34. Fast wave dispersion relation

(a) Consider a plasma with a uniform magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_y$, density ρ_0, and pressure p_0 at rest. Using the ideal MHD equations, derive the dispersion relation for a fast mode wave with it's wave vector $\mathbf{k}$ entirely along the x direction. What is the physical cause for this effect?

(b) Assume a magnetic field perturbation $\delta B_y = h(x,t)\,B_0$ with the wave shape function (or profile) $h(x,t) = h(x - x_0 + v_f t)$ and the fast mode speed $v_f = \sqrt{(\gamma p_0 + B_0^2/\mu_0)/\rho_0}$. Note that $h(s)$ can have any shape but it maximum should be smaller than 1 - otherwise a shock will form immediately. Using the MHD equations show that

$$
\begin{aligned}
u_x &= h(x,t)\,v_f \\
\rho &= (1 + h(x,t))\,\rho_0 \\
p &= (1 + \gamma h(x,t))\,p_0
\end{aligned}
$$

Solution:

Ideal MHD Equations (Section 2.5):

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla p + \mathbf{j} \times \mathbf{B} \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) \\
\frac{\partial p}{\partial t} &= -\nabla \cdot p\mathbf{u} - (\gamma - 1)\,p \nabla \cdot \mathbf{u} \\
\mathbf{j} &= \frac{1}{\mu_0} \nabla \times \mathbf{B}
\end{aligned}
$$

where we have combined Faraday's law and Ohm's law in the third equation.

With uniform equilibrium properties $\mathbf{B}_0 = B_0\mathbf{e}_y$, density ρ_0, pressure p_0, $\mathbf{u}_0 = 0$ and $\mathbf{j}_0 = 0$ (because of the uniform $\mathbf{B}_0$) the linearized equations are

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\nabla \cdot \rho_0 \mathbf{u}_1 \\
\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} &= -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \\
\frac{\partial p_1}{\partial t} &= -\gamma p_0 \nabla \cdot \mathbf{u}_1 \\
\mathbf{j}_1 &= \frac{1}{\mu_0} \nabla \times \mathbf{B}_1
\end{aligned}
$$

Using $\mathbf{B}_0 = B_0\mathbf{e}_y$, $\mathbf{k} = k\mathbf{e}_x$, perturbations $\sim \exp i(kx - \omega t)$, and noting that $\mathbf{j}_1 = \frac{ik}{\mu_0} B_{y1}\mathbf{e}_z - \frac{ik}{\mu_0} B_{z1}\mathbf{e}_y$ the equations become

$$
\begin{aligned}
-i\omega \rho_1 &= -ik\rho_0 u_{x1} \\
-i\omega \rho_0 u_{x1} &= -ikp_1 - \frac{ik}{\mu_0} B_{y1} B_0 \\
-i\omega B_{y1} &= -ik(u_{x1}B_{y0} - u_{y1}B_{x0}) \\
-i\omega B_{z1} &= ik(u_{z1}B_{x0} - u_{x1}B_{z0}) \\
-i\omega p_1 &= -ik\gamma p_0 u_{x1}
\end{aligned}
$$

and $i\omega u_{y1} = 0$, $i\omega u_{z1} = 0$, $i\omega B_{x1} = 0$. Substituting $B_{x0} = B_{z0} = 0$, $B_{x1} = 0$, $u_{y1} = u_{z1} = 0$, j_{y1}, and j_{z1} yields

$$
\omega \rho_1 = k\rho_0 u_{x1}
$$

$$\begin{aligned}
\omega\rho_0 u_{x1} &= kp_1 + \frac{k}{\mu_0} B_{y1} B_0 \\
\omega B_{y1} &= k u_{x1} B_0 \\
\omega B_{z1} &= 0 \\
\omega p_1 &= k\gamma p_0 u_{x1}
\end{aligned}$$

this implies $B_{z1} = 0$ and $j_{y1} = 0$. Since the density perturbation is not present in equations 2 and 3 the dispersion relation is given by

$$\begin{aligned}
\omega\rho_0 u_{x1} &= \frac{1}{\omega} kk\gamma p_0 u_{x1} + \frac{k}{\mu_0} B_{y1} B_0 \\
\omega B_{y1} &= k u_{x1} B_0
\end{aligned}$$

where we have replaced p_1 from the pressure equation. Multiplication with $\omega/\left(k^2\rho_0\right)$ and substituting the second equation into the first equation

$$u_{x1}\frac{\omega^2}{k^2} = u_{x1} c_s^2 + \frac{\omega}{k} v_A^2 \frac{B_{y1}}{B_0} = u_{x1}\left(c_s^2 + v_A^2\right)$$

With the fast speed $v_f^2 = \gamma p_0/\rho_0 + B_0^2/\mu_0\rho_0 = c_s^2 + v_A^2$ the dispersion relation becomes

$$\frac{\omega^2}{k^2} = v_f^2$$

(b) The derivation of the dispersion relation for the given geometry shows that $u_{y1} = u_{z1} = 0$, $B_{x1} = B_{z1} = 0$. A k vector only in the x direction implies for the derivatives $\partial_y = \partial_z = 0$. Furthermore the current density carried by the wave has only a z component $j_{z1} = \partial_x B_{y1}$. Denoting $\partial_t = \partial/\partial t$

$$\begin{aligned}
\partial_t \rho_1 &= -\rho_0 \partial_x u_{x1} \\
\rho_0 \partial_t u_{x1} &= -\partial_x p_1 + j_{y1} B_{z0} - j_{z1} B_{y0} = -\partial_x p_1 - B_0 \partial_x B_{y1} \\
\partial_t B_{y1} &= -\partial_x \left(u_{x1} B_{y0} - u_{y1} B_{x0}\right) = -B_0 \partial_x u_{x1} \\
\partial_t p_1 &= -\gamma p_0 \partial_x u_{x1}
\end{aligned}$$

Assuming that all perturbation variables are functions of $h\left(x,t\right) = h\left(X\right)$ with $X = x - x_0 + v_f t$ such that, for instance, $\partial_t \rho_1 = \left(d_h \rho_1\right) \partial_t h = v_f \left(d_h \rho_1\right) d_X h$ and $\partial_x \rho_1 = \left(d_h \rho_1\right) d_X h$ with $d_h = d/dh$ and $d_X = d/dX$, which implies that $\partial_t \rho_1 = v_f \partial_x \rho_1$.

Continuity: $v_f d_h \rho_1 = -\rho_0 d_h u_{x1}$; Integration with respect to h yields: $v_f \rho_1\left(h\right) = -\rho_0 u_{x1}\left(h\right)$

Induction equation: $v_f d_h B_{y1} = -B_0 d_h u_{x1}$; Integration: $v_f B_{y1}\left(h\right) = -B_0 u_{x1}\left(h\right)$

Pressure equation: $v_f d_h p_1 = -\gamma p_0 d_h u_{x1}$; Integration: $v_f p_1 = -\gamma p_0 u_{x1}$.

With $\delta B_y = B_{y1} = h\left(x,t\right) B_0$ we obtain:

$$\begin{aligned}
u_{x1} &= -v_f h\left(X\right) \\
\rho_1 &= \rho_0 h\left(X\right) \\
p_1 &= \gamma p_0 h\left(X\right)
\end{aligned}$$

with $X = x - x_0 + v_f t$. Here $X = const$ implies wave propagation in the $-x$ direction. Using $X = x - x_0 - v_f t$ implies propagation into the $+x$direction and changes $u_{x1} = v_f h\left(X\right)$.

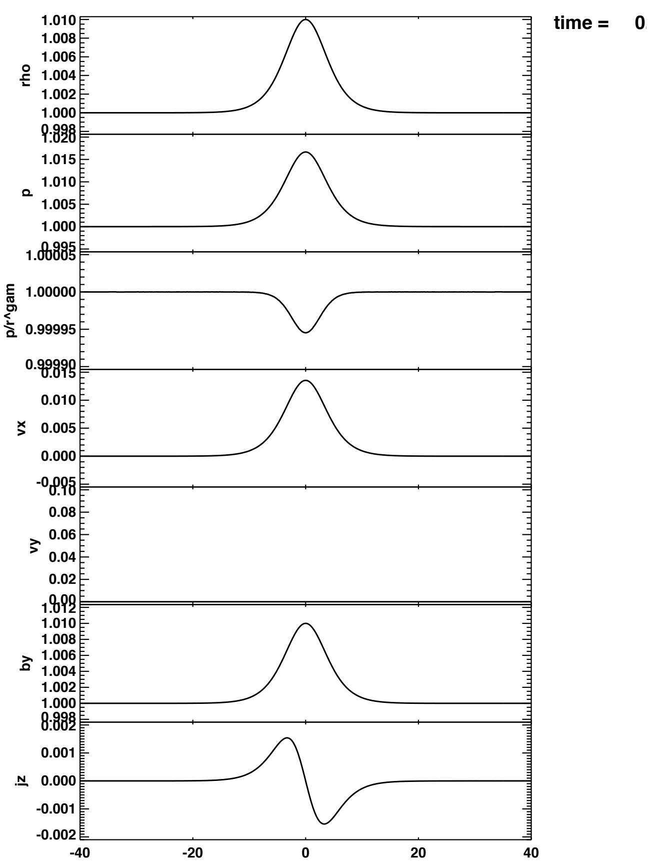
34. Fast wave propagation + interaction with a plasma boundary

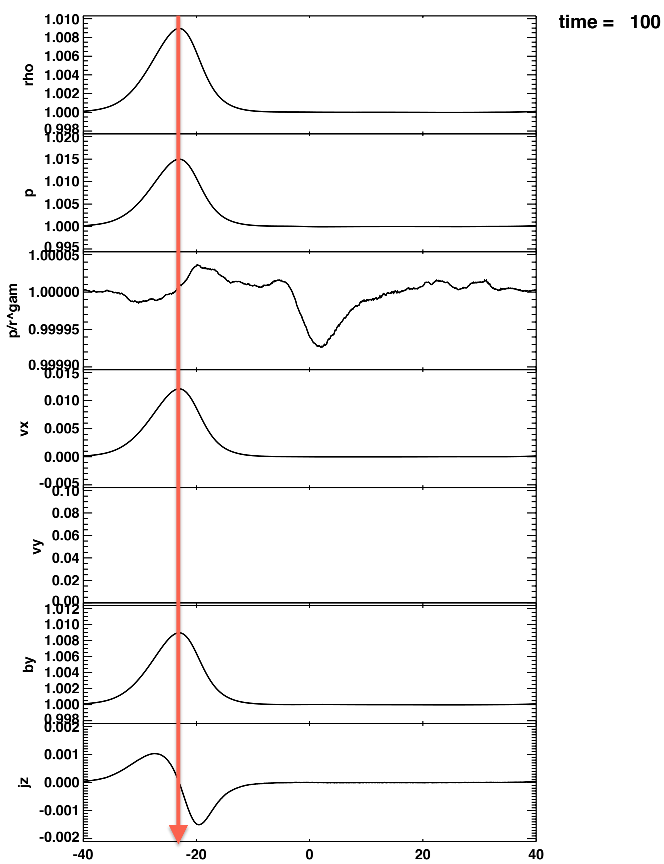
(a) Use the 1D MHD code with the initial condition 5. This condition uses a bell shaped waveform with $h = \widehat{h}/\cosh^2\left[\left(x - v_f t\right)/l_0\right]$ with the width l_0 and the amplitude $\widehat{h}$. Run this for $l_0 = 5$ and amplitude between $0.01, 0.1,$ and 1. Does the wave for small amplitudes satisfy the dispersion relation? How fast do you observe the steepening of the fast wave to a shock for the different perturbation? For the amplitude of $\widehat{h} = 1$ check the jump conditions for the fast perpendicular shock for the integration methods $\texttt{intu} = 0$ and 2.

(b) Now change the initial conditions to center the perturbation at $x = -20$ with $l_0 = 4$ and introduce a density boundary where the density decreases to 0.25 for $x > 0$ (this boundary can represent the magnetopause). Run the program for an amplitude of $\widehat{h} = 0.5$. What happens to the wave and the density boundary when the wave pulse interacts with this boundary? Repeat this for an amplitude of $\widehat{h} = 1$. Explain why the density boundary is displaced in this experiment. How would you compute this displacement?

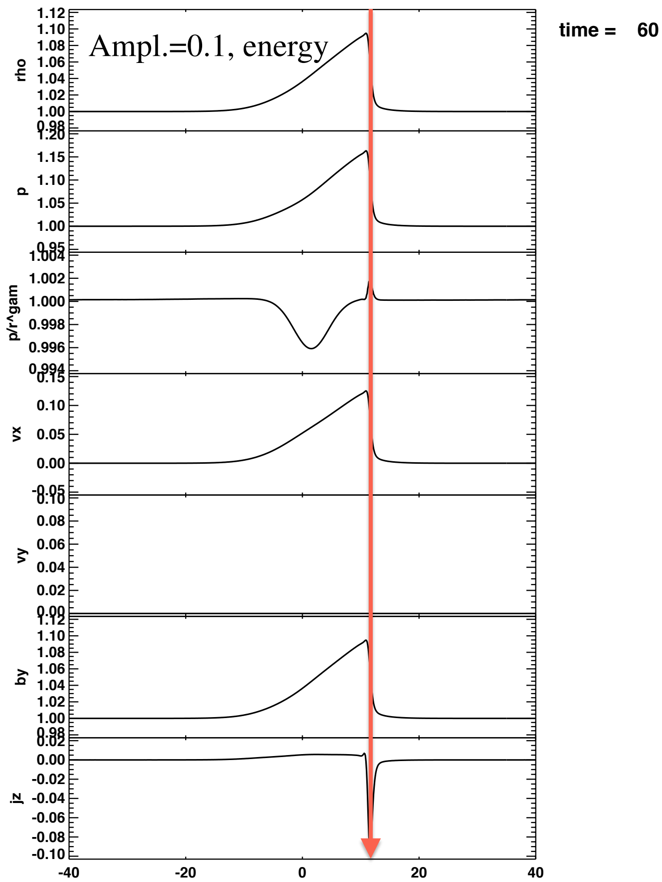
Solution:

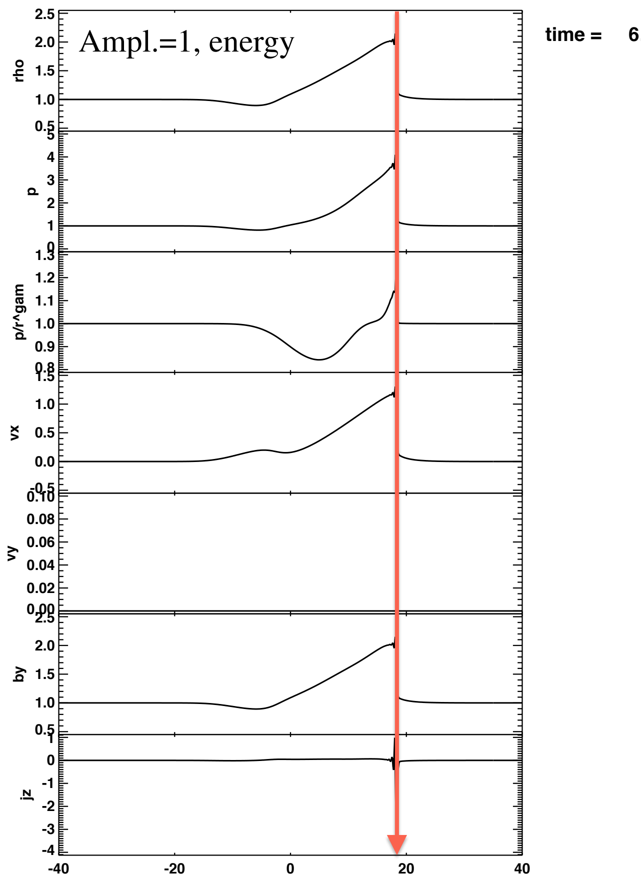
(a) Initial condition 5 has already implemented with the $1/\cosh^2$ profile. The following plots show the wave for an amplitude of 0.01 and a width of $l_0 = 5$ at time $t = 0$ and $t = 100$. At $t = 100$ the wave has moved through the periodic system almost twice or a distance of $s = 137$ which yields a velocity of about $v_{w0.01} = 1.37$ compared to the theoretical speed of $v_f = \sqrt{\left(\gamma p/2 + B^2\right)/\rho} = 1.354$ (with $p_0 = 1$, $\rho_0 = 1$, and $B_0 = 1$; $\texttt{intu} = 2$). At this point in time the amplitude of the wave is reduced by about 10% and the wave has slightly steepened even though the amplitude is only 0.01.



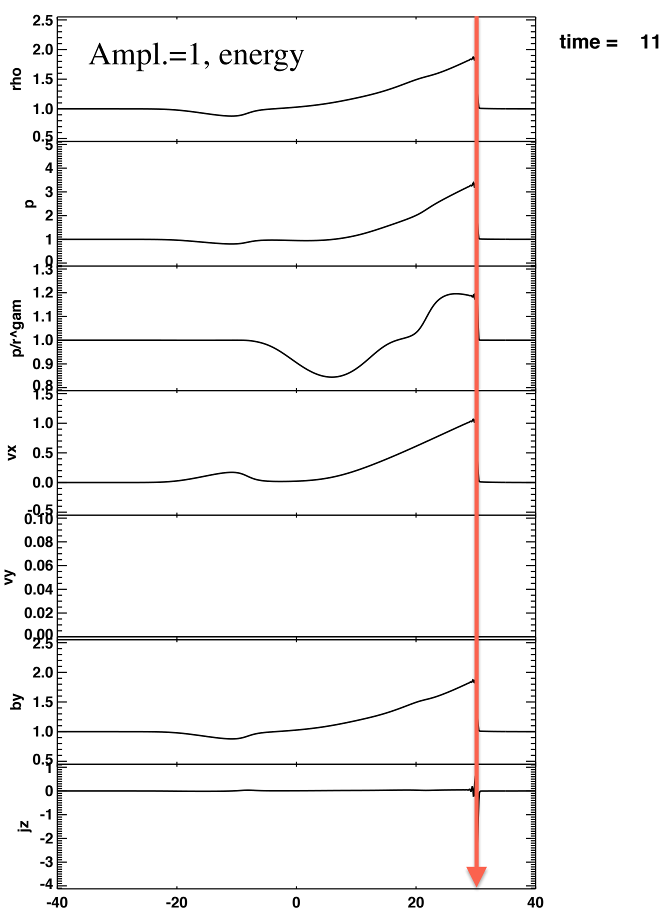


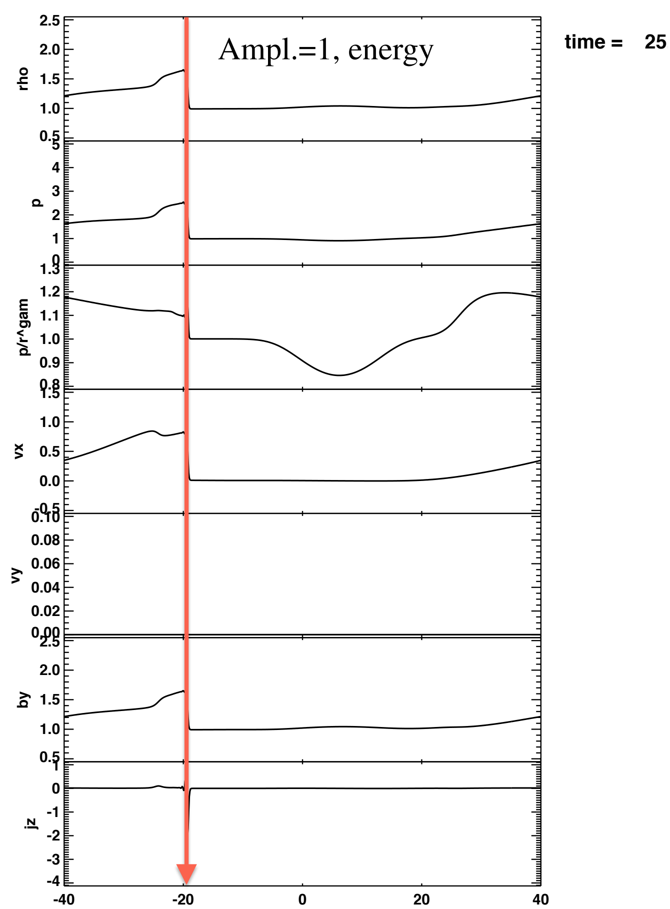
The following two plots show the fast wave for amplitudes 0.1 and 1 just at the time when the wave has steepened to a shock.





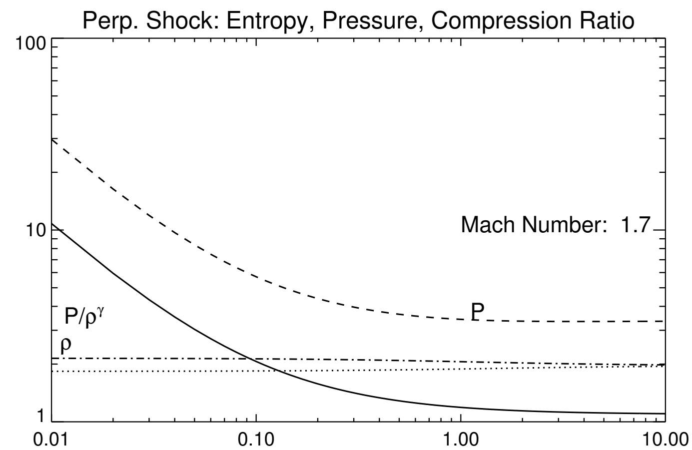
The next two plots show the further propagation of the shock wave to determine the speed and mach number:

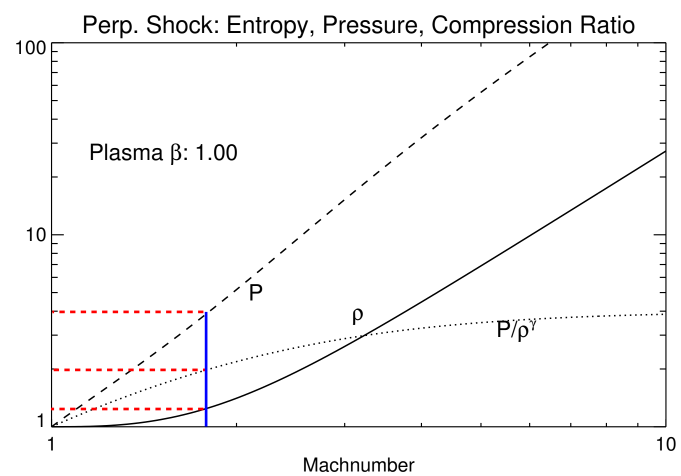




For the amplitude of $\hat{h} = 0.1$ and $t = 60$ the wave has just steepened to a shock and has traveled a distance of $s = 91$ corresponding to speed of $v_{w0.1} = 1.53$ considerably faster (because of the steepening) than the linear wave speed of $v_f = 1.354$. More extreme is the case with $\hat{h} = 1$. The wave becomes a shock at $t = 6$ after a distance of only s=18 yielding and initial speed of $v_{w1} = 3$. At $t = 11$ the shock is a s=30 which implies a speed of the shock of $w_{s1} = \Delta s/\Delta t = 12/5 = 2.4$. In the frame moving with the shock the upstream velocity is therefore $v_u = w_{s1}$ and the downstream velocity is $v_d = w_{s1} - v_{x,wave} = 2.4 - 1.2 = 1.2$, where $v_{x,wave}$ is the velocity v_x just behind the shock. The fast Mach number is $M_f = v_u/v_f = 1.6$. Note that compression ratio, and $v_{x,wave}$ are smaller by 10 to 20% at $t = 11$ indicating that the shock is damped and that therefore the Mach number at $t = 6$ may be about 10% higher.

To examine the shock conditions we need to identify the shock parameters. The velocity at the shock ramp changes from about 1.2 at $t = 4$ to about 1.0 at $t = 14$. The location of the ramp is $x = 13.7$ at $t = 4$ and $x = 37$ at $t = 14$. This yields a shock velocity of about $v_{shock} = v_{up} = 2.3$ (note that the upstream velocity is measured in the rest frame of the shock and in this frame the plasma is stream at a speed of v_{shock}) which yields a fast Mach number of $M_f = 1.75$. The following plot shows compression ratio, amplification of plasma pressure, and change of p/ρ^γ (similar to problem 25). For a fast Mach number of 1.7 and plasma $\beta = 1$ we expect a compression ratio of about 1.9 and a pressure increase of about 3.5. These values agree reasonably well with the density and magnetic field increase of about $X \simeq 2$ and a pressure increase of $p_d/p_u \simeq 3.7$ from the simulation. Note, the lower compression of about $X = 1.8$ later at $t = 11$ indicates that the shock has initially moved a bit faster and slowed down. This is due to the dissipation which lowers the shock ramp of the wave during its evolution which implies a slowing down of the shock.

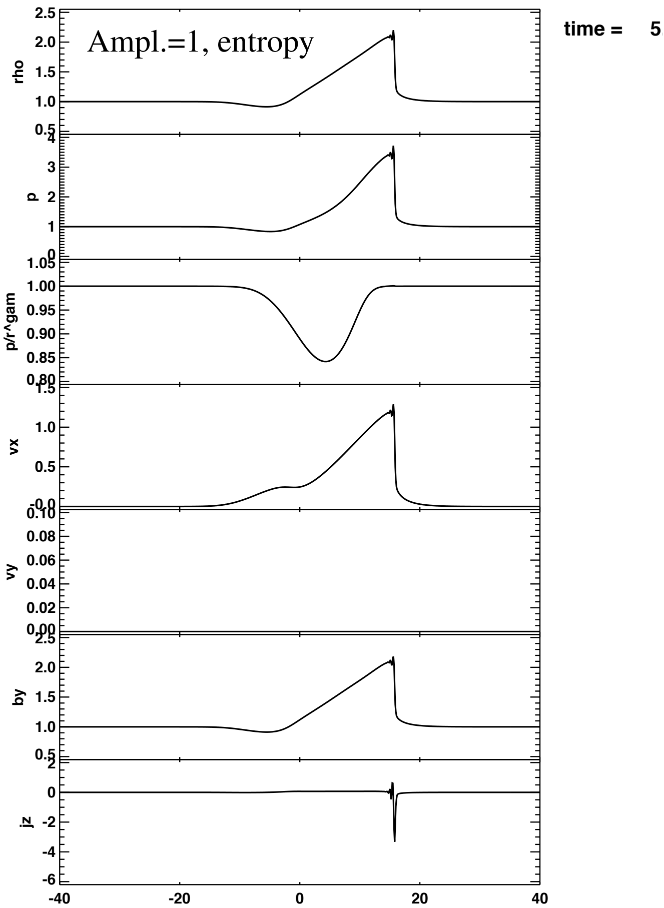


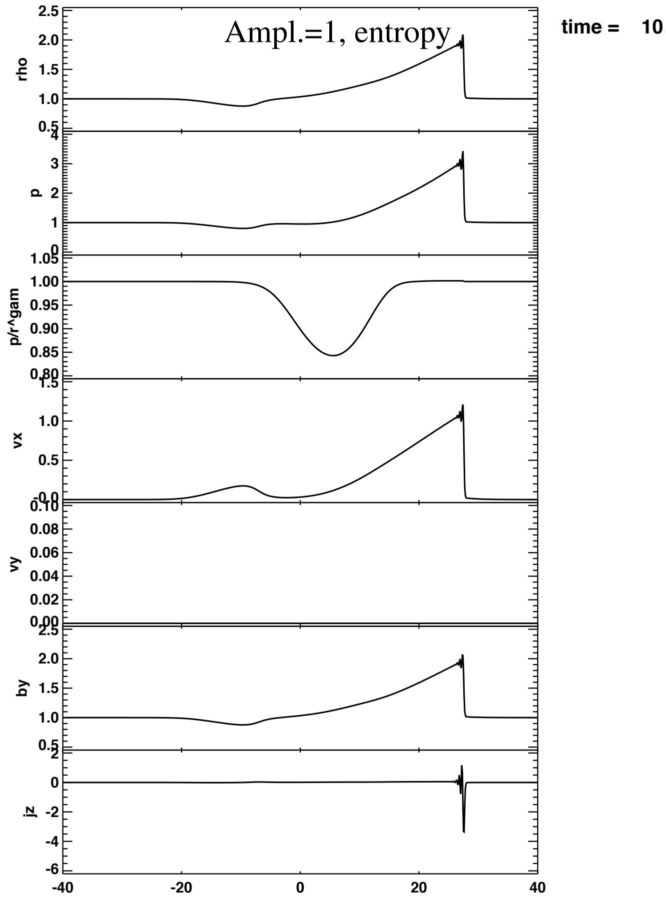


This is confirmed by the last plot at $t = 25$. Here all shock properties, compression ratio, shock speed, entropy production are reduced because the shock ramp is reduced by the dissipation in the shock. This is nicely visible in the entropy trail of the shock. The entropy has a maximum at about $x = 30$ and decreases toward the shock demonstrating that the entropy production by the shock has been decreasing as the shock propagated.

Finally it should be remarked that the time it takes to form a shock depends approximately inversely on the initial profile. This is consistent with the variation of the fast mode speed within the wave profile which depends linearly on the wave amplitude.

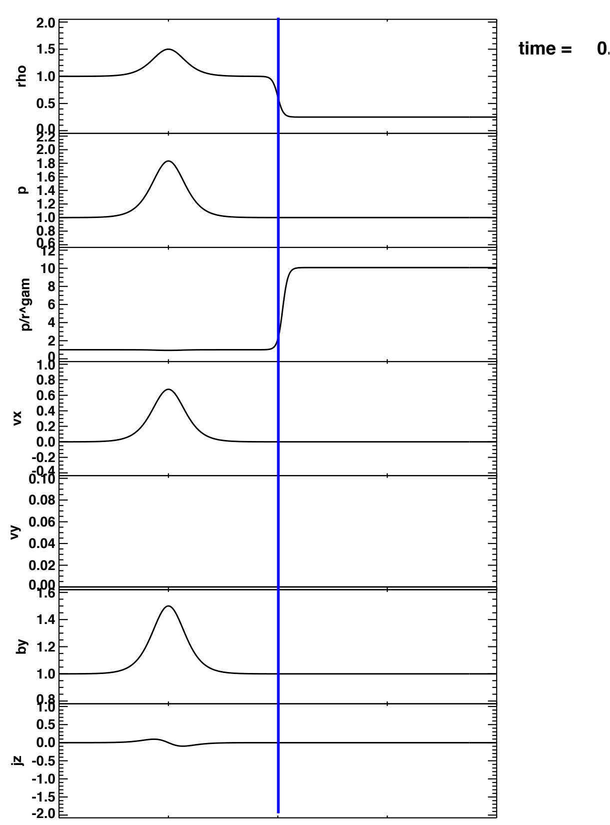
The following plots show the results for amplitude $\hat{h} = 1$ using the equation for entropy conservation (intu=0).



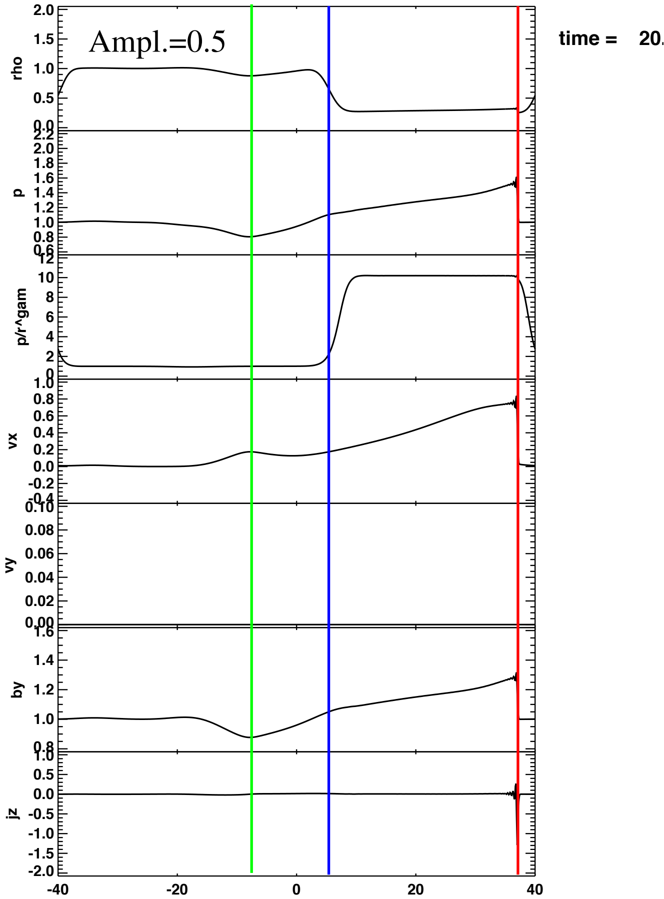


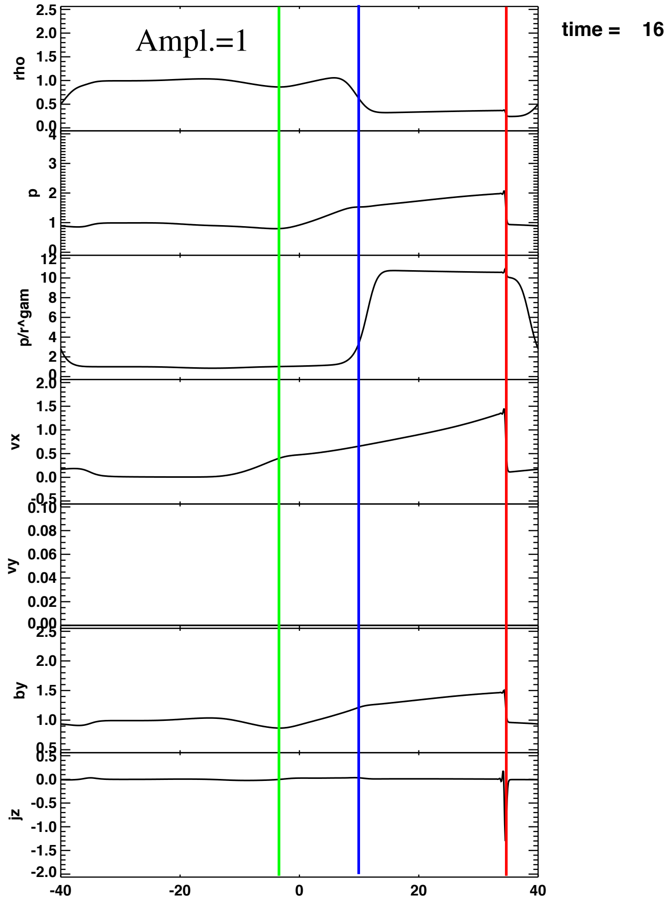
Using entropy conservation yields very similar results. Shock formation occurs at about the same time and shock properties (except for the entropy increase) are slightly larger but very close to the case using energy conservation. The the reason for the similarity is the relatively small nonadiabatic heating for the chosen parameters. The figures on shock properties indicate that this changes for much larger Mach numbers and especially for much lower plasma β.

(b) Fast wave interaction with a density gradient:
The plot on the right shows the initial configuration for an amplitude of $\hat{h} = 0.5$. It appears surprising to consider this case as an example for the magnetopause but in terms of the wave transmition and reflection the density gradient is the most dominant magnetopause property. It is also a highly characteristic property of the actual magnetopause boundary with a density ratio that ranges from about 10 to 100.



The interaction of the wave with the density boundary is illustrated in the following two plots for the wave amplitudes of $\hat{h} = 0.5$ and 1.





Here the vertical blue line indicates the location of the density boundary, the red line the location of the transmitted wave/shock, and the vertical green line the maximum perturbation of a reflected wave. The simulation is stopped at times $t = 20$ and $t = 16$ for the two perturbation amplitudes because the transmitted wave reaches the boundary (which is still periodic and therefore any results after the wave moving into the boundary are nonsense). The wave propagation and steepening with the density boundary are shown in the results from part a and are unaltered up to the time when the wave interacts with the boundary.

Boundary interaction: During the boundary interaction the boundary is moved into the positive x direction the cause for this is simple considering conservation laws. After the wave has interacted with the boundary basic boundary properties such as density or magnetic field on both sides of the boundary should be same as for the initial conditions. The solution after the interaction must contain a transmitted wave and a reflected fast wave (only fast waves are possible for this geometry). The wave after the interaction must conserve mass, momentum, and energy (note that one could also use magnetic flux conservation, which, however, is in this case equivalent to mass conservation). Thus conservation laws impose 3 conditions for the transmitted and reflected waves and thus overdetermine the problem because there are only two waves. This issue is resolved by the boundary motion, which is determined by the condition to maintain the same mass (or magnetic flux) on the two sides of the density boundary as for the initial condition with the incoming fast wave. In the plots the transmitted wavefront is marked by the red line. It is still a shock but with a smaller mach number. The reflected waves is a rarefaction wave because it has a positive plasma velocity but is moving in the negative direction. The distance of the boundary motion is determine by initial wave amplitude and size. The actual velocity of the motion is determined by wave amplitude alone.