r^3}$$

The magnetic field in cylindrical coordinates is then determined by

$$\begin{aligned}\mathbf{B} &= B_r\mathbf{e}_r + B_\lambda\mathbf{e}_\lambda \\ &= -2\kappa\frac{\sin\lambda}{r^3}\left(\mathbf{e}_z\sin\lambda + \mathbf{e}_R\cos\lambda\right) + \kappa\frac{\cos\lambda}{r^3}\left(\mathbf{e}_z\cos\lambda - \mathbf{e}_R\sin\lambda\right) \\ &= -3\kappa\frac{\sin\lambda\cos\lambda}{r^3}\mathbf{e}_R + \kappa\frac{1}{r^3}\left(\cos^2\lambda - 2\sin^2\lambda\right)\mathbf{e}_z \\ &= -3\kappa\frac{zR}{r^5}\mathbf{e}_R + \kappa\frac{1}{r^5}\left(R^2 - 2z^2\right)\mathbf{e}_z\end{aligned}$$

such that

$$B_z = \kappa \frac{R^2 - 2z^2}{\left(z^2 + R^2\right)^{5/2}} \qquad B_R = -3\kappa \frac{zR}{\left(z^2 + R^2\right)^{5/2}}$$

Derivation from the vectorpotential $A_\varphi\left(z, R\right) = -\kappa R / \left(z^2 + R^2\right)^{3/2}$ using $\mathbf{B} = \nabla \times A_\phi \mathbf{e}_\phi$

$$\begin{aligned}
B_R &= -\frac{\partial A_\phi}{\partial z} = -3\kappa \frac{zR}{\left(z^2 + R^2\right)^{5/2}} \\
B_z &= \frac{1}{R}\frac{\partial\left(RA_\phi\right)}{\partial R} = -\frac{\kappa}{R}\left[\frac{2R}{\left(z^2 + R^2\right)^{3/2}} - \frac{3R^2 R}{\left(z^2 + R^2\right)^{5/2}}\right] \\
&= -\kappa \frac{2\left(z^2 + R^2\right) - 3R^2}{\left(z^2 + R^2\right)^{5/2}} = -\kappa \frac{2z^2 - R^2}{\left(z^2 + R^2\right)^{5/2}}
\end{aligned}$$

which is identical to the magnetic field from the coordinate transformation.

(c) The gradient operator in cylindrical coordinates is

$$\nabla f = \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$$

such that

$$\begin{aligned}
\mathbf{B} \cdot \nabla f &= -\frac{\partial A_\phi}{\partial z}\frac{\partial f}{\partial R} + \frac{1}{R}\frac{\partial\left(RA_\phi\right)}{\partial R}\frac{\partial f}{\partial z} = 0 \\
&\text{implies} \\
\frac{1}{R}\frac{\partial\left(RA_\phi\right)}{\partial z}\frac{\partial f}{\partial R} &= \frac{1}{R}\frac{\partial\left(RA_\phi\right)}{\partial R}\frac{\partial f}{\partial z}
\end{aligned}$$

This is solved by the function $f = RA_\phi$ (or any power of RA_ϕ) such that magnetic field lines are given by

$$RA_\phi = -\frac{\kappa R^2}{\left(z^2 + R^2\right)^{3/2}} = const \tag{4}$$

Using a constant $-\kappa c_0$ with $\kappa c_0 > 0$ the field line equations are

$$\begin{aligned}
R^2 &= c_0\left(z^2 + R^2\right)^{3/2} \\
&or \\
z^2 &= c_0^{-2/3} R^{4/3} - R^2
\end{aligned}$$

Note that that this equation has a maximum value of R which is determined by $R = c_0^{-1}$. We can substitute $R = r\cos\lambda$ into (4) and the field line equations become

$$-\frac{\cos^2\lambda}{r} = \text{const}$$

which is the same as in spherical coordinates!

13. Dipole magnetic field: (a) Assume the magnetic field of the Earth to be dipolar. Two magnetic field lines are radially separated by 1000 km in the magnetic equator at a distance of 5 Earth radii (5 R_E). What is the separation at the Earth's surface?

(b) Consider the superposition of a constant IMF $\mathbf{B}_{IMF} = -B_0\mathbf{e}_z$ to the dipole field of the Earth. Compute the radial distance of the X-line that separates open and closed field for $B_0 = 3 \cdot 10^{-9}T$ and $B_0 = 3 \cdot 10^{-8}T$. Determine the latitude on the Earth's surface of the boundary between closed and open magnetic field (polar cap) for the two IMF values. (Assume as illustrated in class that the IMF and the Earth's dipole field can be linearly superimposed. Hint: Use the field line equation to map the magnetic field lines connected to the X-line to the Earth's surface.)

Solution:

(a) Fieldline equation: $\widetilde{r} = L\cos^2\lambda$

Latitude at the Earth's surface: $\lambda = \arccos\sqrt{1/L}$

For $L = 5$: 63.435°

For $L = 5 + 1000/6400 = 5.156$: 63.872°

This corresponds to a distance of about 48.8 km or 30 miles (Note 1° corresponds to about 111.7 km).

(b) Estimate the latitude on the Earth's surface from which the magnetic field is open into space for $\mathbf{B}_{IMF} = -B_0\mathbf{e}_z$ with $B_0 > 0$.

Equatorial magnetic field: $B_{eq} = -B_0 + B_E R_E^3/r^3$ with $B_E = 3.11 \cdot 10^{-5}T$

X line, $B_{eq} = 0$ => $r_{xl} = (B_E/B_0)^{1/3} R_E$ or for L-shell: $L_{xl} = (B_E/B_0)^{1/3}$

For $B_0 = 3 \cdot 10^{-9}T$: $r_{xl,1} = 21.8\, R_E$

For $B_0 = 3 \cdot 10^{-8}T$: $r_{xl,1} = 10.1\, R_E$

Field line equation:

$$r^2\cos^2\lambda\left(-\frac{B_0}{2} - \frac{B_E R_E^3}{r^3}\right) = const = r_{xl}^2\left(-\frac{B_0}{2} - \frac{B_E R_E^3}{r_{xl}^3}\right)$$

On Earth's surface $r = 1\, R_E$:

$$\begin{aligned}\cos^2\lambda_E\left(\frac{B_0}{2} + B_E\right) &= L_{xl}^2\left(\frac{B_0}{2} + \frac{B_E}{L_{xl}^3}\right)\\ &= \left(\frac{B_E}{B_0}\right)^{2/3}\left(\frac{B_0}{2} + B_0\right)\end{aligned}$$

or

$$\cos^2\lambda_E = \left(\frac{B_E}{B_0}\right)^{2/3}\frac{3B_0}{B_0 + 2B_E} = 3\left(\frac{B_E}{B_0}\right)^{2/3}\frac{1}{1 + 2B_E/B_0}$$

For $B_E \gg B_0$:

$$\cos^2\lambda_E \simeq 3\left(\frac{B_E}{B_0}\right)^{2/3}\frac{B_0}{2B_E} = \frac{3}{2}\left(\frac{B_0}{B_E}\right)^{1/3}$$

and

$$\lambda_E \simeq \arccos\left[\frac{3}{2}\left(\frac{B_0}{B_E}\right)^{1/3}\right]^{1/2} = \arccos\left[\frac{3}{2L_{xl}}\right]^{1/2}$$

For $B_0 = 3 \cdot 10^{-9} T$: $\quad \lambda_e = 74.79^\circ$

For $B_0 = 3 \cdot 10^{-8} T$: $\quad \lambda_e = 67.36^\circ$