24. Simulation and Normalization

The MHD simulation uses a normalized set of equations where all quantities are measured in typical units, i.e., the magnetic induction $\mathbf{B}$ is normalized to a typical magnetic field B_0, density to a typical density ρ_0, length to L_0, velocity to $v_A = B_0/\sqrt{\mu_0\rho_0}$, time to $t_0 = L_0/v_A$, and pressure to $p_0 = B_0^2/(2\mu_0)$, and current density to $j_0 = B_0/(\mu_0 L_0)$. The resulting ideal ($\eta = 0$) MHD equations for the normalized quantities are:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho\mathbf{u} \\
\frac{\partial \rho\mathbf{u}}{\partial t} &= -\nabla \cdot \left[(\rho\mathbf{u}\mathbf{u}) + \frac{1}{2}\left(p + B^2\right)\underline{\underline{\mathbf{1}}} - \mathbf{B}\mathbf{B}\right] \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times [\mathbf{u} \times \mathbf{B}] \\
\frac{\partial p}{\partial t} &= -\nabla \cdot p\mathbf{u} - (\gamma - 1)\, p\nabla \cdot \mathbf{u}
\end{aligned}
$$

(a) Demonstrate for this normalization that the Alfvén speed is simply $v_{A,sim} = B_{sim}/\sqrt{\rho_{sim}}$ and the speed of sound is $c_{s,sim} = \sqrt{\gamma p_{sim}/(2\rho_{sim})}$ in simulation units where B_{sim}, ρ_{sim}, and p_{sim} are the (equilibrium values for magnetic field, density, and pressure specified in the simulation code.

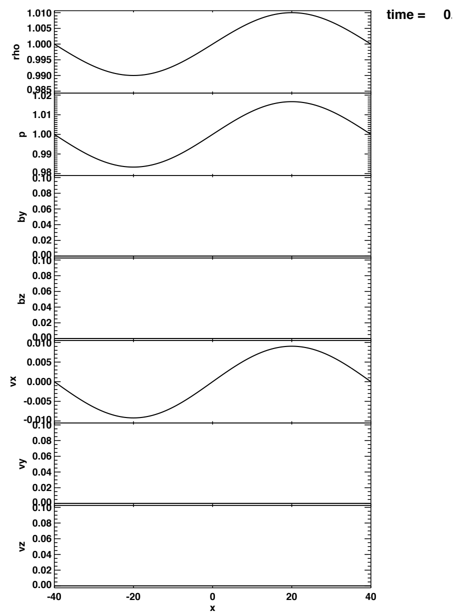
(b) Using the 1D MHD code run the cases of the Alfvén and of the sound wave and examine whether these waves indeed propagate with the correct phase velocity. Vary the density, magnetic field, and pressure for this test and report your results.

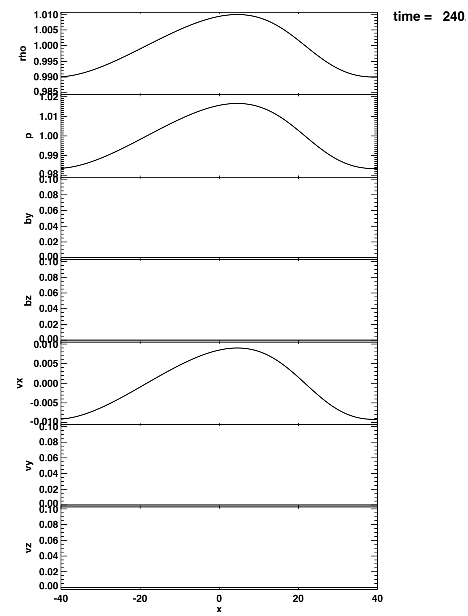
Solution:

(a) Let us denote normalization quantities by an index 0 and equilibrium quantities with an index e to distinguish between normalization quantities and equilibrium quantities. Further, quantities specified in the simulation code (in normalized units) are denoted by an index sim such that B_0 is the normalization of the magnetic field, B_e is the equilibrium magnetic field in physical units, and B_{sim} is the magnetic field in normalized units as specified in the simulation code. In physical units the Alfvén speed for a given equilibrium is $v_{A,e} = B_e/\sqrt{\mu_0\rho_e}$ and the sound speed is $c_{s,e} = \sqrt{\gamma p_e/(2\rho_e)}$. The normalization for the velocity is $v_{A0} = B_0/\sqrt{\mu_0\rho_0}$ and for the pressure is $p_0 = B_0^2/(2\mu_0)$. Thus the Alfvén and sound speeds in normalized units are

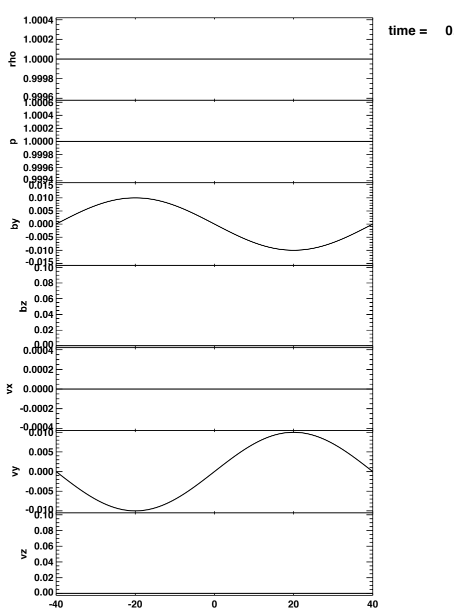
$$
\begin{aligned}
v_{A,sim} = \frac{v_{A,e}}{v_{A0}} &= \frac{B_e}{\sqrt{\mu_0\rho_e}}\frac{\sqrt{\mu_0\rho_0}}{B_0} = \frac{B_e/B_0}{\sqrt{\mu_0\rho_e/(\mu_0\rho_0)}} = \frac{B_{sim}}{\sqrt{\rho_{sim}}} \\
c_{s,sim} = \frac{c_{s,e}}{v_{A0}} &= \sqrt{\frac{\gamma p_e}{\rho_e}}\frac{\sqrt{\mu_0\rho_0}}{B_0} = \sqrt{\frac{\gamma p_e/(B_0^2/\mu_0)}{\rho_e/\rho_0}} = \sqrt{\frac{\gamma p_e/(2p_0)}{\rho_e/\rho_0}} = \sqrt{\frac{\gamma p_{sim}}{2\rho_{sim}}}
\end{aligned}
$$

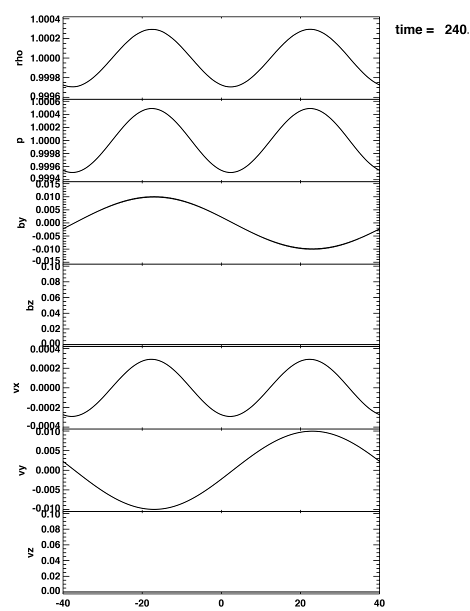
(b) Sound wave: the following plot shows snapshot of the sound wave for $t = 0$and $t = 240$ for the reference case from homework 21. The theoretical speed of sound for a normalized pressure and density of $p = 1$ and $\rho = 1$ is $c_s = \sqrt{\gamma p/2\rho} = \sqrt{5/6} = 0.913$. The 2nd snapshot illustrates that the wave has traveled almost 3 time its wavelength or $d = 222$ unit length scales which yields a velocity of $c_{s,sim} = 222/240 = 0.925$ such that the error is about 1%. Note that compared to the reference case the wave amplitude is chosen as 0.01 (i.e., a tenth of the default because this lowers the wave steepening due to nonlinearity. Changing parameter such as density or pressure has the expected effects. For an increase of 4 in the density the wave indeed travels with almost exactly half the expected velocity.





Alfvén wave: Here we use initial condition 2 with a wave amplitude of 0.01 to reduce nonlinear effects. The theoretical Alfvén speed for $B = 1$ and $\rho = 1$ is $v_A = 1$. The following results show the wave at $t = 0$ and $t = 240$ when is has travelled approximately 3 time through the entire system. The distance traveled is $d = 242$ such that the inferred Alfvén speed is 1.008 such that again the error is about 1%. Note that the plots not only show perturbations in u_y and B_y as expected for the Alfvén wave but also in density, pressure, and u_x which is a compressional perturbation driven by the Alfvén wave. Note, however, that this perturbation is smaller than the Alfvénic perturbations and scales approximately as the square of the Alfvénic perturbations such that it is negligible for sufficiently small amplitudes of the wave.





25. Alfven waves

Consider a homogeneous plasma with a magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_x$, pressure p_0, and density ρ_0. If the velocity and magnetic field perturbations are only along one perpendicular direction the only wave solution is the Alfvén mode.

(a) Derive the dispersion relation of Alfvén waves for this case.

(b) Assume a perturbation of the form $\mathbf{B}_1 = b\,(x - v_A t)\,\mathbf{e}_y$ and derive the velocity perturbation, and the current density associated with the wave.

(c) Assume $b\,(x - v_A t) = b_1 \cosh^{-1}\left[(x - v_A t)/L\right]$. Choose reasonable values for the constants b_1 and L and modify the initial condition for the Alfvén wave in the 1D simulation and run the modified code for this case. Do you observe any wave steepening and how do the results change when you switch on resistivity? Plot and report your results.

Solution:

Starting from the normalized MHD equations (one can as well use the original equations and assuming the velocity and magnetic field perturbation in the y direction. The wave travels along x such that perturbations have the general form $f\,(x,t) = \widehat{f}\exp\left(ik_x x - iwt\right)$ and terms such as $\nabla\cdot\mathbf{u} = ik_x u_x$ but $u_x = 0$ such that these terms are 0. For this reason the continuity and the pressure equation imply that $\rho_1 = 0$ and $p_1 = 0$. Linearization of the momentum and induction equations yields

$$\begin{aligned}\rho_0\frac{\partial\mathbf{u}_1}{\partial t} &= -\frac{1}{2}\nabla\left(p_1 + 2\mathbf{B}_0\cdot\mathbf{B}_1\right) + \nabla\cdot\left[\mathbf{B}_0\mathbf{B}_1 + \mathbf{B}_1\mathbf{B}_0\right]\\ \frac{\partial\mathbf{B}_1}{\partial t} &= \nabla\times\left[\mathbf{u}_1\times\mathbf{B}_0 - \eta\nabla\times\mathbf{B}_1\right]\end{aligned}$$

The x component of the induction equation demonstrates that $B_{1x} = 0$ and the the x component of the momentum equation yields $u_{1x} = 0$. Since perturbation are assumed in the y direction we can ignore the z components of the induction and momentum equations. and the y -components become:

$$\begin{aligned}-i\omega\rho_0 u_{y1} &= +ik_x\left[B_{x0}B_{y1} + B_{x1}B_{y0}\right]\\ -i\omega B_{y1} &= -ik_x\left[u_{x1}B_{y0} - u_{y1}B_{x0} - ik_x\eta B_{y1}\right]\\ &= -ik_x\left[-u_{y1}B_{x0} - ik_x\eta B_{y1}\right]\end{aligned}$$

Since $\mathbf{B}_0 = B_0\mathbf{e}_x$

$$\begin{aligned}-\omega\rho_0 u_{y1} &= k_x B_0 B_{y1}\\ -\omega B_{y1} &= k_x B_0 u_{y1} + ik_x^2\eta B_{y1}\end{aligned}$$

Multiplication of the 1st equation with $k_x B_0$ and the 2nd equation with $\omega\rho_0$ and taking the sum eliminates u_{y1} to yield:

$$-\omega^2\rho_0 B_{y1} = -k_x^2 B_0^2 B_{y1} + i\omega\eta\rho_0 k_x^2 B_{y1}$$

Division by ρ_0 and B_{y1} and sorting this equation yields

$$\frac{\omega}{k_x} = -i\frac{\eta}{2}k_x \pm \frac{B_0}{\sqrt{\rho_0}}\sqrt{1-\frac{\eta^2\rho_0}{4B_0^2}k_x^2}$$

For $B_0/\sqrt{\rho_0} \gg \eta k_x/2$ the mode is weakly damped but propagating with the Alfvén speed $B_0/\sqrt{\rho_0}$. For $B_0/\sqrt{\rho_0} \leq \eta k_x/2$ the frequency is purely imaginary and the mode is strongly damped with a rate of about $\omega \simeq \eta k_x^2/2$.

(b) For $\eta = 0$ and $\mathbf{B}_1 = b\,(X)\,\mathbf{e}_y$ with $X = x - v_A t$ the momentum equation is

$$\rho_0 \frac{\partial u_{y1}}{\partial t} = B_0 \frac{\partial B_{y1}}{\partial x} \qquad \text{or}$$

$$\frac{\partial u_{y1}}{\partial t} = \frac{B_0}{\rho_0}\frac{\partial b}{\partial x} = \frac{B_0^2}{\rho_0}\frac{1}{B_0}\frac{\partial b}{\partial X} = v_A^2\frac{1}{B_0}\frac{\partial b}{\partial t}\left(\frac{dX}{dt}\right)^{-1} = -v_A\frac{1}{B_0}\frac{\partial b}{\partial t}$$

Integration then yields:

$$u_{y1}\,(x,t) = -\frac{v_A}{B_0}b\,(x - v_A t) + \text{const}$$

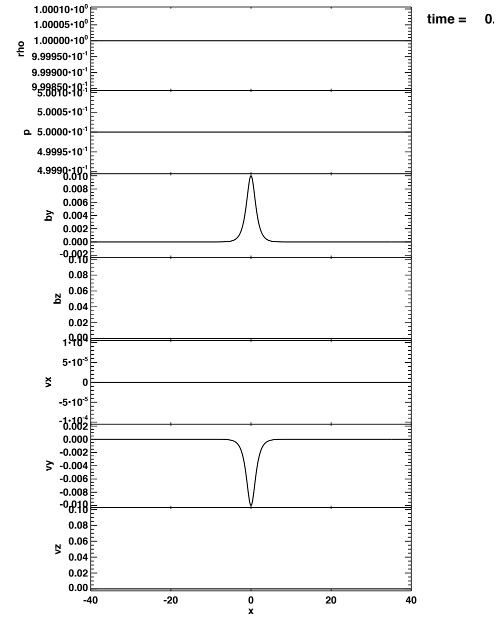
with a $+$ sign for $X = x + v_A t$ or

$$\frac{\mathbf{u}_1}{v_A} = \pm\frac{\mathbf{B}_1}{B_0}$$

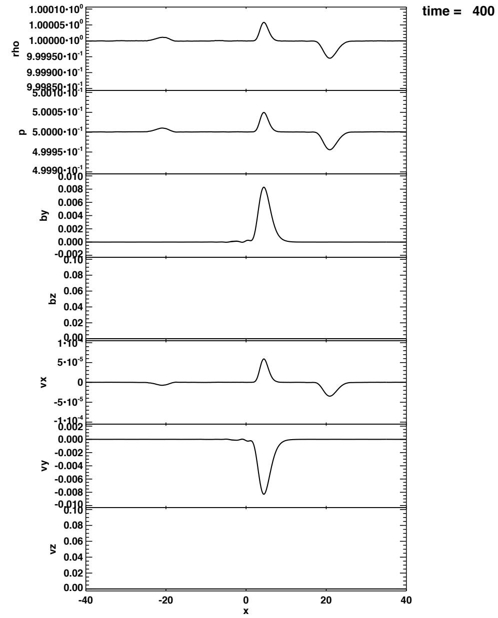
Current density:

$$\mathbf{j}_1 = \nabla \times \mathbf{B}_1 = \frac{\partial}{\partial x}B_{y1}\mathbf{e}_z = \frac{\partial b}{\partial x}\mathbf{e}_z = \frac{\partial b}{\partial X}\mathbf{e}_z$$

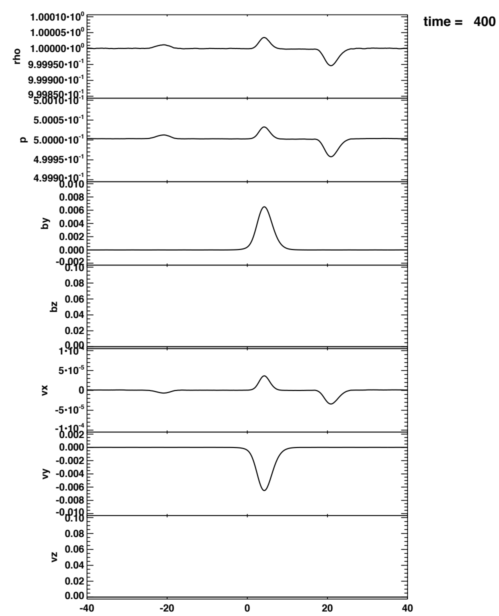
(c) Assume $b\,(x - v_A t) = b_1 \cosh^{-1}\left[(x - v_A t)/L\right]$ such that $B_{y1}\,(x,t) = b_1 \cosh^{-1}\left[(x - v_A t)/L\right]$ and $u_{y1}\,(x,t) = -\frac{v_A}{B_0}b_1 \cosh^{-1}\left[(x - v_A t)/L\right]$. The following for plots show the initial condition for $l_0 = 1$ and 3 additional snapshots each at $t = 400$ for resistivity $\eta = 0.001$, 0.01, and 0.1. The case with $\eta = 0.001$ is close to the case for $\eta = 0$. The slight decrease in amplitude is actually mostly due to viscosity (viscous coefficient are all set to 0.05) and numerical resistivity. There is less such damping for $l_0 = 2$ or even better $l_0 = 4$. The wave amplitude is $b_1 = 0.01$. For larger amplitudes such as 0.1 nonlinear effects (coupling to sound wave) lead to steepening and additional energy loss/dissipation. The cases for increasing η demonstrate that the increasing resistivity leads to considerable damping. Note that this damping is stronger for smaller values of l_0 because current densities are larger for smaller l_0.



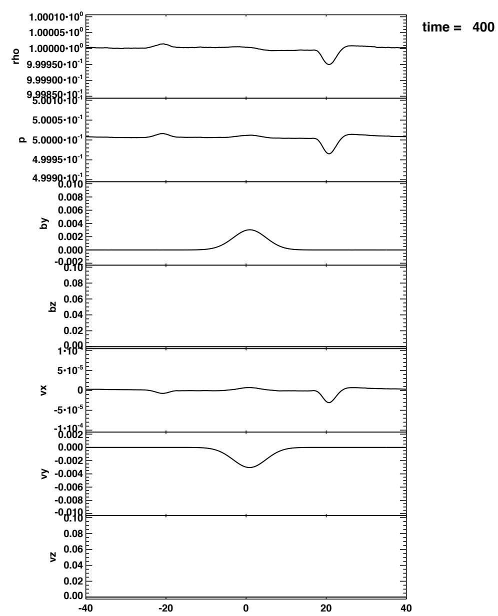
Initial configuration



$\eta = 0.001$



$\eta = 0.01$



$\eta = 0.1$

26. Gas-dynamic Shock

(a) Derive the compression ratio

$$X = \frac{(\gamma+1)\,M_u^2}{2+(\gamma-1)\,M_u^2}$$

for the pure gas-dynamic shock.

(b) Using the relations from class for this shock derive the downstream sound speed (in terms of the upstream parameters and sound speed) and show that the downstream Mach number is always smaller than 1 for $M_u > 1$.

Solution:

(a) Equations for the hydrodynamic shock:

$$\begin{aligned} n_d u_d &= n_u u_u \\ p_d + m n_d u_d^2 &= p_u + m n_u u_u^2 \\ \left(\frac{1}{2} m n_d u_d^2 + \frac{\gamma}{(\gamma-1)} p_d\right) u_d &= \left(\frac{1}{2} m n_u u_u^2 + \frac{\gamma}{(\gamma-1)} p_u\right) u_u \end{aligned}$$

Introduce: Mach number $M = u/c_s$ with $c_s^2 = \gamma p/mn$ and the relations

$$\begin{aligned} \frac{n_d}{n_u} &= X \\ \frac{u_d}{u_u} &= \frac{1}{X} \end{aligned}$$

Re-writing the pressure equations:

$$\begin{aligned} \frac{p_d}{p_u} &= 1 + \frac{\rho_u u_{nu}^2}{p_u}\left(1 - \frac{\rho_d u_{nd}^2}{\rho_u u_{nu}^2}\right) \\ &= 1 + \gamma M^2 \left(1 - X^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{p_d}{p_u} &= \frac{u_{nu}}{u_{nd}} + \frac{\gamma-1}{2\gamma}\frac{\rho_u u_u^2}{p_u}\left(\frac{u_{nu}}{u_{nd}} - \frac{\rho_d u_{nd}^2}{\rho_u u_{nu}^2}\right) \\ &= X + \frac{\gamma-1}{2} M^2 \left(X - X^{-1}\right) \end{aligned}$$

Equating the two expressions

$$1 + \gamma M^2 \left(1 - X^{-1}\right) = X + \frac{\gamma-1}{2} M^2 \left(X - X^{-1}\right)$$

yields

$$\left(2 + (\gamma-1)\,M^2\right) X^2 - 2\left(1+\gamma M^2\right) X + (\gamma+1)\,M^2 = 0 \qquad \text{or}$$

$$X^2 - \frac{2\,(1+\gamma M^2)}{2+(\gamma-1)\,M^2} X + \frac{(\gamma+1)\,M^2}{2+(\gamma-1)\,M^2} = 0$$

Note that $X = 1$ is a solution to the equation. Thus we can factorize it as $(X - A)(X - 1)$ which yields the solution

$$
\begin{aligned}
X = \frac{n_d}{n_u} &= \frac{(\gamma + 1) M_u^2}{2 + (\gamma - 1) M_u^2} \\
\frac{u_d}{u_u} &= \frac{n_u}{n_d} \\
\frac{p_d}{p_u} &= \frac{2\gamma M_u^2 - (\gamma - 1)}{\gamma + 1}
\end{aligned}
$$

(b) Downstream sound speed

$$
c_{sd}^2 = \frac{\gamma p_d}{\rho_d} = \frac{2\gamma M_u^2 - (\gamma - 1)}{\gamma + 1} \frac{1}{X} \frac{\gamma p_u}{\rho_u}
$$

Downstream Machnumber

$$
\begin{aligned}
\frac{u_d^2}{c_{sd}^2} &= \frac{u_u^2}{X^2} X \frac{\gamma + 1}{2\gamma M_u^2 - (\gamma - 1)} \frac{1}{\gamma p_u / \rho_u} \\
&= M^2 \frac{\gamma + 1}{2\gamma M_u^2 - (\gamma - 1)} \frac{2 + (\gamma - 1) M_u^2}{(\gamma + 1) M_u^2} = \frac{2 + (\gamma - 1) M_u^2}{2\gamma M_u^2 - (\gamma - 1)} \\
&= \frac{2\gamma M_u^2 - \gamma + 1 + \gamma + 1 - \gamma M_u^2 - M_u^2}{2\gamma M_u^2 - (\gamma - 1)} = 1 + \frac{(\gamma + 1)(1 - M_u^2)}{2\gamma M_u^2 - (\gamma - 1)}
\end{aligned}
$$

For $M_u > 1$ we have $1 - M_u^2 < 0$ such that the second expression in the last equation is negative for $M_u > 1$ and thus $u_d^2 / c_{sd}^2 < 1$.