27. Fast wave, fast shock, perpendicular shock, entropy, jump relation.

Consider the fast perpendicular shock.

(a) Determine the positive root of the solution for the compression ratio. How does the compression ratio behave for very large and very small upstream plasma β?

(b) Derive the downstream Mach number as a function of upstream Mach number and for small and large plasma β. How does the downstream Mach number change as a function of β for fixed upstream Mach number?

(c) An important quantity for space plasma processes is entropy. A measure of entropy is $s = p/n^\gamma$. Determine the downstream to upstream ratio of s as a function of upstream Machnumber and plasma β. Plot this ratio and the compression ratio as a function of plasma β for a fixed upstream Machnumber of 20. What is the value of the minimum upstream plasma β to sustain a shock? Why?

Solution:

(a) Relations:

$$M = \frac{u_u}{c_s} \qquad \beta = \frac{p_{thu}}{p_{Bu}} = \frac{2\mu_0 p_u}{B_u^2} = \frac{2}{\gamma}\frac{c_s^2}{u_A^2}$$
$$c_s = \sqrt{\frac{\gamma p}{\rho}} \qquad u_A = \frac{B_u}{\sqrt{\mu_0 \rho_u}}$$

The equation for the compression ratio $X = \rho_d/\rho_u$ is

$$P_X = 2\left(2-\gamma\right)X^2 + \left[2\beta + \left(\gamma - 1\right)\beta M^2 + 2\right]\gamma X - \gamma\left(\gamma+1\right)\beta M^2 = 0$$

Substitute of the sonic Machnumber by the fast Machnumber:

$$M^2 = \frac{u^2}{c_s^2} = \frac{u^2}{c_s^2 + v_A^2}\frac{c_s^2 + v_A^2}{c_s^2} = M_f^2\left(1 + \frac{B^2\rho}{\mu_0 \rho\gamma p}\right) = M_f^2\left(1 + 2\frac{B^2}{2\mu_0\gamma p}\right) = M_f^2\frac{2+\gamma\beta}{\gamma\beta}$$

yields

$$P_X = 2\left(2-\gamma\right)X^2 + \left[2\gamma\beta + \left(\gamma-1\right)\left(2+\gamma\beta\right)M_f^2 + 2\gamma\right]X - \left(\gamma+1\right)\left(2+\gamma\beta\right)M_f^2 = 0$$

Formal (exact) solution: Division by $2\left(2-\gamma\right)$

$$\begin{aligned} X^2 + 2aX - b &= 0 \\ \text{with} \quad a &= \frac{2\gamma\beta + \left(\gamma-1\right)\left(2+\gamma\beta\right)M_f^2 + 2\gamma}{4\left(2-\gamma\right)} \\ b &= \frac{\left(\gamma+1\right)\left(2+\gamma\beta\right)M_f^2}{2\left(2-\gamma\right)} \\ \text{with the solution :} \quad X &= -a \pm \sqrt{b + a^2} \end{aligned}$$

Note (not necessary) expansion into Taylor series (valid for large M_f):

$$X = -a + a\sqrt{1 + \frac{b}{a^2}} = -a + a\left(1 + \frac{b}{2a^2} - \frac{b^2}{8a^4}\right) = \frac{b}{2a} - \frac{b^2}{8a^3}$$

Here only the positive solution is physical. For both $\beta \ll 1$ and $\beta \gg 1$ the compression goes to 1 for $M_f \to 1$ such that we consider $M_f \gg 1$ here:

For $\beta \ll 1$:

$$P_X = (2-\gamma) X^2 + \left[(\gamma-1) M_f^2 + \gamma\right] X - (\gamma+1) M_f^2 = 0$$

$$\begin{aligned}
a &= \frac{(\gamma-1) M_f^2 + \gamma}{2(2-\gamma)} \\
b &= \frac{(\gamma+1) M_f^2}{(2-\gamma)} \\
X &= -\frac{(\gamma-1) M_f^2 + \gamma}{2(2-\gamma)} + \frac{1}{2(2-\gamma)}\sqrt{4(2-\gamma)(\gamma+1) M_f^2 + \left[(\gamma-1) M_f^2 + \gamma\right]^2} \\
&\simeq \frac{(\gamma+1) M_f^2}{(\gamma-1) M_f^2 + 2\gamma} \quad for \quad M_f^2 \gg 1
\end{aligned}$$

For $\beta \gg 1$:

$$P_X = 2(2-\gamma) X^2 + \gamma\beta\left[2 + (\gamma-1) M_f^2\right] X - \gamma\beta(\gamma+1) M_f^2 = 0$$

$$\begin{aligned}
a &= \gamma\beta\frac{2 + (\gamma-1) M_f^2}{4(2-\gamma)} \\
b &= \gamma\beta\frac{(\gamma+1) M_f^2}{2(2-\gamma)}
\end{aligned}$$

such that

$$\begin{aligned}
P_X &= 2(2-\gamma) X^2 + \left[2\gamma\beta + (\gamma-1)(1+\gamma\beta) M_f^2 + 2\gamma\right] X - (\gamma+1)(1+\gamma\beta) M_f^2 \\
&\simeq \gamma\beta\left[2 + (\gamma-1) M_f^2\right] X - \gamma\beta(\gamma+1) M_f^2 = 0 \\
\text{with} \quad X &= \frac{(\gamma+1) M_f^2}{2 + (\gamma-1) M_f^2}
\end{aligned}$$

Note that this is the solution for the gasdynamic shock because the magnetic field is negligible for $\beta \gg 1$.

(b) Derive the downstream Mach number as a function of upstream Mach number and for small and large plasma β. How does the downstream Mach number change as a function of β for fixed upstream Mach number? This illustrates the computation for the fast mode Machnumber but a solution for the sonic Machnumber is also acceptable since this was not specified in the problem.

Pressure ratio:

$$\frac{p_d}{p_u} = 1 + \beta^{-1}\left(1 - X^2\right) + \gamma M^2\left(1 - X^{-1}\right) = 1 + \beta^{-1}\left(1 - X^2\right) + \beta^{-1}(1+\gamma\beta) M_f^2\left(1 - X^{-1}\right)$$

Machnumber ratio:

$$\begin{aligned}\frac{M_d^2}{M_u^2} &= \frac{u_{nd}^2}{u_{nu}^2}\frac{v_{Au}^2+c_{su}^2}{v_{Ad}^2+c_{sd}^2} = \frac{u_{nd}^2}{u_{nu}^2}\frac{1+c_{su}^2/v_{Au}^2}{1+c_{sd}^2/v_{Ad}^2}\frac{v_{Au}^2}{v_{Ad}^2} \\ &= X^{-3}\frac{1+\gamma\beta_u/2}{1+\gamma\beta_d/2}\end{aligned}$$

so we need to compute β_d/β_u:

$$\begin{aligned}\frac{\beta_d}{\beta_u} &= \frac{2\mu_0 p_d}{B_d^2}\frac{B_u^2}{2\mu_0 p_u} = \frac{p_d}{p_u}\frac{B_u^2}{B_d^2} \\ &= \frac{1}{X^2}\left[1+\beta^{-1}\left(1-X^2\right)+\beta^{-1}\left(1+\gamma\beta\right)M_f^2\left(1-X^{-1}\right)\right]\end{aligned}$$

Substituting:

$$\begin{aligned}\frac{M_d^2}{M_u^2} &= X^{-3}\frac{2+\gamma\beta_u}{2+\gamma\beta_d} \\ &= X^{-3}\frac{2+\gamma\beta_u}{2+\gamma\beta_u X^{-2}\left[1+\beta_u^{-1}\left(1-X^2\right)+\beta_u^{-1}\left(1+\gamma\beta_u\right)M_f^2\left(1-X^{-1}\right)\right]} \\ &= X^{-1}\frac{2+\gamma\beta_u}{2X^2+\gamma\beta_u+\gamma\left(1-X^2\right)+\gamma\left(1+\gamma\beta_u\right)M_f^2\left(1-X^{-1}\right)}\end{aligned}$$

Note that $X=1$ yields $M_d^2 = M_u^2$. Also the derivative of the denominator is positive for $X \geq 1$ such that the denominator is only increasing for $X \geq 1$ while the numerator is constant. Therfore M_d^2/M_u^2 is monotonically decreasing for $X \geq 1$.

For large β (and large M_f:

$$\frac{M_d^2}{M_u^2} = X^{-1}\frac{\beta_u}{\beta_u+\gamma\beta_u M_f^2\left(1-X^{-1}\right)}$$

For small β:

$$\frac{M_d^2}{M_u^2} = X^{-1}\frac{2}{2X^2+\gamma\left(1-X^2\right)+\gamma M_f^2\left(1-X^{-1}\right)}$$

(c) Entropy change: $H = p/\rho^\gamma$

Relations:

$$X = \frac{\rho_d}{\rho_u} \qquad \frac{u_{nd}}{u_{nu}} = \frac{1}{X}, \qquad \frac{B_{yd}}{B_{yu}} = X$$

and $$\frac{p_d}{p_u} = 1+\beta^{-1}\left(1-X^2\right)+\gamma M^2\left(1-X^{-1}\right) = 1+\beta^{-1}\left(1-X^2\right)+\beta^{-1}\left(1+\gamma\beta\right)M_f^2\left(1-X^{-1}\right)$$

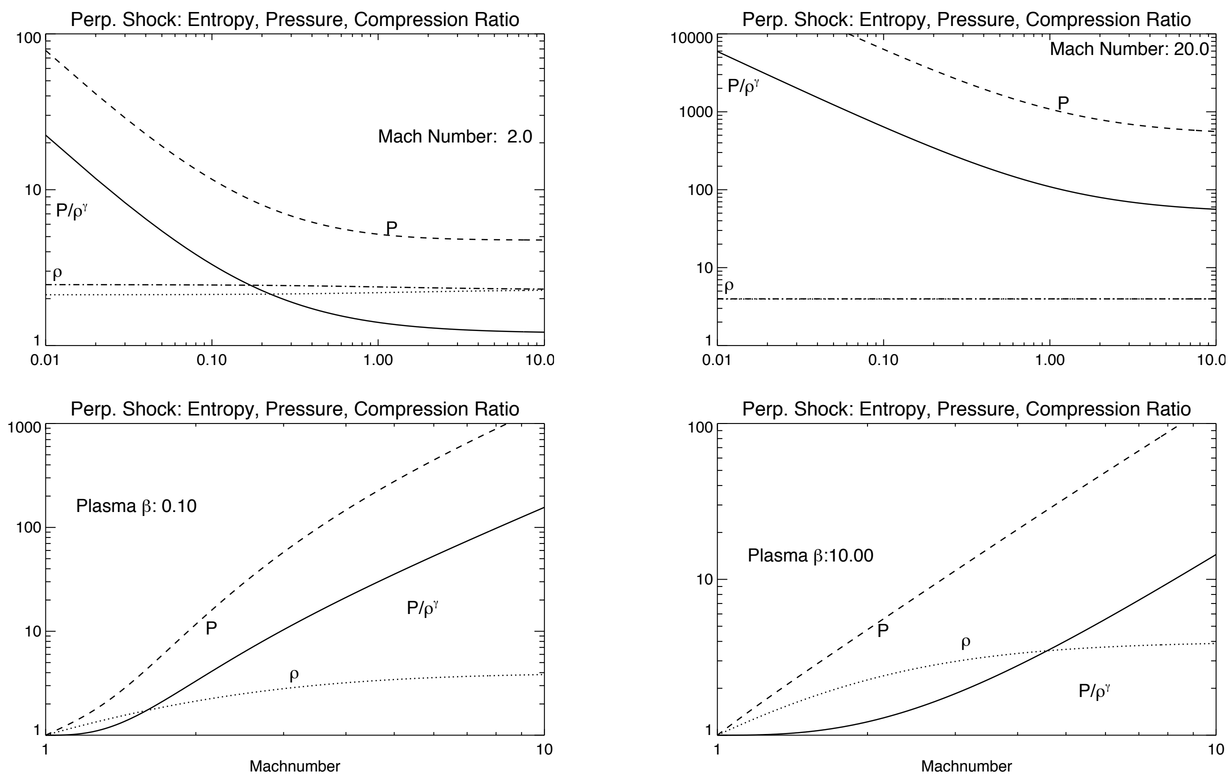
Entropy:

$$\begin{aligned}\frac{S_d}{S_u} &= \frac{p_d}{p_u}\left(\frac{\rho_u}{\rho_d}\right)^\gamma \\ &= \left[1+\beta^{-1}\left(1-X^2\right)+\beta^{-1}\left(1+\gamma\beta\right)M_f^2\left(1-X^{-1}\right)\right]X^{-\gamma}\end{aligned}$$

Note: The simulation treats the energy equation usually as $\partial p^{1/\gamma}/\partial t = -\nabla \cdot \left(p^{1/\gamma}\mathbf{u}\right)$ which use a one-dimensional discontinuity implies $p^{1/\gamma}\mathbf{u} = \text{const}$ or

$$\left(\frac{p_d}{p_u}\right)^{1/\gamma} = \frac{u_{nu}}{u_{nd}} \quad \text{or} \quad \frac{p_d}{p_u} = X^\gamma$$

In this case $S_d/S_u = 1$. For the actual solution of the compression ratio, the down- to upstream pressure ratio, and the ratio of the entropy function S the following 4 plots show these quantities as functions of plasma β and of fast mode Machnumber for a small and a large value of plasma β and Mach Number.



28. Simulation of a hydrodynamic shock.

The initial condition 8 in the simulation code is an example for a hydrodynamic shock. Note, in order to run this stable and without large oscillation you need to increase the viscosity parameters significantly.

(a) For what values of the viscosities do you get a reasonably stable solution?

(b) Run the code for two different upstream Machnumbers and use methods 0, 1, and 2 for the treatment of the energy equation (parameter intu). Describe any differences in the results. Compare the analytic jump conditions for a hydrodynamic shock with your results for the two Machnumbers. Discuss these results. (Note, that for larger Machnumbers you will need higher viscosity and possibly also a wider transition region parameterized with $l0$ in the program.

Solution:

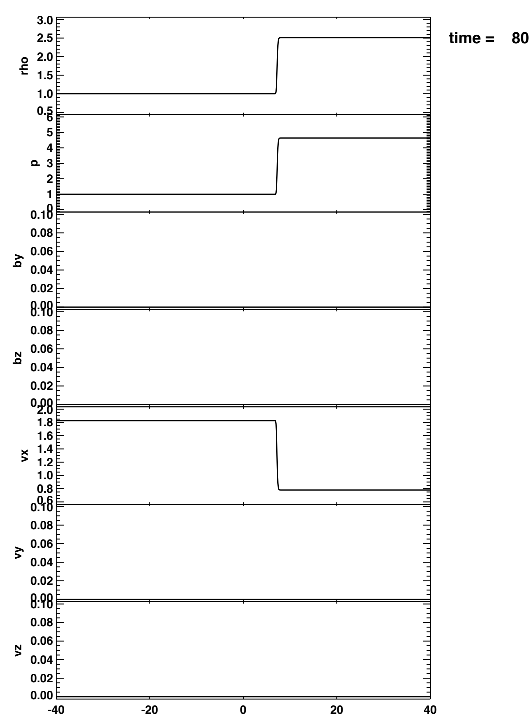
(a) The required viscosity depends somewhat on the upstream Machnumber. For the new version of the code smaller Machnumbers up to 3 or 4 require diffusion and viscosity of about 0.1 and they should be increased to about 0.2 for stronger shocks (Mach numbers as high as 20). Of equal or even higher importance is the time step. Theoretically the maximum time step is limited by $\Delta t \leq \Delta x / \max(v)$. The grid spacing is 0.1 for the default values. The maximum velocity is the maximum speed of information transport which in the case of a flowing plasma is the bulk velocity plus the fastest wave mode. In this case it is the upstream velocity plus the upstream speed of sound which for the current parameters is $\max(v) \simeq M_u c_s + c_s$. Here we have chosen upstream Machnumbers of 2 and 8 and the sound speed is $c_s = \sqrt{\gamma p / 2\rho} \simeq 0.91$ such that for $M_u = 2$ the limiting time step is approximately 0.36 and for $M_u = 8$ it is 0.012. Indeed the simulation for $M_u = 2$ runs stable for $\Delta t = 0.035$ but is unstable for $\Delta t = 0.04$. For $M_u = 2$ runs stable for $\Delta t = 0.01$ but is unstable for $\Delta t = 0.015$ consistent with the teoretical expectation.

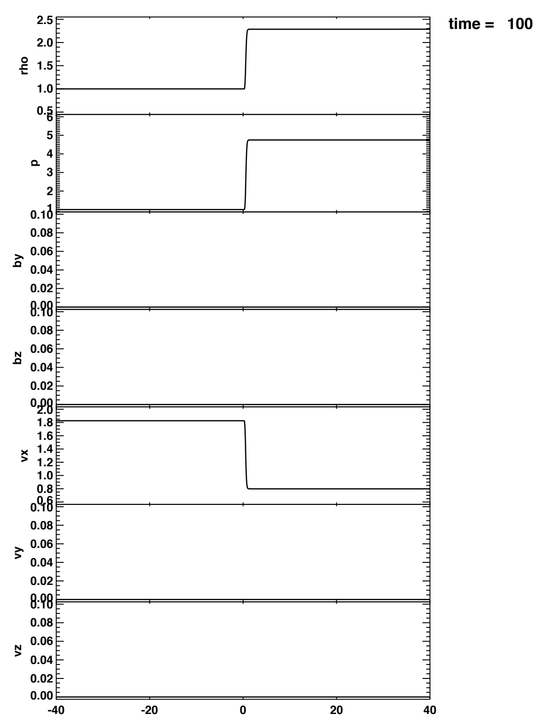
(b) The following 4 plots show the results for the 4 cases. The first two plots show the cases with $M_u = 2$ and the next two plots show the results for $M_u = 8$.

There are two apparent differences (although small for $M_u = 2$. The density compression is sightly larger and the shock moves in the positive $x$direction (with a velocity of 0.07) for `intu` $= 0$. The theoretical prediction for the shock are

$$X = \frac{(\gamma + 1) M_u^2}{2 + (\gamma - 1) M_u^2} \qquad \frac{p_d}{p_u} = \frac{2\gamma M_u^2 - (\gamma - 1)}{\gamma + 1}$$

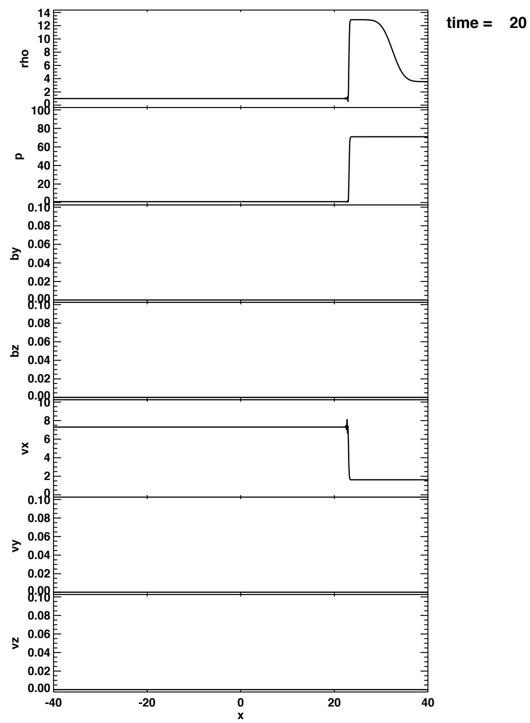
which yields $X = 2.3$ and $p_d/p_u = 4.75$ which is almost exactly satisfied by the 2nd method (energy conservation). However, for small Machnumbers the error usinf the 1st method is only about 10 %. (Note that the fact that the shock move in this case implies that the actual Machnumber is slightly less (by the shock velocity) than in the initial conditions.



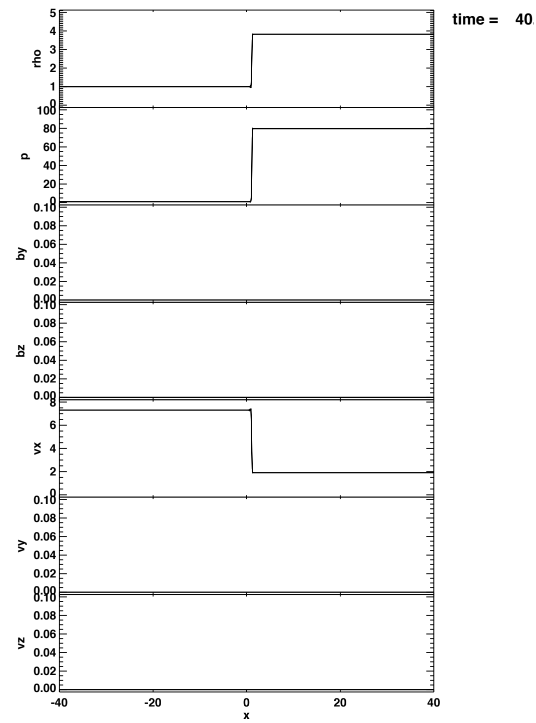


$M_u = 2$, energy equation using $p^{1/\gamma}$

$M_u = 2$, energy conservation



$M_u = 8$, energy equation using $p^{1/\gamma}$



$M_u = 8$, energy conservation

For the 2nd case with $M_u = 8$ the differences are much larger. The theoretical predictions are $X = 3.9$ and $p_d/p_u = 79.8$. Again the energy consevation method is highly accurate. The method for $\texttt{intu} = 0$ achieves almost the same results for the pressure and velocity change, but it fails strongly regarding the density change which is almost 13 for the compression here. Also, the shock moves now with considerable velocity (about 1.2) into the downstream direction, thus altering the actually applicable Machnumber.

27. Project

Decide a project topic, collect the material, and make a first attempt to understand the basic motivation, methodology, and impact of the results of the article.

In case you choose a simulation topic, outline your goals for this topic and formulate an approach how to achieve those.

Solution: Individual