14. Local expansion of $\mathbf{B}$: Consider the matrix

$$\nabla \mathbf{B} = \begin{pmatrix} \alpha_x & \gamma & \delta_x \\ \gamma' & \alpha_y & \delta_y \\ \beta_x & \beta_y & \alpha_z \end{pmatrix} \qquad (1)$$

with constant parameters α_i, β_i, δ_i, γ, and γ' for the expansion of the magnetic field at the origin.

(a) Assume the magnetic field at the origin $\mathbf{r} = 0$ has only a B_z component B_0. Determine the magnetic field in the vicinity of the origin to first order in x, y, and z.

(b) Derive the most general form of the matrix elements for a static vacuum magnetic field $\mathbf{j} = 0$ and satisfying $\nabla \cdot \mathbf{B} = 0$.

(c) Using the above result, demonstrate that the presence of nonzero curvature in a static vacuum field always implies the presence of a nonzero magnetic gradient and vice versa.

Solution:

a) At the origin $\mathbf{r} = 0$ has only a B_z component B_0. Magnetic field in the vicinity of the origin to first order in x, y, and z.

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \alpha_x x + \gamma' y + \beta_x z \\ \gamma x + \alpha_y y + \beta_y z \\ B_0 + \delta_x x + \delta_y y + \alpha_z z \end{pmatrix}$$

b) $\nabla \cdot \mathbf{B} = 0$:

=> $\alpha_x + \alpha_y + \alpha_z = 0$ => $\alpha_z = -\alpha_x - \alpha_y$

$\mathbf{j} = \nabla \times \mathbf{B} = 0$

x Component: $\delta_y - \beta_y = 0$ => $\delta_y = \beta_y$

y Component: $\beta_x - \delta_x = 0$ => $\delta_x = \beta_x$

z Component: $\gamma - \gamma' = 0$

Most general form

$$\nabla \mathbf{B} = \begin{pmatrix} \alpha_x & \gamma & \beta_x \\ \gamma & \alpha_y & \beta_y \\ \beta_x & \beta_y & -\alpha_x - \alpha_y \end{pmatrix}$$

c) Demonstrate that the presence of nonzero curvature in a static vacuum field always implies the presence of a nonzero magnetic gradient and vice versa.

Nonzero curvature implies either β_x or β_y nonzero. However, this implies that either δ_x or δ_y is nonzero.

15. Current layer magnetic field: Consider a magnetic field given by $\mathbf{B} = y\mathbf{e}_x + \alpha \sin(kx)\,\mathbf{e}_y$ and assume $\alpha,\ k > 0$.

(a) Derive the equations for the field lines.

(b) Determine the vector potential and the two Euler potentials.

(c) Sketch and discuss the field lines (Hint: determine the location of X and O lines first).

(d) Determine the condition for α and k such that the the current density at X lines is 0.

Solution:

a) Derive the equations for the field lines.

$$
\begin{aligned}
B_x = \partial_y A_z = y \quad &=> \quad A_z = \frac{1}{2}y^2 + f(x) \\
B_y = -\partial_x A_z = \alpha \sin(kx) \quad &=> \quad A_z = \frac{\alpha}{k}\cos(kx) + g(y)
\end{aligned}
$$

Combination of the two yields

$$A_z = \frac{1}{2}y^2 + \frac{\alpha}{k}\cos(kx)$$

Fields lines $A_z = const$: $\frac{1}{2}y^2 + \frac{\alpha}{k}\cos(kx) = C$

b) Determine the vector potential and the two Euler potentials.

The vector potential is already computed. The Euler potentials are

from $\mathbf{B} = \nabla\alpha \times \nabla\beta$. Comparison with the vector potential $\mathbf{B} = \nabla A \times \mathbf{e}_z$ yields

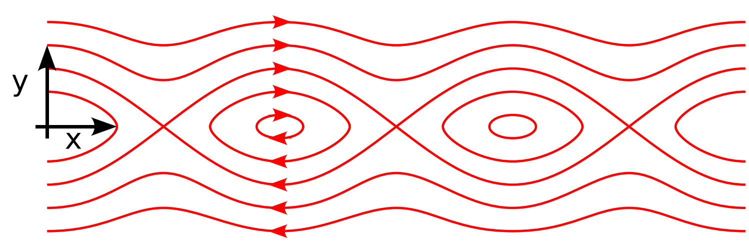
$$\nabla\beta = \mathbf{e}_z \quad or \quad \frac{\partial\beta}{\partial z} = 1$$

which yields

$$
\begin{aligned}
\alpha &= A_z \\
\beta &= z
\end{aligned}
$$

c) Sketch and discuss the field lines.

Sketch:



For $C \geq \frac{\alpha}{k}$ field lines have maxima and minima for $\sin kx = 0$. Maxima are for $x = 2n\pi/k$, minima for $x = (2n+1)\pi/k$ (for the upper branch $y = +\sqrt{2\left(C - \frac{\alpha}{k}\cos(kx)\right)}$ and opposite for the lower branch). Field lines are periodic in x with the period $2\pi/k$.

Expansion for locations with $\mathbf{B} = 0$:

i. $C = \frac{\alpha}{k}$, $x = 0$:

$\frac{1}{2}y^2 + \frac{\alpha}{k}(1 - k^2x^2) = \frac{\alpha}{k}$ => $y = \pm\alpha k x$ **Magnetic X lines**

ii. $C < \frac{\alpha}{k}$, $x = \pi + \widetilde{x}$:

$\frac{1}{2}y^2 - \frac{\alpha}{k}(1 - k^2\widetilde{x}^2) = C$ => $y^2 + 2\alpha k\widetilde{x}^2 = C$ **Ellipsoids - Magnetic O lines**

d) Determine the condition for α and k such that the the current density at X lines is 0.

$J_z = \frac{1}{\mu_0}(\partial_x B_y - \partial_y B_x) = \frac{1}{\mu_0}(\alpha k - 1)$

=> $J_z = 0$ for $\alpha = 1/k$

16. Radius of curvature: A magnetic field is given by $B_z = \epsilon B_0$, $B_x = B_0 z/L$.

(a) Compute the radius of curvature as a function of z, ϵ and L and show that the radius of curvature is $r_c = \epsilon L$ at $z = 0$.

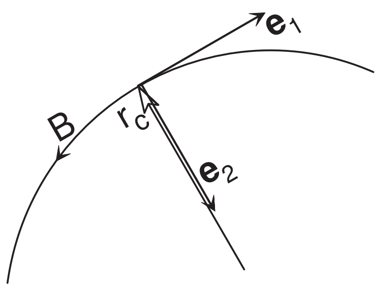
(b) Use $L = 2\,R_E$, and $\epsilon = 0.1$ to determine the centrifugal acceleration for 10 keV (parallel energy) electrons and protons at $z = 0$ in the magnetotail.

(c) Compute the resulting curvature drift velocity for these particles for $B_0 = 20\,nT$.

Solution:

With a local coordinate system $\mathbf{e}_1 = \mathbf{B}/B$ and $\mathbf{e}_2 = -\mathbf{r}_c/r_c$ and noting that $\mathbf{e}_2/r_c = \partial\mathbf{e}_1/\partial s$ where s is the line element along the field line we can rewrite the curvature term as

$$\frac{\mathbf{r}_c}{r_c^2} = -\frac{\partial}{\partial s}\left(\frac{\mathbf{B}}{B}\right) = -\frac{1}{B}\frac{\partial \mathbf{B}}{\partial s} + \frac{\mathbf{B}}{B^2}\frac{\partial B}{\partial s} \tag{2}$$



With $\partial/\partial s = \frac{1}{B}\mathbf{B}\cdot\nabla$

$$\frac{\mathbf{r}_c}{r_c^2} = -\frac{\partial}{\partial s}\left(\frac{\mathbf{B}}{B}\right) = -\frac{1}{B^2}\mathbf{B}\cdot\nabla\mathbf{B} + \frac{\mathbf{B}}{B^2}\mathbf{B}\cdot\nabla B$$

and

$$\begin{aligned} B_x &= B_0 z/L \\ B_y &= 0 \\ B_z &= \epsilon B_0 \\ B &= B_0\sqrt{\epsilon^2 + z^2/L^2} \\ g &= \sqrt{\epsilon^2 + z^2/L^2} \end{aligned}$$

we obtain:

$$\begin{aligned}
\frac{\mathbf{r}_c}{r_c^2} &= -\frac{B_0}{B^2}\left(B_x\partial_x + B_z\partial_z\right)\begin{pmatrix} z/L \\ 0 \\ \epsilon \end{pmatrix} + \frac{B_0^2}{B^3}\begin{pmatrix} z/L \\ 0 \\ \epsilon \end{pmatrix}\left(B_x\partial_x + B_z\partial_z\right)\sqrt{\epsilon^2 + z^2/L^2} \\
&= -\epsilon\frac{B_0^2}{B^2}\begin{pmatrix} 1/L \\ 0 \\ 0 \end{pmatrix} + \epsilon\frac{B_0^3}{B^3}\begin{pmatrix} z/L \\ 0 \\ \epsilon \end{pmatrix}\frac{z/L^2}{\sqrt{\epsilon^2 + z^2/L^2}} \\
&= -\epsilon\frac{1}{Lg^4}\begin{pmatrix} \epsilon^2 + z^2/L^2 \\ 0 \\ 0 \end{pmatrix} + \epsilon\frac{1}{Lg^4}\begin{pmatrix} z^2/L^2 \\ 0 \\ \epsilon z/L \end{pmatrix} \\
&= -\epsilon\frac{1}{Lg^4}\begin{pmatrix} \epsilon^2 \\ 0 \\ -\epsilon z/L \end{pmatrix} = -\epsilon^2\frac{1}{Lg^4}\begin{pmatrix} \epsilon \\ 0 \\ -z/L \end{pmatrix}
\end{aligned}$$

Absolute value of $\mathbf{r}_c/r_c^2$:

$$\frac{1}{r_c} = \left|\frac{\mathbf{r}_c}{r_c^2}\right| = \epsilon^2 \frac{1}{Lg^4}\sqrt{\epsilon^2 + z^2/L^2} = \frac{\epsilon^2}{Lg^3}$$

=>

$$r_c = \frac{L}{\epsilon^2}\left(\epsilon^2 + z^2/L^2\right)^{3/2}$$

(b) Centrifugal acceleration for 10 keV (parallel energy):

$$\begin{aligned}
\mathbf{F}_c\,(z=0) &= m v_\parallel^2 \frac{\mathbf{r}_c}{r_c^2} = -2\frac{m}{2} v_\parallel^2 \frac{1}{\epsilon L}\mathbf{e}_x \\
\frac{d\mathbf{v}}{dt} &= -\frac{2e}{m} 10^4 \frac{1}{0.1\cdot 6.4\cdot 10^6}\mathbf{e}_x \\
&= -\frac{1.6\cdot 10^{-19}}{1.67\cdot 10^{-27}} 1.56\cdot 10^{-2}\mathbf{e}_x = -3\cdot 10^6 m s^{-2}\mathbf{e}_x \qquad for\ protons \\
&= -\frac{1.6\cdot 10^{-19}}{9.11\cdot 10^{-31}} 1.56\cdot 10^{-2}\mathbf{e}_x = -5.5\cdot 10^9 m s^{-2}\mathbf{e}_x \qquad for\ electrons
\end{aligned}$$

(c) Curvature drift:

$$\begin{aligned}
\mathbf{v}_F &= \frac{\mathbf{F}\times\mathbf{B}}{qB^2} = \frac{1}{qB^2} m v_\parallel^2 \frac{\mathbf{r}_c}{r_c^2}\times\mathbf{B} \\
&= \frac{2}{qB}\frac{m v_\parallel^2}{2}\frac{1}{\epsilon L}\mathbf{e}_y = \frac{2}{e2\cdot 10^{-9}} e 10^4 \frac{1}{0.1\cdot 6.4\cdot 10^6}\mathbf{e}_y \\
&= 1.56\cdot 10^4 km s^{-1}\mathbf{e}_y \qquad for\ protons
\end{aligned}$$

The drift velocity is the same for electrons but in the opposite (-y) direction. Note, that the proton gyroradius is about 500 km. This is almost the curvature radius such that one can expect the protons to be nonadiabatic and the drift approximation is not applicable for protons. The electron gyroradius is by a factor of 40 ($\sqrt{m_p/m_e}$) smaller such that electron are still adiabatic and carry out this drift.

17. Loss cone: Calculate the size of the loss cone at the geomagnetic equator for particles on a dipole magnetic field line whose equatorial crossing distance is $5R_E$ ($1\ R_E = 6400km$). Assume the particles are mirrored in the ionosphere at an altitude of 100 km from the surface. How large is the difference in the loss cone if particles are lost at 1000 km altitude?

Solution:

a) Magnetic field line equation: $\cos^2\lambda = r/r_{eq}$ or $\sin^2\lambda = 1 - r/r_{eq}$

Variation of the magnetic field strength along the field line:

$$B = B_E \frac{R_E^3}{r^3}\left(1 + 3\sin^2\lambda\right)^{1/2} = B_E \frac{R_E^3}{r^3}\left(4 - 3r/r_{eq}\right)^{1/2}$$

Magnetic field at the equator for $L = L_0 = 5$: $B_{eq} = B_E/L_0^3$

Magnetic field at the mirror point with geocentric distance $r_m = xR_E$: $B_{mirror} = B_E \frac{R_E^3}{r_m^3}\left(4 - 3r_m/r_{eq}\right)^{1/2}$

Loss cone:

$$\sin^2\alpha_{eq} = \frac{B_{eq}}{B_{mirror}} = \frac{r_m^3}{L_0^3 R_E^3}\left(4 - 3r_m/r_{eq}\right)^{-1/2} = \frac{x^3}{L_0^3}\left(4 - 3x/L_0\right)^{-1/2}$$

- First altitude: $x = 6500/6400$: $\alpha = 3.87°$
- Second altitude $x = 7400/6400$: $\alpha = 4.73°$

b) Probability for a particle to be lost: $p = 1 - \cos\alpha$

- First altitude: $p = 0.0023$
- Second altitude: $p = 0.034$