24. Simulation and Normalization

The MHD simulation uses a normalized set of equations where all quantities are measured in typical units, i.e., the magnetic induction $\mathbf{B}$ is normalized to a typical magnetic field B_0, density to a typical density ρ_0, length to L_0, velocity to $v_A = B_0/\sqrt{\mu_0 \rho_0}$, time to $t_0 = L_0/v_A$, and pressure to $p_0 = B_0^2/(2\mu_0)$, and current density to $j_0 = B_0/(\mu_0 L_0)$. The resulting ideal ($\eta = 0$) MHD equations for the normalized quantities are:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot \left[(\rho \mathbf{u}\mathbf{u}) - \frac{1}{2}\left(p + B^2\right) \underline{\underline{\mathbf{1}}} - \mathbf{B}\mathbf{B} \right] \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times [\mathbf{u} \times \mathbf{B}] \\
\frac{\partial p}{\partial t} &= -\nabla \cdot p\mathbf{u} - (\gamma - 1)\, p \nabla \cdot \mathbf{u}
\end{aligned}
$$

(a) Demonstrate for this normalization that the Alfvén speed is simply $v_{A,sim} = B_{sim}/\sqrt{\rho_{sim}}$ and the speed of sound is $c_{s,sim} = \sqrt{\gamma p_{sim}/(2\rho_{sim})}$ in simulation units where B_{sim}, ρ_{sim}, and p_{sim} are the values for magnetic field, density, and pressure in the simulation.

(b) Using the 1D MHD code run the cases of the Alfvén and of the sound wave and examine whether these waves indeed propagate with the correct phase velocity. Vary the density, magnetic field, and pressure for this test and plot select results. Provide commensts on any difficulties and on the accuracy of the results.

25. Alfven waves

Consider a homogeneous plasma with a magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_x$, pressure p_0, and density ρ_0. If the velocity and magnetic field perturbations are only along one perpendicular direction the only wave solution is the Alfvén mode.

(a) Derive the dispersion relation of Alfvén waves for this case.

(b) Assume a perturbation of the form $\mathbf{B}_1 = b(\tilde{x})\, \mathbf{e}_y$ where $b(\tilde{x})$ can be any function of $\tilde{x} = x - t/v_A$ and derive the velocity perturbation, and the current density associated with the wave.

(c) Assume $b(x - t/v_A) = b_1 \cosh^{-1}[(x - t/v_A)/L]$. Choose reasonable values for the constants b_1 and L and modify the initial condition for the Alfvén wave (initc2) in the 1D simulation. Run the modified code for this case. Do you observe any wave steepening and how do the results change when you switch on a small resistivity (in the range from 10^{-3} to 10^{-1})? Plot, report and comment your results.

26. Gas-dynamic Shock

(a) Derive the compression ratio

$$X = \frac{(\gamma + 1) M_u^2}{2 + (\gamma - 1) M_u^2}$$

for the pure gas-dynamic shock.

(b) Using the relations from class for this shock derive the downstream sound speed (in terms of the upstream parameters and sound speed) and show that the downstream Mach number is always smaller than 1 for $M_u > 1$.

27. Project

Decide a project topic, collect material for the topic, and make a first attempt to understand the basic motivation, methodology, and impact of the results of the article.

In case you choose a simulation topic, outline your goals for this topic and formulate an approach how to achieve those.

Please turn in the solutions to the homework on Friday, 4/04/2014