21. Energy conservation

Derive the conservative form of the energy equation in MHD.

Solution:

$$\frac{\partial w_{tot}}{\partial t} = -\nabla \cdot \left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1} + \frac{1}{\mu_0} B^2 \right) \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{B}}{\mu_0} \mathbf{B} + \frac{\eta}{\mu_0} \mathbf{j} \times \mathbf{B} \right]$$

with

$$w_{tot} = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} + \frac{1}{2\mu_0} B^2$$

MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \quad (1)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} \quad (2)$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \quad (3)$$

$$\frac{1}{\gamma - 1} \left(\frac{\partial}{\partial t} p + \nabla \cdot p \mathbf{u} \right) = -p \nabla \cdot \mathbf{u} + \eta \mathbf{j}^2 \quad (4)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (5)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (6)$$

Bulk flow energy density: $\frac{1}{2}\rho \mathbf{u}^2$

Internal energy with $\gamma = 5/3$: $p/(\gamma - 1)$

Field energy: $\frac{1}{2\mu_0}\mathbf{B}^2$

The internal energy equation is already available. For the bulk flow we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial \rho \mathbf{u}^2}{\partial t} &= \frac{1}{2}\rho^2 \mathbf{u}^2 \frac{\partial \rho^{-1}}{\partial t} + \frac{1}{2\rho} 2\rho \mathbf{u} \cdot \frac{\partial \rho \mathbf{u}}{\partial t} \\
&= -\frac{1}{2}\mathbf{u}^2 \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \frac{\partial \rho \mathbf{u}}{\partial t} \\
&= \frac{1}{2}\mathbf{u}^2 \nabla \cdot \rho \mathbf{u} + \mathbf{u} \cdot [-\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla p + \mathbf{j} \times \mathbf{B}] \\
&= \frac{1}{2}\mathbf{u}^2 \nabla \cdot \rho \mathbf{u} + \mathbf{u} \cdot [-\mathbf{u} \nabla \cdot (\rho \mathbf{u}) - \rho \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \mathbf{j} \times \mathbf{B}] \\
&= -\frac{1}{2}\mathbf{u}^2 \nabla \cdot \rho \mathbf{u} + \mathbf{u} \cdot \left[-\frac{1}{2}\rho \nabla \mathbf{u}^2 - \nabla p + \mathbf{j} \times \mathbf{B} \right] \\
&= -\frac{1}{2}\nabla \cdot \left[\left(\rho \mathbf{u}^2 \right) \mathbf{u} \right] - \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\mathbf{j} \times \mathbf{B})
\end{aligned}$$

Magnetic energy:

$$
\begin{aligned}
\frac{1}{2\mu_0}\frac{\partial \mathbf{B}^2}{\partial t} &= \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} \\
&= -\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\times\mathbf{E} \\
&= -\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\times\left(-\mathbf{u}\times\mathbf{B}+\eta\mathbf{j}\right) \\
&= -\frac{1}{\mu_0}\left[\nabla\cdot\left[\left(-\mathbf{u}\times\mathbf{B}+\eta\mathbf{j}\right)\times\mathbf{B}\right]-\frac{1}{\mu_0}\left(-\mathbf{u}\times\mathbf{B}+\eta\mathbf{j}\right)\cdot\nabla\times\mathbf{B}\right] \\
&= -\frac{1}{\mu_0}\nabla\cdot\left[\mathbf{u}\mathbf{B}^2-\mathbf{B}\mathbf{u}\cdot\mathbf{B}+\eta\mathbf{j}\times\mathbf{B}\right]-\left(-\mathbf{u}\times\mathbf{B}+\eta\mathbf{j}\right)\cdot\mathbf{j}
\end{aligned}
$$

And the internal energy:

$$
\frac{1}{\gamma-1}\frac{\partial p}{\partial t} = -\frac{1}{\gamma-1}\nabla\cdot p\mathbf{u}-p\nabla\cdot\mathbf{u}-\eta\mathbf{j}^2
$$

Adding the three equations:

$$
\begin{aligned}
\frac{1}{\gamma-1}\frac{\partial p}{\partial t}+\frac{1}{2}\frac{\partial\rho\mathbf{u}^2}{\partial t}+\frac{1}{2\mu_0}\frac{\partial\mathbf{B}^2}{\partial t} &= -\frac{1}{\gamma-1}\nabla\cdot p\mathbf{u}-p\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla p \\
&\quad -\frac{1}{2}\nabla\cdot\left[\left(\rho\mathbf{u}^2\right)\mathbf{u}\right]-\frac{1}{\mu_0}\nabla\cdot\left[\mathbf{u}\mathbf{B}^2-\mathbf{B}\mathbf{u}\cdot\mathbf{B}+\eta\mathbf{j}\times\mathbf{B}\right]
\end{aligned}
$$

and all other terms on the rhs cancel. Combining the remaining terms and re-arranging:

$$
\frac{1}{\gamma-1}\frac{\partial p}{\partial t}+\frac{1}{2}\frac{\partial\rho\mathbf{u}^2}{\partial t}+\frac{1}{2\mu_0}\frac{\partial\mathbf{B}^2}{\partial t} = -\nabla\cdot\left[\frac{\gamma}{\gamma-1}p\mathbf{u}+\frac{1}{2}\left(\rho\mathbf{u}^2\right)\mathbf{u}+\frac{1}{\mu_0}\left(\mathbf{u}\mathbf{B}^2-\mathbf{B}\mathbf{u}\cdot\mathbf{B}+\eta\mathbf{j}\times\mathbf{B}\right)\right]
$$

22. Sound waves

a) Reduce the MHD equations to a non-magnetic fluid. Determine the equilibrium conditions.

b) Why does $\gamma = \infty$ correspond to incompressibility (γ is the ratio of specific heats)?

c) Assume a homogeneous system (equilibrium without flow) and derive the dispersion relation by linearizing the equations and assuming waves of the form $f(x,t) = f_0 \exp\{i(kx - \omega t)\}$. The corresponding waves are sound waves. What is the wave speed?

Solution: (a) Starting from the MHD equations (1) - (6) in Problem 19 and deleting all terms which contain magnetic field, electric field, and current density yields:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla p \\
\frac{\partial p}{\partial t} &= -\nabla \cdot p\mathbf{u} - (\gamma - 1)\, p \nabla \cdot \mathbf{u} = -\mathbf{u} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{u}
\end{aligned}$$

Equilibrium conditions: $\partial/\partial t = 0$ and $\mathbf{u} = 0$ yields $\nabla p = 0$ or $p = \text{const}$. There is no condition for ρ or T but they have to satisfy the ideal gas law $p = nk_B T$.

(b) In the limit of $\gamma \to \infty$ the last term of the pressure equation requires $\nabla \cdot \mathbf{u} \to 0$ otherwise the term would diverge and pressure changes become infinitely large. Going back to the continuity equation we obtain with $\nabla \cdot \rho \mathbf{u} = \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u}$ such that

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \frac{d\rho}{dt} = \rho \nabla \cdot \mathbf{u} \hookrightarrow 0$$

and $d\rho/dt = 0$ means there is no density change along the path of the fluid element.

For $\gamma = 1$ and $p = nk_B T = \frac{k_B}{m} \rho T$

$$\begin{aligned}
\frac{k_B}{m}\frac{dT}{dt} &= \frac{d}{dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\frac{dp}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} \\
&= -\frac{1}{\rho}\gamma p \nabla \cdot \mathbf{u} + \frac{p}{\rho^2}\rho \nabla \cdot \mathbf{u} = -\frac{p}{\rho}(\gamma - 1) \nabla \cdot \mathbf{u} = 0
\end{aligned}$$

(c) With $\rho_0 = \text{const}$, $p_0 = \text{const}$, and $\mathbf{u}_0 = 0$ and applying a linearization $\rho = \rho_0 + \rho_1$ where ρ_1 is a small perturbation and all nonlinear term in the perturbations are neglected. Also we assume $\mathbf{u}_1 = u\mathbf{e}_x$ because the medium is isotropic. Note, since $\mathbf{u}_0 = 0$ we can omit the index 1 for the velocity:

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\rho_0 \partial u/\partial x \\
\rho_0 \frac{\partial u}{\partial t} &= -\partial p_1/\partial x \\
\frac{\partial p_1}{\partial t} &= -\gamma p_0 \partial u/\partial x
\end{aligned}$$

We can now assume perturbations of the form $f_1 = \widehat{f_1} \exp(i\omega t - ikx)$ such that one can replace time derivatives with $\partial/\partial t \hookrightarrow i\omega$ and spatial derivatives with $\partial/\partial x \hookrightarrow -ik$. This leads to a set of algebraic equations for the relation of ω and k. The expression $\omega(k)$ is the dispersion relation and ω/k is the phase velocity:

$$
\begin{aligned}
i\omega\rho_1 &= i\rho_0 k u \\
i\omega\rho_0 u &= ikp_1 \\
i\omega p_1 &= i\gamma p_0 k u
\end{aligned}
$$

Division by i and substitution of equ. 3 into 2 yields: $\omega^2\rho_0 u = \gamma p_0 k^2 u$ or $\omega^2/k^2 = \gamma p_0/\rho_0 = c_s^2$ for the dispersion relation.

Alternatively one can make this step a bit later and take first the time derivative of the 2nd equation and substitute the pressure equation:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\frac{\partial p_1}{\partial t} = \gamma p_0 \frac{\partial^2 u}{\partial x^2}$$

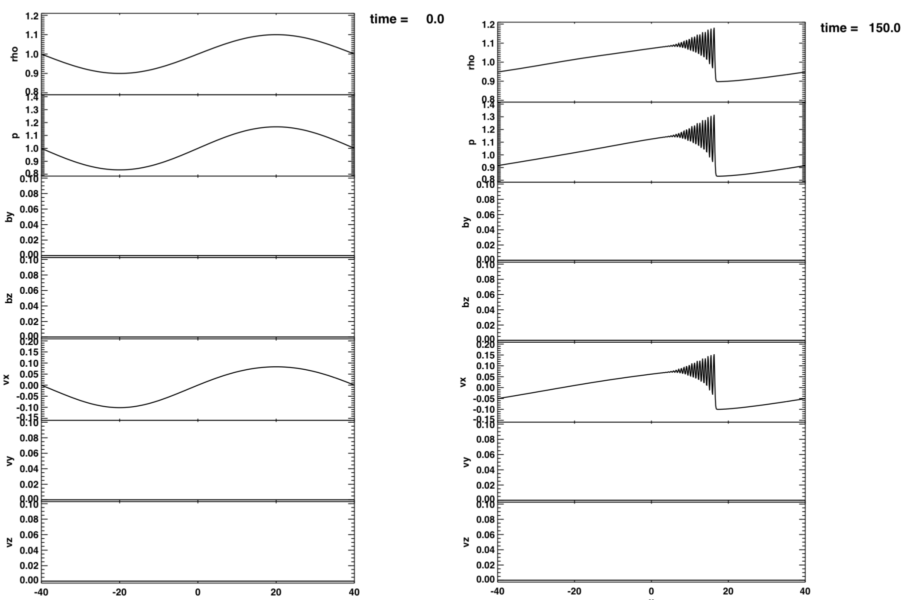
and with the definition of the speed of sound $c_s^2 = \gamma p_0/\rho_0$ on obtains $\partial^2 u/\partial t^2 - c_s^2 \partial^2 u/\partial x^2 = 0$. This is the wave equation for sound waves. Any function of the type $f(x,t) = h(x \pm c_s t)$ solves the above equation.

23. Simulation of sound waves

The compressed file mhd1d_code on the class website contains a one-dimensional MHD simulation code, files to run the program, an idl program to visualize results, a readme file that explains how to run the code, and a few page of background on the MHD simulation. The one-dimensional MHD code contains various initial conditions. Make yourself familiar with the program by running it for the case of a sound wave (the preset initial condition 3 -> subroutine initc3). Compile and run the program in the distributed version. Examine the results using the provided idl program (Note the system is periodic such that the wave that exits to the right re-enters from the left). (1) Is it possible to get rid of the oscillations at later times by increasing one of the viscosity parameters in m1in (ivisrho, ivisv, ivisu)? (2) What happens when you change the wave velocity amplitude to 0.01 or the density to 4 (in the program)? (3) How large can you make the time step before you encounter an instability? Note that you may have to use a shorter run (smaller iend) and more frequent outputs (smaller iout) to see this. Report your results with selected plots that document your findings and attempt an interpretation of your findings.

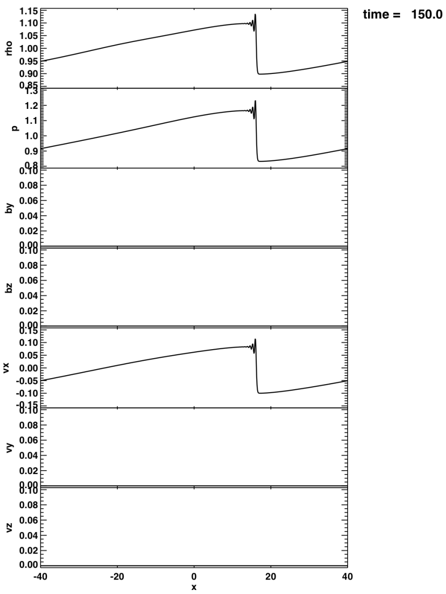
Solution:

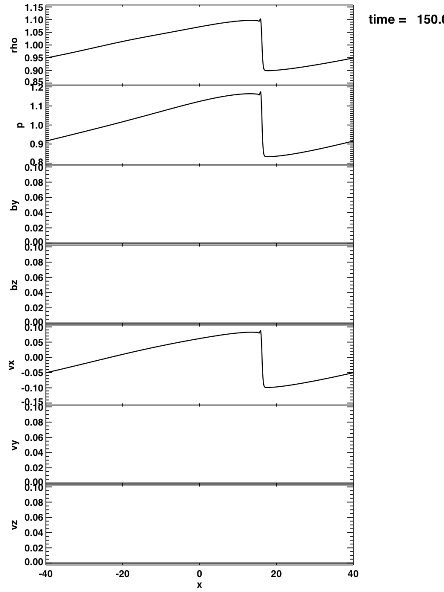
(1) Case 1: The following 2 figures show results with the default values for the sound wave (initial condition 3) for the times 0, 150. The time step is 0.05.



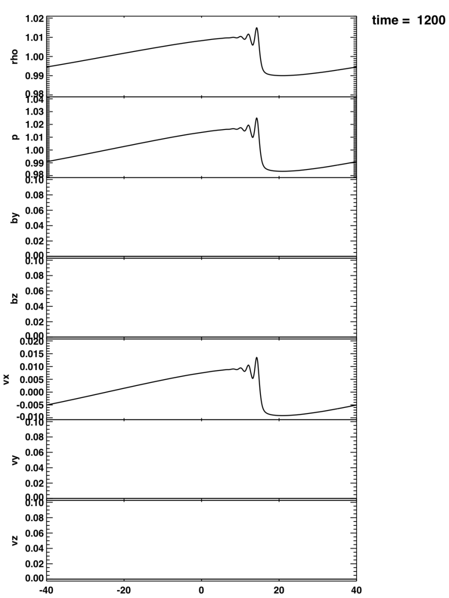
After about $t = 100$ the wave steepens much and develops increasing oscillations. Wave steepening is often the case when wave amplitudes are large such that the wave modifies the speed of sound (travels faster in regions of higher pressure and sound speed and slower in region of lower pressure).

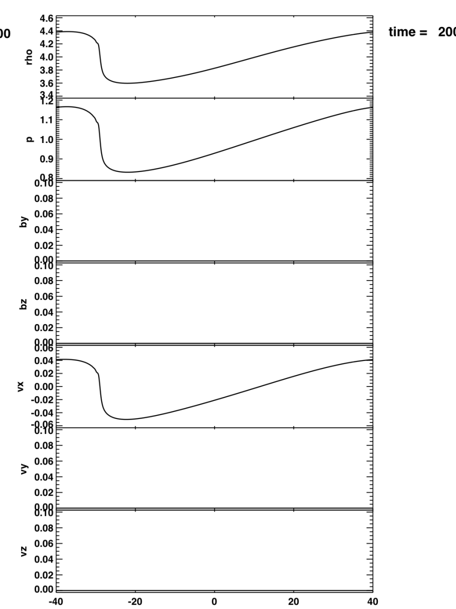
Case 2: Here we have increased the viscosity parameter for pressure to 0.1 (left) and to 0.3 (right) for the same time shown in case 1. The increase in the viscosity (diffusion) clearly damps the oscillations where 0.1 is almost sufficient and with a value of 0.3 the oscillations are gone. Note that the chosen viscosity does not alter the propagation of the sepwave noticably.





(2) Cases 3 and 4: Decreasing the amplitude of the wave to 0.01 (below left) leads to a much slower evolution of the wave steepening consistent with the explanation given for case 1. A larger density of 4 leads to a larger absolute density perturbation and a smaller velocity perturbation. The pressure perturbation appears the same. The wave travels slower (visible clearly in the animation) and steepening is also a bit slower. However steepening appears to be proportional to the distance travelled (for everything else fixed).





(3) Case 5: Increasing the time step leads to an instability. The next plot show the wave for a time step of 0.098. For a slightly smaller time (0.97) the wave develops oscillations associated with the step a bit earlier but the oscillations do not grow large whereas for time step 0.98 oscillation occur even without the wave steepening and grow fast in time.