**18 Gradient and curvature drift**

(a) A single proton has a parallel and perpendicular energy of 10 keV. Compute $(\mathbf{B} \times \nabla B)/B^3$ and determine the instantaneous curvature and gradient drift velocity for the Earth's dipole field in the magnetic equator at a radial distance of 5 $R_E$

(b) Consider an isotropic Maxwell plasma distribution for protons with a temperature $T = 10^8$ K in the equatorial plane. Evaluate the instantaneous bulk (average) velocity of the distribution at a radial distance of 5 $R_\mathrm{E}$ based on the gradient and curvature drift.

**Solution:**

**(a)** Gradient and curvature drifts in the absence of electric currents (dipole field):

$$\begin{aligned}
\mathbf{v}_\nabla &= \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B) \\
\mathbf{v}_C &= \frac{mv_\parallel^2}{qB^3}\mathbf{B} \times (\nabla B)
\end{aligned}$$

With the magnetic field ($B_E = 3.11 \cdot 10^{-5}$T)

$$\begin{aligned}
B_r &= -2B_E R_E^3 \frac{\cos\theta}{r^3} \\
B_\theta &= -B_E R_E^3 \frac{\sin\theta}{r^3} \\
B &= \frac{B_E R_E^3}{r^3}\left(1 + 3\cos^2\theta\right)^{1/2}
\end{aligned}$$

In the equatorial plane $\nabla B$ has only a radial component:

$$\nabla B = \partial B/\partial r\, \mathbf{e}_r = -3\frac{B_E R_E^3}{r^4}\,\mathbf{e}_r = -3\frac{B}{r}\,\mathbf{e}_r$$

$$\begin{aligned}
\frac{1}{B^3}(\mathbf{B} \times \nabla B)_{\theta=90^\circ} &= -3\frac{B}{r}\frac{B_\theta}{B^3}(\mathbf{e}_\theta \times \mathbf{e}_r) \\
&= -\frac{3}{r}\frac{B_E R_E^3}{r^3}\left(\frac{r^3}{B_E R_E^3}\right)^2 \mathbf{e}_\phi = -\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi
\end{aligned}$$

Gradient drift of an individual particle with $mv_\perp^2/2 = 10keV$:

$$\begin{aligned}
\mathbf{v}_\nabla &= \frac{mv_\perp^2}{2qB^3}(\mathbf{B} \times \nabla B) = -\frac{mv_\perp^2}{2q}\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi \\
&= -\frac{10^4 e}{e}\frac{3 \cdot 5^2}{3.11 \cdot 10^{-5} \cdot 6.4 \cdot 10^6}\mathbf{e}_\phi\, ms^{-1} = -10^3\frac{3 \cdot 5^2}{3.11 \cdot 6.4}\mathbf{e}_\phi\, ms^{-1} \\
&= -10^3\frac{3 \cdot 5^2}{3.11 \cdot 6.4}\mathbf{e}_\phi\, ms^{-1} = 3.77\, kms^{-1}
\end{aligned}$$

Curvature drift of an individual particle with $mv_\parallel^2/2 = 10keV$:

$$\mathbf{v}_C = \frac{mv_\parallel^2}{qB^3}(\mathbf{B}\times\nabla B) = -2\frac{mv_\parallel^2}{2q}\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi = 7.54\,kms^{-1}$$

such that the combined gradient and curvature drift is $\mathbf{v}_{D,10keV} = 11.3\,kms^{-1}$.

**(b)** Gradient and curvature drift for a Maxwell distribution with $T = 10^8K$:

Gradient drift (using z along the parallel direction):

$$\mathbf{u}_\nabla = \frac{1}{n}\int^\infty \left(\frac{mv_\perp^2}{2qB^3}\mathbf{B}\times\nabla B\right) f d^3v = \frac{1}{2nq}\frac{\mathbf{B}\times\nabla B}{B^3}\int^\infty m\left(v_x^2+v_y^2\right) f d^3v$$

From homework problem 8 we have for a Maxwell distribution $m\int^\infty v_x^2 f d^3v = m\int^\infty v_y^2 f d^3v = nk_BT$ such that

$$\begin{aligned}\mathbf{u}_\nabla &= \frac{1}{2nq}\frac{\mathbf{B}\times\nabla B}{B^3}2nk_BT = \frac{k_BT_\perp}{q}\frac{\mathbf{B}\times\nabla B}{B^3}\\ &= -\frac{k_BT_\perp}{q}\frac{3L^2}{B_ER_E}\mathbf{e}_\phi\end{aligned}$$

Curvature drift:

$$\begin{aligned}\mathbf{u}_C &= \frac{1}{n}\int^\infty \left(\frac{mv_\parallel^2}{qB^3}\mathbf{B}\times\nabla B\right) f d^3v = \frac{1}{nq}\frac{\mathbf{B}\times\nabla B}{B^3}\int^\infty mv_z^2 f d^3v\\ &= \frac{k_BT_\parallel}{q}\frac{\mathbf{B}\times\nabla B}{B^3} = -\frac{k_BT_\parallel}{q}\frac{3L^2}{B_ER_E}\mathbf{e}_\phi\end{aligned}$$

Such that the drifts are identical for the same parallel and perpendicular temperatures:

$$\begin{aligned}\mathbf{u}_\nabla &= \mathbf{u}_C = -\frac{k_BT}{q}\frac{3L^2}{B_ER_E}\mathbf{e}_\phi\\ &= -\frac{1.38\cdot10^{-23}10^8\cdot3}{1.6\cdot10^{-19}3.11\cdot10^{-5}\cdot6.4\cdot10^6}L^2\mathbf{e}_\phi\mathrm{ms}^{-1} = -130\,L^2\mathbf{e}_\phi\mathrm{ms}^{-1}\\ &= -3.25\,kms^{-1}\mathbf{e}_\phi\end{aligned}$$

and for the combined drift $\mathbf{u}_D = \mathbf{u}_\nabla + \mathbf{u}_C = -6.5\,kms^{-1}\mathbf{e}_\phi$

### 19. Loss cone distribution

Consider an initial Maxwell distribution function $f(v) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right)$ with density $n$ and temperature $T$ in the equatorial plane.

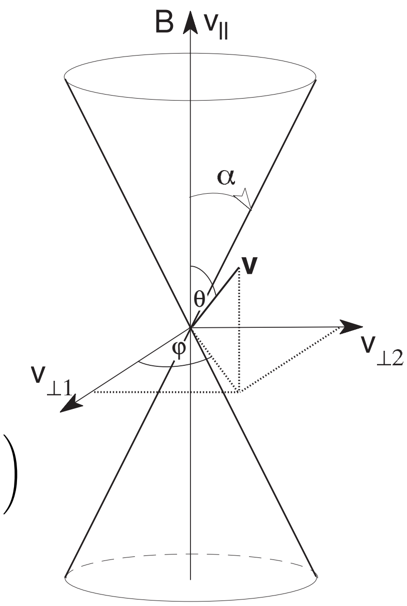
(a) For a given loss cone with angle $\alpha$ determine the fraction of particles lost from the isotropic distribution function. Determine the number density $\tilde{n}$ for the new distribution function $\widetilde{f}$ (distribution without particles in the loss cone; hint: Represent the distribution in velocity space in spherical velocity coordinates.).

(b) Compute the parallel and perpendicular energy density for the distribution function with the loss cone as a function of temperature $T$, density $n$, and angle $\alpha$.

(c) What is the angle $\alpha$ for the loss cone if the energy ratio is $W_{\|}/W_{\perp} = 1/4$?

**Solution:**

**(a)** For a given loss cone with angle $\alpha$ determine the fraction of particles lost from the originally isotropic distribution function. Determine the number density $\widetilde{n}$ for the new distribution function $\widetilde{f}$.With




$$\int_0^\infty x^2 \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{4a^3}$$

and using sperical coordinates for the velocity integration the number of particles in the loss cone is

$$\begin{aligned} n_L &= 2\int_0^{2\pi} d\varphi \int_0^\alpha \sin\vartheta d\vartheta \int_0^\infty v^2 f(v) dv \\ &= 4\pi\left[-\cos\alpha\right]_0^\alpha n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty dv v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) \\ &= 4\pi\left(1-\cos\alpha\right) n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \frac{\sqrt{\pi}}{4}\left(\frac{2k_B T}{m}\right)^{3/2} \end{aligned}$$

where $n$ is the original number of particles. Total Number of Particles in the loss cone:

$$n_L = n\left(1-\cos\alpha\right)$$

Particles remaining in the distribution: $\tilde{n} = n - n_L = n\cos\alpha$

Fraction of particles lost: $r = n_L/n = 1 - \cos\alpha$

The loss cone distribution function (= distribution function with the loss cone) does actually not change outside of the loss cone but should be re-written in the form

$$\widetilde{f}(\vartheta, v) = \frac{\widetilde{n}}{\cos\alpha}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) S(\vartheta, \alpha)$$

to account for the normalization to $\tilde{n}$. Here $S(\vartheta, \alpha)$ is a step function with satisfies $S(\vartheta, \alpha) = 1$ for $\alpha \leq \vartheta \leq \pi - \alpha$.

**(b)** Parallel and perpendicular energy:

With

$$\int_0^\infty x^4 \exp\left(-a^2x^2\right) dx = \frac{1}{2a^2}\int_0^\infty 3x^2 \exp\left(-a^2x^2\right) dx = 3\frac{\sqrt{\pi}}{8a^5}$$

and $v_\parallel = v\cos\vartheta$ and $v_\perp = v\sin\vartheta$ the energy densities are

$$\begin{aligned}
w_\parallel &= \int_0^{2\pi} d\varphi \int_\alpha^{\pi-\alpha} \sin\vartheta d\vartheta \int_0^\infty \frac{m}{2} v_\parallel^2 \tilde{f}(v) v^2 dv \\
&= 2\pi \frac{m}{2}\frac{\tilde{n}}{\cos\alpha}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_\alpha^{\pi-\alpha} \sin\vartheta\cos^2\vartheta d\vartheta \int_0^\infty v^4 \exp\left(-\frac{mv^2}{2k_BT}\right) dv \\
&= \pi m \frac{\tilde{n}}{\cos\alpha}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \left.\frac{-\cos^3\vartheta}{3}\right|_\alpha^{\pi-\alpha} \frac{3\sqrt{\pi}}{8}\left(\frac{2k_BT}{m}\right)^{5/2} \\
&= m\frac{\tilde{n}}{\cos\alpha}\frac{2\cos^3\alpha}{3}\frac{3}{8}\frac{2k_BT}{m} = \frac{1}{2}\tilde{n}k_BT\cos^2\alpha
\end{aligned}$$

$$\begin{aligned}
w_\perp &= \int_0^{2\pi} d\varphi \int_\alpha^{\pi-\alpha} \sin\vartheta d\vartheta \int_0^\infty \frac{m}{2} v_\perp^2 \tilde{f}(v) dv \\
&= 2\pi \frac{m}{2}\frac{\tilde{n}}{\cos\alpha}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_\alpha^{\pi-\alpha} \sin^3\vartheta d\vartheta \int_0^\infty v^4 \exp\left(-\frac{mv^2}{2k_BT}\right) dv \\
&= \pi m \frac{\tilde{n}}{\cos\alpha}\left(\frac{m}{2\pi k_B T}\right)^{3/2} \left[-\cos\vartheta + \frac{\cos^3\vartheta}{3}\right]_\alpha^{\pi-\alpha} \frac{3\sqrt{\pi}}{8}\left(\frac{2k_BT}{m}\right)^{5/2} \\
&= m\frac{\tilde{n}}{\cos\alpha} 2\left[\cos\alpha - \frac{\cos^3\alpha}{3}\right]\frac{3}{8}\frac{2k_BT}{m} = \frac{3}{2}\tilde{n}k_BT\left(1-\frac{1}{3}\cos^2\alpha\right)
\end{aligned}$$

The average energies per particle (density for the losscone distribution is $\tilde{n}$) are

$$\begin{aligned}
W_\parallel &= \frac{1}{2}k_BT\cos^2\alpha \\
W_\perp &= \frac{3}{2}k_BT\left(1-\frac{1}{3}\cos^2\alpha\right)
\end{aligned}$$

Note anothe way to express this would be through parallel and perpendiculat 'effective' temperature defined through $p_\perp = \tilde{n}k_BT_\perp$ and $p_\parallel = \tilde{n}k_BT_\parallel$ with $p_\perp = w_\perp$ and $p_\parallel = 2w_\parallel$ such that $T_\perp = \frac{3}{2}T\left(1-\frac{1}{3}\cos^2\alpha\right)$ and $T_\parallel = T\cos^2\alpha$.

**(c)** Angle $\alpha$ for the loss cone if the energy ratio is $W_\parallel/W_\perp = 1/4$?

$$W_\parallel/W_\perp = \frac{\cos^2\alpha}{3-\cos^2\alpha} = \frac{1}{4}$$

such that $\cos^2\alpha = 3/5$ or $\alpha = 39.2°$.

**20 Particle drifts**

(a) What perpendicular particle energy is required to compensate the co-rotational drift in the Earth's magnetosphere through the gradient B drift at the magnetic equator? Sketch or plot the required perpendicular velocity as a function of r. What is this energy for particles at $L = 4$, 6, and 8?

(b) Express the perpendicular energy through the magnetic moment. Assume that the particle is on field lines with $L = 8$ and is mirrored just above the ionosphere. What are the latitude of the mirror point and what is the parallel energy of the particle? What is wrong with this problem?

**Solution:**

**(a)** Gradient drift and angular velocity:

$$\begin{aligned}
\mathbf{v}_\nabla &= \frac{mv_\perp^2}{2qB^3}\left(\mathbf{B}\times\nabla B\right) = -\frac{mv_\perp^2}{2q}\frac{3r^2}{B_E R_E^3}\mathbf{e}_\phi \\
\omega_\nabla &= \frac{v_\nabla}{r} = -\frac{mv_\perp^2}{2q}\frac{3r}{B_E R_E^3} = -\frac{mv_\perp^2}{4\pi q}\frac{3L}{B_E R_E^2}
\end{aligned}$$

Corotational angular velocity with $T_d = 1$ day:

$$\omega_{cor} = \frac{2\pi}{T_d}$$

such that

$$\begin{aligned}
\frac{mv_\perp^2}{4\pi q}\frac{3L}{B_E R_E^2} &= \frac{2\pi}{T_d} \\
w_\perp = \frac{mv_\perp^2}{2} &= q\frac{2\pi}{T_d}\frac{B_E R_E^2}{3L} \\
&= e\frac{2\pi}{24\cdot 3600}\frac{3.11\cdot 10^{-5}\cdot 6.4^2\cdot 10^{12}}{3L} = \\
&= \frac{e}{L}\frac{2\pi}{2.4\cdot 3.600}\frac{3.11\cdot 6.4^2\cdot 10^3}{3} = \frac{30.9keV}{L}
\end{aligned}$$

which corresponds to 7.7 keV, 5.1 keV, and 3.9 keV for $L = 4,\ 6,\ \text{and}\ 8$.

**(b)** Magnetic moment in the equatorial plane and at the mirror point:

$$\mu = \frac{mv_{\perp eq}^2}{2B_{eq}} = \frac{mv_{\perp m}^2}{2B_m}$$

such that

$$\frac{mv_{\perp m}^2}{2} = \frac{mv_{\perp eq}^2}{2}\frac{B_m}{B_{eq}}$$

with

$$\begin{aligned}
B &= B_E \frac{R_E^3}{r^3}\left(1+3\sin^2\lambda\right)^{1/2} = B_E \frac{R_E^3}{r^3}\left(4-3\cos^2\lambda\right)^{1/2} \\
and & \\
r &= r_{eq}\cos^2\lambda
\end{aligned}$$

such that the mirror latitude is $\lambda_m = \arccos\sqrt{r_m/r_{eq}} = \arccos\sqrt{1/L} = 69.3°$.
The magnetic field magnitude along the field line with the equatorial crossing point $r_{eq} = LR_E$ is

$$\begin{aligned}
B &= B_E \frac{R_E^3}{r^3}\left(4-3\frac{r}{r_{eq}}\right)^{1/2} \\
& such\; that \\
B_{eq} &= B_E \frac{1}{L^3} \\
B_{m,1R_E} &= B_E\left(4-3\frac{1}{L}\right)^{1/2}
\end{aligned}$$

and the perpendicular energy at the mirror point for $L = 8$ is

$$\begin{aligned}
\frac{mv_{\perp m}^2}{2} &= \frac{mv_{\perp eq}^2}{2}\frac{B_m}{B_{eq}} = \frac{mv_{\perp eq}^2}{2}L^3\left(4-3\frac{1}{L}\right)^{1/2} \\
&= 974\frac{mv_{\perp eq}^2}{2} = 3.797MeV
\end{aligned}$$

Energy conservation implies that the total energy in the equatorial plane is the same as at the mirror point such that the parallel energy is

$$w_{\|} = \frac{mv_{\perp m}^2}{2} - \frac{mv_{\perp eq}^2}{2} = 973\frac{mv_{\perp eq}^2}{2} = 3.794MeV$$

The problem assumed a particle for which the gradient drift in the equator compensates corotation. However, a particle that mirrors just above the ionosphere has a parallel energy that is almost a 1000 times the perpendicular energy in the equatorial plane. For such a particle the curvature drift is about a 1000 times faster than the gradient drift in the equator such that the particle actually does not compensate corotation.