30. Switch-off shock - Properties

Switch-off shocks are an important structure in simple models for magnetic reconnection.

(a) Compute the compression ratio for the limiting cases of the shock angle $\theta = 0$ and $\theta = \pi/2$.

(b) The dimensionless reconnection rate r is typically of $O(0.1)$ and is approximately $r = \cos\theta$ (i.e., θ is close to $\pi/2$). Expand the solution for the compression ratio around $\theta = \pi/2$ to include the lowest order correction for r and show that the compression ratio is approximately

$$X = 1 + \frac{1}{\gamma\beta + \gamma - 1}\left(1 - \frac{(\gamma\beta - 1)}{\gamma(\beta + 1)}r^2\right)$$

(c) Show that the amplification of the thermal pressure is given by

$$\frac{p_d}{p_u} = X\left[1 + \frac{\gamma - 1}{\gamma\beta}\left(1 - r^2\right)\right]$$

and compute the ratio of the entropy function $s = p/\rho^\gamma$.

(d) What are the compression, the pressure, and the entropy ratio in the limit of $\beta = 0$ and $\beta \gg 1$? Can you get large ratios for the entropy function in the case of high plasma β?

Solution:

(a) General Shock Relations (From class):

$$\frac{\rho_d}{\rho_u} = X \quad (1)$$

$$\frac{u_{nd}}{u_{nu}} = \frac{1}{X} \quad (2)$$

$$\frac{u_{yd}}{u_{yu}} = \frac{u_u^2 - u_{Au}^2}{u_u^2 - Xu_{Au}^2} \quad (3)$$

$$\frac{B_{yd}}{B_{yu}} = \frac{u_u^2 - u_{Au}^2}{u_u^2 - Xu_{Au}^2}X \quad (4)$$

$$\frac{p_d}{p_u} = X + \frac{\gamma - 1}{2}\frac{Xu_u^2}{c_{su}^2}\left(1 - \frac{u_d^2}{u_u^2}\right) \quad (5)$$

For the slow switch-off shock $B_{yd} = 0$ and $\mathbf{u}_u \parallel \mathbf{B}_u$ such that

$$u_u = u_{Au}$$

$$u_{nu} = \frac{B_{nu}}{\sqrt{\mu_0\rho_u}} = u_{An}$$

One can factorize this expression for the Compression ratio by noting that $X = 1$ is a solution (from class)

$$P_X = (X - 1)\left[2X\frac{c_s^2}{u_A^2} + \cos^2\theta\left\{X(\gamma - 1) - (\gamma + 1)\right\}\right] + X\sin^2\theta\left((\gamma - 1)X - \gamma\right) = 0$$

With $c_s^2/v_A^2 = \gamma\mu_0 p/B^2 = \gamma\beta/2$ and re-arranging yields the equation which has one positive solution

$$\begin{aligned} P_X &= (X-1)\left[X\gamma\beta + \cos^2\theta\left\{X(\gamma-1)-(\gamma+1)\right\}\right] + X\sin^2\theta\left((\gamma-1)X-\gamma\right) \\ &= (\gamma\beta+(\gamma-1))X^2 + \left(-\gamma\beta - \cos^2\theta\left((\gamma-1)+(\gamma+1)\right) - \gamma\sin^2\right)X + (\gamma+1)\cos^2\theta \\ &= (\gamma\beta+\gamma-1)X^2 - \gamma\left(\beta+1+\cos^2\theta\right)X + (\gamma+1)\cos^2\theta = 0 \end{aligned}$$

General solution:

$$\begin{aligned} AX^2 - BX + C &= 0 \\ X &= \frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2-4AC} \\ A &= \gamma\beta+\gamma-1 \\ B &= \gamma\left(\beta+1+\cos^2\theta\right) \\ C &= (\gamma+1)\cos^2\theta \end{aligned}$$

For $\cos^2\theta = 0$:

$$X_0 = \frac{\gamma(\beta+1)}{\gamma\beta+\gamma-1} = 1 + \frac{1}{\gamma\beta+\gamma-1}$$

For $\cos^2\theta = 1$:

$$\begin{aligned} X &= \frac{\gamma(\beta+2)}{2(\gamma\beta+\gamma-1)} + \frac{1}{2(\gamma\beta+\gamma-1)}\sqrt{\gamma^2(\beta+2)^2 - 4(\gamma+1)(\gamma\beta+\gamma-1)} \\ &= \frac{\gamma(\beta+2)}{2(\gamma\beta+\gamma-1)} + \frac{\pm(\gamma\beta-2)}{2(\gamma\beta+\gamma-1)} \\ &= \begin{cases} 1 & \text{for} \quad \gamma\beta > 2 \\ \frac{\gamma+1}{\gamma\beta+\gamma-1} & \text{for} \quad \gamma\beta < 2 \end{cases} \end{aligned}$$

(b) General solution with the reconnection rate $r \approx \cos\theta$

$$X = \frac{\gamma(\beta+1+r^2)}{2(\gamma\beta+\gamma-1)} + \frac{1}{2(\gamma\beta+\gamma-1)}\sqrt{\gamma^2(\beta+1+r^2)^2 - 4(\gamma+1)(\gamma\beta+\gamma-1)r^2}$$

With

$$\begin{aligned} \gamma^2\left(\beta+1+r^2\right)^2 - 4(\gamma+1)(\gamma\beta+\gamma-1)r^2 &= \\ \gamma^2(\beta+1)^2 + \gamma^2\left(2(\beta+1)r^2 + r^4\right) - 4\left(\gamma^2\beta+\gamma^2-\gamma+\gamma\beta+\gamma-1\right)r^2 &= \\ \gamma^2(\beta+1)^2 - 2\left(\gamma^2\beta+\gamma^2+2\gamma\beta-2\right)r^2 + r^4 & \end{aligned}$$

Taylor expansion for $x \ll 1$: $\sqrt{1-x} = 1 - x/2 + O(x^2)$ such that for $r^2 \ll 1$:

$$\begin{aligned} X &\simeq \frac{\gamma(\beta+1+r^2)}{2(\gamma\beta+\gamma-1)} + \frac{1}{2(\gamma\beta+\gamma-1)}\sqrt{\gamma^2(\beta+1)^2 - 2(\gamma^2\beta+\gamma^2+2\gamma\beta-2)r^2} \\ &= \frac{\gamma(\beta+1+r^2)}{2(\gamma\beta+\gamma-1)} + \frac{\gamma(\beta+1)}{2(\gamma\beta+\gamma-1)}\sqrt{1 - 2\frac{\gamma^2\beta+\gamma^2+2\gamma\beta-2}{\gamma^2(\beta+1)^2}r^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma\left(\beta+1+r^2\right)}{2\left(\gamma\beta+\gamma-1\right)}+\frac{\gamma\left(\beta+1\right)}{2\left(\gamma\beta+\gamma-1\right)}\left(1-\frac{\gamma^2\beta+\gamma^2+2\gamma\beta-2}{\gamma^2\left(\beta+1\right)^2}r^2\right)\\
&= \frac{\gamma\left(\beta+1\right)}{\gamma\beta+\gamma-1}+\frac{\gamma}{2\left(\gamma\beta+\gamma-1\right)}r^2-\frac{1}{2\left(\gamma\beta+\gamma-1\right)}\frac{\gamma^2\beta+\gamma^2+2\gamma\beta-2}{\gamma\left(\beta+1\right)}r^2\\
&= \frac{\gamma\left(\beta+1\right)}{\gamma\beta+\gamma-1}+\left(\frac{\gamma^2\left(\beta+1\right)-\left(\gamma^2\beta+\gamma^2+2\gamma\beta-2\right)}{2\gamma\left(\gamma\beta+\gamma-1\right)\left(\beta+1\right)}\right)r^2\\
&= 1+\frac{1}{\gamma\beta+\gamma-1}-\frac{2\left(\gamma\beta-1\right)}{2\gamma\left(\gamma\beta+\gamma-1\right)\left(\beta+1\right)}r^2\\
X &\simeq 1+\frac{1}{\gamma\beta+\gamma-1}\left(1-\frac{\gamma\beta-1}{\gamma\left(\beta+1\right)}r^2\right)
\end{aligned}
$$

The result shows that the reconnection rate has only a minor influence on the compression ratio. For $\beta > 1/\gamma$ the reconnection rate will lower the compression ratio and for smaller β reconnection will increase the compression ratio.

(c) Plasma pressure ratio

$$
\frac{p_d}{p_u}=X+\frac{\gamma-1}{2}\frac{Xu_u^2}{c_{su}^2}\left(1-\frac{u_d^2}{u_u^2}\right)
$$

From $B_{yd}=0$ we obtain $u_{nu}^2=u_{An}^2$ and $u_u^2=u_A^2$. In addition $u_d^2=u_{nd}^2=u_{nu}^2/X^2$. With $c_s^2=\gamma p/\left(mn\right)$ and $u_A^2=B^2/\left(\mu_0 mn\right)$and $c_s^2/u_A^2=\gamma\mu_0 p/B^2=\gamma\beta/2$ such that

$$
\begin{aligned}
\frac{p_d}{p_u} &= X\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{1}{X^2}\frac{u_{An}^2}{u_A^2}\right)\right]\\
&= X\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{1}{X^2}\frac{B_n^2}{B_u^2}\right)\right]\\
&= X\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{r^2}{X^2}\right)\right]
\end{aligned}
$$

For the entropy function we obtain:

$$
\frac{p_d}{p_u}\left(\frac{\rho_u}{\rho_d}\right)^\gamma=X^{1-\gamma}\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{r^2}{X^2}\right)\right]
$$

(d) Limit of $\beta=0$:

$$
\begin{aligned}
\lim_{\beta\to0}X &= \lim_{\beta\to0}\left[1+\frac{1}{\gamma\beta++\gamma-1}\left(1-\frac{\gamma\beta-1}{\gamma\left(\beta+1\right)}r^2\right)\right]=\frac{\gamma}{\gamma-1}+\frac{r^2}{\gamma-1}\\
\lim_{\beta\to0}\frac{p_d}{p_u} &= \left(\frac{\gamma}{\gamma-1}+\frac{r^2}{\gamma-1}\right)\lim_{\beta\to0}\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{r^2}{X^2}\right)\right]=\infty\\
\lim_{\beta\to0}\frac{p_d}{p_u}\left(\frac{\rho_u}{\rho_d}\right)^\gamma &= \left(\frac{\gamma}{\gamma-1}+\frac{r^2}{\gamma-1}\right)^{1-\gamma}\lim_{\beta\to0}\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{r^2}{X^2}\right)\right]=\infty
\end{aligned}
$$

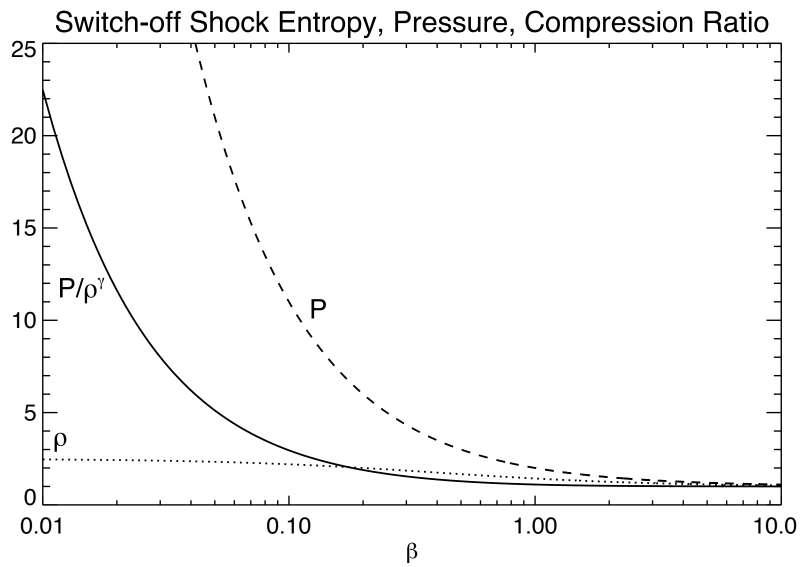
Limit of $\beta=\infty$:

$$
\lim_{\beta\to\infty}X \simeq \lim_{\beta\to\infty}\left[1+\frac{1}{\gamma\beta+\gamma-1}\left(1-\frac{\gamma\beta-1}{\gamma\left(\beta+1\right)}r^2\right)\right]=1
$$

$$\lim_{\beta\to\infty}\frac{p_d}{p_u} = \lim_{\beta\to\infty} X\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{r^2}{X^2}\right)\right]=1$$

$$\lim_{\beta\to\infty}\frac{p_d}{p_u}\left(\frac{\rho_u}{\rho_d}\right)^\gamma = \lim_{\beta\to\infty} X^{1-\gamma}\left[1+\frac{\gamma-1}{\gamma\beta}\left(1-\frac{r^2}{X^2}\right)\right]=1$$

The following is a plot of the compression X, the pressure and the entropy ratio for a slow switch-off shock as a function of the plasma β.



31. Switch-off shock - Simulation

Modify the the initial condition 1 to introduce a separate equilibrium density, pressure and magnetic field such that the initial configuration becomes $B_y = B_0 \tanh x$, $\rho = \rho_0 + B_0^2/\cosh^2 x$ and use $p = p_0 + B_0^2/\cosh^2 x$ and choose $B_0 = 1$, $\rho_0 = 1$, and $p_0 = 1$ as a reference. Run this case for with $B_x = 0$ and $B_x = 0.1$ (note, you have to run this probably for several 100 simulation times to see the evolution clearly.

(a) The two step waves that propagate in the positive and negative x direction are slow switch-off shock. Examine the properties and compare these with the results for the plasma compression and pressure ratio.

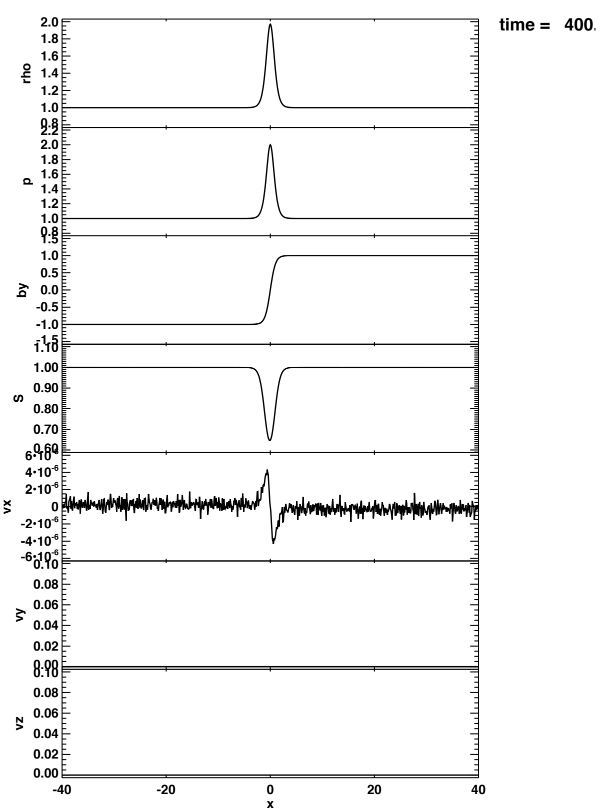
(b) Sketch the situation. What transformation velocity is needed for a transformation into a frame in which the electric field is 0?

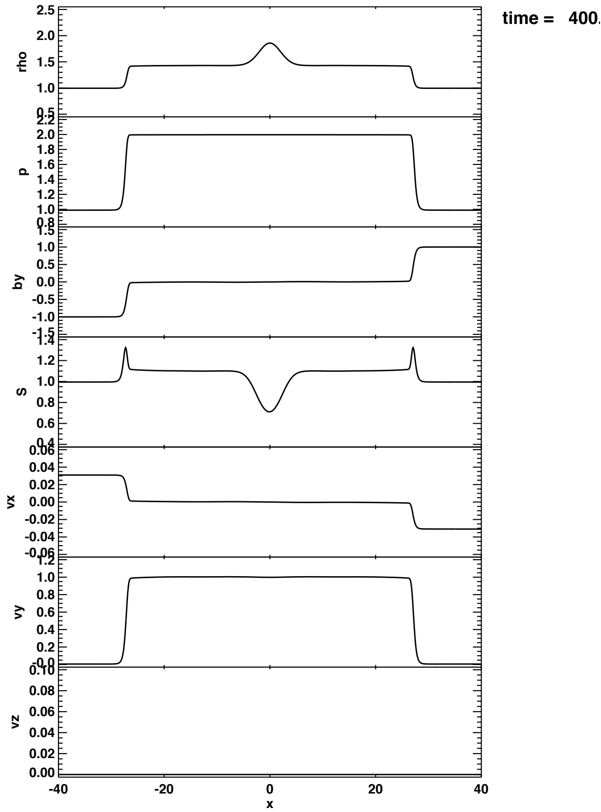
(c) What changes are needed to simulate low and a high β cases. Apply these changes and run one low and one high β case. Are the results consistent with the prediction for the low and high beta limits for the compression and pressure ratio.

(d) Run the low beta case for `intu` $= 0$ and 2. How do the results differ?

Solution:

(a) The following plots show the configuration for $B_0 = 1$, $\rho_0 = 1$, and $p_0 = 1$ at time $t = 400$. The plot on the left shows the case for $B_x = 0$ and on right for $B_x = 0.1$. We have added the entropy ratio to the plot panels.



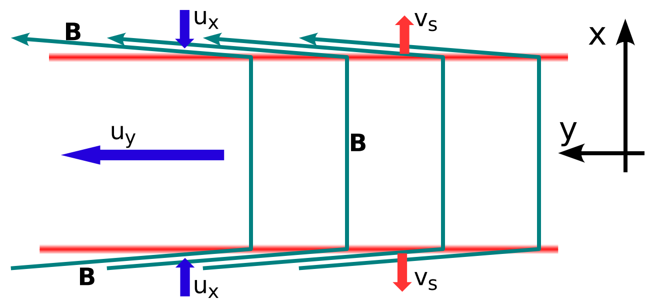


For $B_x = 0$ the configuration is in rather good equilibrium. Note that the scale for the velocity is of order 10^{-6} and the small variation is just background noise. This case is run with viscosities of $5 \cdot 10^{-4}$ to minimize dissipation and variations in the entropy are in the 1% range indicating that the configuration stays in a stable Harris type equilibrium for a long time. For $B_x = 0.1$ the B_y component behind the moving step wave is 0 such that we expect this wave to represent a slow shock. For $\beta = 1$ and $r^2 \simeq \cos^2\theta = B_x^2/B^2 \simeq 0.01$ the switch-off shock relations are

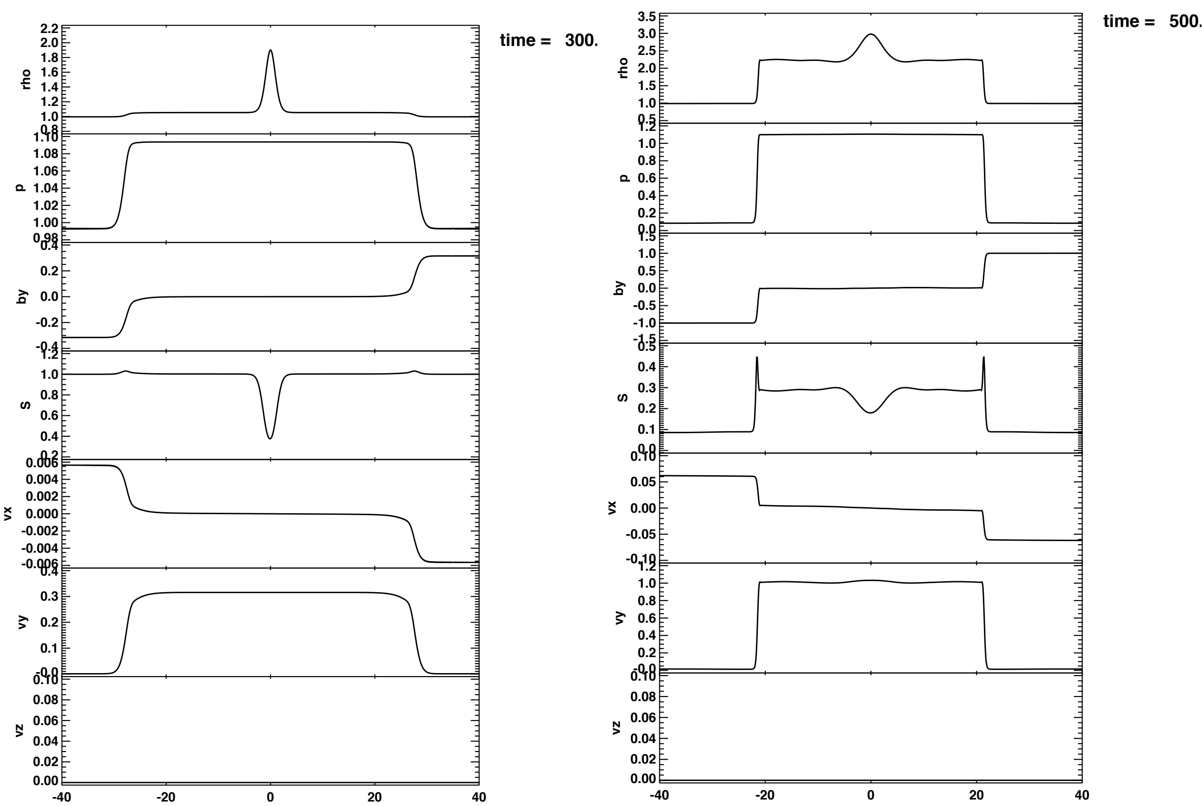
$$
\begin{aligned}
X &\simeq 1 + \frac{1}{\gamma\beta + \gamma - 1}\left(1 - \frac{\gamma\beta - 1}{\gamma(\beta + 1)} r^2\right) \simeq \frac{2\gamma}{2\gamma - 1} = \frac{10}{7} = 1.43 \\
\frac{p_d}{p_u} &= X\left[1 + \frac{\gamma - 1}{\gamma\beta}\left(1 - \frac{r^2}{X^2}\right)\right] \simeq \frac{2\gamma}{2\gamma - 1}\left[\frac{2\gamma - 1}{\gamma}\right] = 2 \\
\frac{p_d}{p_u}\left(\frac{\rho_u}{\rho_d}\right)^\gamma &= X^{1-\gamma}\left[1 + \frac{\gamma - 1}{\gamma\beta}\left(1 - \frac{r^2}{X^2}\right)\right] \simeq 2\left[\frac{2\gamma - 1}{2\gamma}\right]^\gamma = 1.10
\end{aligned}
$$

The step wave has the properties $\Delta\rho \simeq 0.43$, $\Delta p = 1$, and $\Delta S = 0.1$ such that $\rho_d/\rho_u = 1.43$, $p_d/p_u = 2$, and $S_d/S_u = 1.1$ in excellent agreement with the theoretical prediction. Note that the velocity changes are $\Delta v_y = 1.0$ and $\Delta v_x = 0.31$. However, the wave is propagating with a velocity of $v_S = 27.2/400 = 0.069$ such that the upstream normal velocity is $u_u = 0.10$ and the down stream velocity is $u_d = 0.069$ which yields a ratio of $u_u/u_d = 1.45$ which is also in reasonable agreement with the expected ratio of X.

(b) The following sketch depicts the magnetic field and flow configuration. We have inflow indicated by u_x and the wave front is propagating outward with a speed of v_s. Behind the wave the plasma is accelerated to a velocity of u_y which for the simulation is equal to the y component of Alfvén speed in the upstream region. Thus a transformation of $V_y = -u_y = -u_{Ay}$ is need to transform the electric field to 0 in the downstream region. However, in order to transform the Electric field to 0 also in the upstream region an observer would have to move with the wave front which is moving with the velocity v_s which implies a transformation velocity of $V_x = v_s$.



(c) To obtain a simulation for large β we can decrease the B_y magnetic field. Choosing $B_0 = 1/\sqrt{10}$ yields a plasma β of 10 (shown in the next plot on the left). For the case of a small plasma β we choose $p_0 = 0.1$ shown on the right.



For the high β case we have very little compression, a very small pressure change, and the entropy ratio is almost 1. The wave is mostly a hydrodynamic shock, however, with a Mach number only slightly greater than 1. For the low β case we have a compression of $X = 2.2$, a pressure ratio of 11, and an entropy ratio of about 3. Compared to the plot in problem 31 this is again in rather good agreement.

(d) The next plot shows the low β case for the integration method `intu` $= 0$ which conserves the entropy function. Compared to the energy conserving scheme the change in pressure, B_y, and u_y are almost identical. However, entropy is exactly conserved in the `intu` $= 0$ case which yields a much larger change in density, entropy, and v_x. Related to this there is also a much slower propagation of the wave in the x direction. The slower propagation is compensated through a higher upstream velocity such that in the rest frame of the wave the upstream normal velocity is still the normal component of the Alfvén speed.