1. Distribution function

Consider a Bi-Maxwellian distribution function $f\left(v_{\parallel}, v_{\perp}\right) = c\exp\left[-\frac{m}{2}\left(\frac{v_{\parallel}^2}{k_B T_{\parallel}} + \frac{v_{\perp}^2}{k_B T_{\perp}}\right)\right]$ where $v_{\parallel}$ is along the magnetic field (assume the z direction) and $v_{\perp}$ is perpendicular to the field ($v_{\perp}^2 = v_x^2 + v_y^2$).

(a) Demonstrate that the normalization coefficient from $n = \int f d^3v$ is

$$c = n\left(\frac{m}{2\pi k_B}\right)^{3/2}\frac{1}{T_{\parallel}^{1/2}T_{\perp}}$$

(b) Compute the components of the pressure tensor.

(c) What is the ratio of $T_{\parallel}/T_{\perp}$ if the average gradient and the curvature drifts are equal in a vacuum magnetic field.

Solution: (a) Normalization:

$$\begin{aligned} n &= c\int_v d^3v \exp\left[-\frac{m}{2}\left(\frac{v_{\parallel}^2}{k_B T_{\parallel}} + \frac{v_{\perp}^2}{k_B T_{\perp}}\right)\right] \\ &= c\int_v dv_x dv_y dv_z \exp\left(-\frac{mv_z^2}{2k_B T_{\parallel}} - \frac{mv_x^2}{2k_B T_{\perp}} - \frac{mv_y^2}{2k_B T_{\perp}}\right) \\ &= c\int_v dv_z \exp\left(-\frac{mv_z^2}{2k_B T_{\parallel}}\right)\int_v dv_x \exp\left(-\frac{mv_x^2}{2k_B T_{\perp}}\right)\int_v dv_y \exp\left(-\frac{mv_y^2}{2k_B T_{\perp}}\right) \end{aligned}$$

$$\text{with} \qquad \int_0^{\infty}\exp\left(-a^2x^2\right)dx = \frac{\sqrt{\pi}}{2a}$$

$$I_z = \int_{-\infty}^{\infty} dv_z \exp\left(-\frac{mv_z^2}{2k_B T_{\parallel}}\right) = \left(\frac{2\pi k_B T_{\parallel}}{m}\right)^{1/2}$$

Same for the x and y portions of the integral (only difference is $T_{\perp}$ instead of $T_{\parallel}$):

$$I_x = \left(\frac{2\pi k_B T_{\perp}}{m}\right)^{1/2} \quad \text{and} \quad I_y = \left(\frac{2\pi k_B T_{\perp}}{m}\right)^{1/2}$$

$$\text{Thus} \qquad n = c\left(\frac{2\pi k_B T_{\parallel}}{m}\right)^{1/2}\left(\frac{2\pi k_B T_{\perp}}{m}\right) = \left(\frac{2\pi k_B}{m}\right)^{3/2}T_{\parallel}^{1/2}T_{\perp}$$

$$\text{or} \qquad c = n\left(\frac{m}{2\pi k_B}\right)^{3/2}\frac{1}{T_{\parallel}^{1/2}T_{\perp}}$$

(b) The components of the pressure tensor are

$$\Pi_{ij} = m\int_v d^3v\left(v_i - u_i\right)\left(v_j - u_j\right)\exp\left[-\frac{m}{2}\left(\frac{v_{\parallel}^2}{k_B T_{\parallel}} + \frac{v_{\perp}^2}{k_B T_{\perp}}\right)\right]$$

All off-diagonal elements are 0 because of the symmetry (and $\mathbf{u} = 0$ because of symmetry)! The diagonal elements Π_{xx} and Π_{yy} are the same.

$$\begin{aligned}
\Pi_{xx} &= mc\int_v d^3v v_x^2 \exp\left[-\frac{m}{2}\left(\frac{v_z^2}{k_BT_\parallel}+\frac{v_x^2+v_y^2}{k_BT_\perp}\right)\right] \\
&= mc\int_v dv_z \exp\left(-\frac{mv_z^2}{2k_BT_\parallel}\right)\int_v dv_x v_x^2\exp\left(-\frac{mv_x^2}{2k_BT_\perp}\right)\int_v dv_y \exp\left(-\frac{mv_y^2}{2k_BT_\perp}\right) \\
&= mcI_zI_y\int_{-\infty}^{\infty} dv_x v_x^2 \exp\left(-\frac{mv_x^2}{2k_BT_\perp}\right) \\
&= mcI_zI_y\left\{\left[-v_x\frac{k_BT_\perp}{m}\exp\left(-\frac{mv_x^2}{2k_BT_\perp}\right)\right]_{-\infty}^{\infty}+\frac{k_BT_\perp}{m}\int_{-\infty}^{\infty} dv_x\exp\left(-\frac{mv_x^2}{2k_BT_\perp}\right)\right\} \\
&= mcI_zI_y\frac{k_BT_\perp}{m}I_x = mn\left(\frac{m}{2\pi k_B}\right)^{3/2}\frac{1}{T_\parallel^{1/2}T_\perp}\frac{k_BT_\perp}{m}\frac{2\pi k_BT_\perp}{m}\left(\frac{2\pi k_BT_\parallel}{m}\right)^{1/2} \\
&= nk_BT_\perp
\end{aligned}$$

and

$$\begin{aligned}
\Pi_{zz} &= mc\int_v d^3v v_z^2 \exp\left[-\frac{m}{2}\left(\frac{v_z^2}{k_BT_\parallel}+\frac{v_x^2+v_y^2}{k_BT_\perp}\right)\right] \\
&= mc\int_v dv_z v_z^2 \exp\left(-\frac{mv_z^2}{2k_BT_\parallel}\right)\int_v dv_x \exp\left(-\frac{mv_x^2}{2k_BT_\perp}\right)\int_v dv_y \exp\left(-\frac{mv_y^2}{2k_BT_\perp}\right) \\
&= mcI_xI_y\int_{-\infty}^{\infty} dv_z v_z^2 \exp\left(-\frac{mv_z^2}{2k_BT_\parallel}\right) \\
&= mcI_x^2\left\{\left[-v_z\frac{k_BT_\parallel}{m}\exp\left(-\frac{mv_z^2}{2k_BT_\parallel}\right)\right]_{-\infty}^{\infty}+\frac{k_BT_\parallel}{m}\int_{-\infty}^{\infty} dv_z\exp\left(-\frac{mv_z^2}{2k_BT_\parallel}\right)\right\} \\
&= mcI_x^2\frac{k_BT_\parallel}{m}I_z = mn\left(\frac{m}{2\pi k_B}\right)^{3/2}\frac{1}{T_\parallel^{1/2}T_\perp}\frac{k_BT_\parallel}{m}\frac{2\pi k_BT_\perp}{m}\left(\frac{2\pi k_BT_\parallel}{m}\right)^{1/2} \\
&= nk_BT_\parallel
\end{aligned}$$

Therefore we have $\Pi_{xx} = \Pi_{yy} = nk_BT_\perp$ and $\Pi_{zz} = nk_BT_\parallel$

(c) Gradient and curvature drift in a vacuum field:

$$\mathbf{v}_C = \frac{mv_\parallel^2}{qB^3}\mathbf{B}\times(\nabla B) \quad \text{and} \quad \mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}(\mathbf{B}\times\nabla B)$$

Average kinetic energy for gradient and curvature drift:

$$\begin{aligned}
\left\langle v_\parallel^2\right\rangle &= \frac{1}{n}\int_v d^3v v_\parallel^2 f\left(v_\parallel, v_\perp\right) = \frac{\Pi_{zz}}{mn} = \frac{k_BT_\parallel}{m} \\
\left\langle v_\perp^2\right\rangle &= \frac{1}{n}\int_v d^3v \left(v_x^2+v_y^2\right) f\left(v_\parallel, v_\perp\right) = \frac{\Pi_{xx}+\Pi_{yy}}{mn} = \frac{2k_BT_\perp}{m}
\end{aligned}$$

Equal average drifts:

$$\frac{m\left\langle v_\parallel^2\right\rangle}{qB^3}\mathbf{B}\times(\nabla B) = \frac{m\left\langle v_\perp^2\right\rangle}{2qB^3}(\mathbf{B}\times\nabla B) \quad \text{or} \quad \left\langle v_\parallel^2\right\rangle = \frac{1}{2}\left\langle v_\perp^2\right\rangle$$

=> $\quad T_\parallel = 2T_\perp/2 = T_\perp$

2. Magnetic field

Consider the magnetic field $B_x = B_0 \tanh(z/L)$, $B_z = \epsilon B_0 x/L$, and $B_y = 0$ with $\epsilon > 0$.

(a) Compute the y component of the vector potential.

(b) Determine the equation for magnetic field lines.

(c) Discuss and sketch the field lines. What is the separatrix angle at the X line?

(d) Compute the current density associated with the magnetic field.

(e) Compute the acceleration as a function of x along the x axis for a plasma at rest (Use the MHD momentum equation and constant pressure). What is this acceleration for $B_0 = 20$ nT, $\epsilon = 0.1$, $L = 1\,R_E$, and density $n = 1$ cm^{-3} (protons and electrons) at $x = L$?

Solution:

(a) Compute the y component of the vector potential.

$$\begin{aligned} B_x &= \partial_y A_z - \partial_z A_y = -\partial_z A_y \\ B_z &= \partial_x A_y - \partial_y A_x = \partial_x A_y \end{aligned}$$

Intergrating the first equations

$$\begin{aligned} A_y(x,z) &= -B_0 \int^z \tanh(z'/L)\, dz' \\ &= -B_0 L \ln\cosh(z/L) + G(x) \end{aligned}$$

And the second equation

$$\begin{aligned} A_y(x,z) &= \epsilon B_0/L \int^x x' dx' \\ &= \epsilon B_0 x^2/(2L) + H(z) \end{aligned}$$

Which yields for the vector potential:

$$A_y(x,z) = LB_0 \left[\epsilon x^2/\left(2L^2\right) - \ln\cosh(z/L)\right]$$

(b) Determine the equations for the magnetic field lines.

One can either integrate

$$\frac{dx}{dz} = \frac{B_x}{B_z} = \frac{\tanh(z/L)}{\epsilon x/L}$$

or use $A_y = const = B_0 L c$ such that $\quad \epsilon x^2/(2L^2) - \ln\cosh(z/L) = c$

Small arguments: $\ln\cosh u = u^2/2 + u^4/12$ such that for $z/L < 1$:

$$\frac{\epsilon x^2}{2L^2} - \frac{z^2}{2L^2} = c \quad \text{or} \quad z^2 = \epsilon x^2 - 2L^2 c$$

or with $C' = -2L^2 c$

$$z = \pm\left(\epsilon x^2 + C'\right)^{1/2}$$

For $C' > 0$ fieldlines are defined for all values of x. For $C' < 0$ field lines start from $x = \pm\sqrt{-C'/\epsilon}$. For $C' = 0$ field lines are lines with slopes $\pm\sqrt{\epsilon}$

$$z = \pm\sqrt{\epsilon/B_0 x}$$

For large arguments: $\ln\cosh u = |u|$ such that for $z/L \gg 1$:

$$\frac{\epsilon x^2}{2L^2} - z/L = c$$

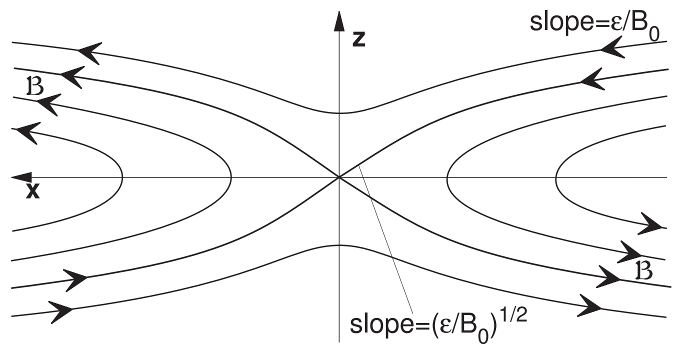
or

$$|z| = \epsilon \frac{x^2}{2L} - cL$$

(c) Discuss and plot the field lines.

- The magnetic field is 0 at $x = z = 0$.
- At this point the separatrix field line has slopes of $\pm\sqrt{\epsilon/B_0}$.
- Along the x axis there is only a B_z component which is $+\epsilon$ for $x > 0$ and is $-\epsilon$ for $x < 0$. Along the z axis there is only a B_x component which is $+B_0$ for $z > 0$ and is $-B_0$ for $z < 0$.
- Far away from the origin along the x axis field lines hae a hyperbolic shape. At large z/L along the z axis field lines assume a parabolc shape.

This yields the following field line plot (for $z/L \leq 1$):



(d) Compute the current density associated with the magnetic field.

Current density:

$$\begin{aligned} j_y &= \frac{1}{\mu_0}\left(\partial_z B_x - \partial_x B_z\right) \\ &= \frac{B_0}{\mu_0 L}\left(\cosh^{-2}\frac{z}{L} - \epsilon\right) \end{aligned}$$

(e) Compute the acceleration as a function of x along the x axis for a plasma at rest (Use the MHD momentum equation and constant pressure). What is this acceleration for $B_0 = 20$ nT, $\epsilon = 0.1$, $L = 1\,R_E$, and density $n = 1$ cm^{-3} (protons and electrons) at $x = L$?

Current density at $z = 0$:

$$j_y = \frac{B_0}{\mu_0 L}(1-\epsilon) = \frac{2 \cdot 10^{-8}}{4\pi \cdot 10^{-7} \cdot 6.4 \cdot 10^6}(1-\epsilon) = 2.24 \cdot 10^{-9} A/m^2$$

Plasma acceleration:

$$\rho \frac{\partial u_x}{\partial t} = \mathbf{j} \times \mathbf{B}|_{x=L} - \partial_x p$$

or

$$\begin{aligned} \frac{\partial u_x}{\partial t}(z=0) &= \frac{1}{\rho} j_y B_z = \frac{\epsilon}{\rho}\frac{B_0^2 x}{\mu_0 L^2}\left(\cosh^{-2}\frac{0}{L} - \epsilon\right) \\ &= \frac{2 \cdot 10^{-9} \cdot 2.24 \cdot 10^{-9}}{1.67 \cdot 10^{-27} \cdot 10^6} ms^{-2} = 2.68 \cdot 10^3 ms^{-2} \end{aligned}$$

3. Ohm's law and induction equation

Assume $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$, Faraday's law, and the field amplitude $B = |\mathbf{B}|$.

(a) What assumption for the flow or for the magnetic field is necessary to obtain

$$\frac{\partial B}{\partial t} + \nabla \cdot (\mathbf{u}B) = 0 \tag{1}$$

(b) Using the continuity equation show that (1) implies

$$\frac{d(\rho/B)}{dt} = 0$$

(c) Explain the physical meaning of these two equations.

Solution:

(a) Using the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (-\mathbf{u} \times \mathbf{B})$$

and $B(\partial B/\partial t) = \mathbf{B} \cdot \partial \mathbf{B}/\partial t$ we obtain

$$\begin{aligned}
\frac{\partial B}{\partial t} &= \frac{\mathbf{B}}{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \\
&= \frac{\mathbf{B}}{B} \cdot \nabla \times (\mathbf{u} \times \mathbf{B}) \\
&= \frac{\mathbf{B}}{B} \cdot [\mathbf{u}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{u} + (\mathbf{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{B}] \\
&= -B\nabla \cdot \mathbf{u} + \frac{\mathbf{B}}{B} \cdot (\mathbf{B} \cdot \nabla)\mathbf{u} - \frac{1}{2B}(\mathbf{u} \cdot \nabla)\mathbf{B}^2 \\
&= -\nabla \cdot (\mathbf{u}B) + \mathbf{B} \cdot \frac{d\mathbf{u}}{ds}
\end{aligned}$$

Thus

$$\mathbf{B} \cdot \frac{d\mathbf{u}}{ds} = 0$$

This equation involves $d\mathbf{u}/ds$ which is the derivative of the vector component of the velocity along a magnetic field line. It is for instance satisfied if $\mathbf{u}$ is constant along any field line. However, it is more general in that $\mathbf{u}$ can in fact change along a field line ($d\mathbf{u}/ds \neq 0$) with the constraint that any such change must occur only perpendicular to the local magnetic field direction.

(b) Using the continuity equation show that (1) implies $\frac{d(\rho/B)}{dt} = 0$

$$\begin{aligned}
\frac{\partial(\rho/B)}{\partial t} &= \frac{1}{B}(-\nabla \cdot \rho\mathbf{u}) - \frac{\rho}{B^2}\frac{\partial B}{\partial t} \\
&= \frac{1}{B}(-\rho\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\rho) + \frac{\rho}{B^2}(B\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla B) \\
&= -\frac{1}{B}\mathbf{u} \cdot \nabla\rho + \frac{\rho}{B^2}\mathbf{u} \cdot \nabla B \\
&= -\frac{1}{B}\mathbf{u} \cdot \nabla\rho - \rho\mathbf{u} \cdot \nabla\frac{1}{B} \\
&= \mathbf{u} \cdot \nabla\left(\frac{\rho}{B}\right)
\end{aligned}$$

or

$$\frac{d(\rho/B)}{dt} = \frac{\partial(\rho/B)}{\partial t} + \mathbf{u} \cdot \nabla\left(\frac{\rho}{B}\right) = 0$$

(c) The first equation $\partial B/\partial t + \nabla \cdot (\mathbf{u}B) = 0$ is a continuity equation. It implies the the total magnetic field in any volume $\int_{vol} B d^3 r$ is constant unless the is an influx of magnetic field through convection into this volume.

The second equation $d\left(\rho/B\right)/dt = 0$ implies that the convective derivative of ρ/B is 0. this derivative represents the change of a quantity along stream or convection lines which represent the path of a fluid element in th evectorfield of $\mathbf{u}$.

4. Single particle drifts

(a) List 3 single particle drifts, **(b)** the basic physical mechanisms which cause the individual drifts, **(c)** the main assumptions which enter into the drift approximations, **(d)** an application for each particular drift where it may play a role.

(a) Single particle drifts:

- General force drift
- $\mathbf{E} \times \mathbf{B}$ drift
- Polarisation drift
- Gradient $\mathbf{B}$ drift
- Curvature $\mathbf{B}$ drift

(b) Mechanism

Force on the particle perpendicular to $\mathbf{B}$leads to a locally larger gyroradius in the direction of the acceleration => deflection of the averge particle motion perpendicular to the force and to $\mathbf{B}$.

- Special case of the general force with $\mathbf{F} = q\mathbf{E}$.
- Caused by a slowly changing electric field. Deflection of the $\mathbf{E} \times \mathbf{B}$ drift.
- Caused by changing local gyroradius due to a gadient in the magnetic field strength. Smaller gyroradius in the region of larger magnetic field => deflection into the direction perpendicular to the gradient and to $\mathbf{B}$.
- Caused by the zentrifugal acceleration of the particle while moving along a curved field line. Special case of the general force drift and frift is perpendicular to the radius of curvature and to $\mathbf{B}$.

(c) Specific assumptions

- General force drift: For general force drift: are that the magnetic field is homogeneous time independent and that the force is steady.
- $\mathbf{E} \times \mathbf{B}$ drift: Same as general force drift.
- Polarisation drift: Sufficiently slow change of the electric field (on a time scale much smaller than the gyro period)
- Gradient $\mathbf{B}$ drift: Sufficiently small gradient of the magnetic field, i.e., change of $\mathbf{B}$ and a length scale much larger than the gyro-scale. Changes of the magnetic field due to
- Curvature $\mathbf{B}$ drift: Sufficiently large radius of curvature (much larger than the gyro radius) for the magnetic field.

(d) Applications:

- General force drift: $\mathbf{E} \times \mathbf{B}$ drift, drift of charged particle as a result of strong gravity (neutron stars, black hole, but also some effect from terrestrial gravitation in the ionosphere. Centrifugal acceleration due to corotation
- $\mathbf{E} \times \mathbf{B}$ drift: Many regions where we have a steady plasma convection, e.g., solar wind.
- Polarisation drift: Occurs for a changing electric field for instance the current associated with wave can be carried by the polarization drift .

- Gradient $\mathbf{B}$ drift: Important in many space physics applications particularly in cases with large magnetic fields and fairly energetic particle populations. Carries a current and specific application is the contribution to the terrestrial ring current.
- Curvature $\mathbf{B}$ drift: Important in many space physics applications particularly in cases with large curved magnetic fields and fairly energetic particle populations. Carries a current and specific application is the contribution to the terrestrial ring current.