Chapter 3

Magnetohydrodynamics

Fluid equations are probably the most widely used equations for the description of inhomogeneous plasmas. While the phase fluid which is governed by the Boltzmann equation represents a first example, many applications do not require the precise velocity distribution at any point in space. Ordinary fluid equations for gases and plasmas can be obtained from the Boltzmann equation or can be derived using properties like the conservation of mass, momentum, and energy of the fluid. The following chapter we will derive a single set of ordinary fluid equations for a plasma and examine properties such a equilibria and waves for these equations.

3.1 Derivation of the Fluid Plasma Equations

3.1.1 Definitions

The equations of ordinary fluids and gases as well as those for magnetofluids (plasmas) can be obtained from equation 1.23 in a systematic manner. Defining the 0th, 1st, and 2nd moment of the integral over the distribution function f_s as mass density ρ_S, fluid bulk velocity $\mathbf{u}_s$, and pressure tensor $\underline{\Pi}_s$

$$\rho_s(\mathbf{x},t) = m_s \int_{-\infty}^{\infty} d^3v f_s(\mathbf{x},\mathbf{v},t) \tag{3.1}$$

$$\mathbf{u}_s(\mathbf{x},t) = \frac{1}{n_s} \int_{-\infty}^{\infty} d^3v \mathbf{v} f_s(\mathbf{x},\mathbf{v},t) \tag{3.2}$$

$$\underline{\Pi}_s(\mathbf{x},t) = m_s \int_{-\infty}^{\infty} d^3v (\mathbf{v}-\mathbf{u}_s)(\mathbf{v}-\mathbf{u}_s) f_s(\mathbf{x},\mathbf{v},t). \tag{3.3}$$

where the index s indicates the particle species (electrons and different ion species if present). With these definitions one also obtains number density $n_s(\mathbf{x},t) = \rho_s/m_s$, charge density $\rho_{c,s}(\mathbf{x},t) = q_s n_s$, momentum density $\mathbf{p}_s(\mathbf{x},t) = \rho_s \mathbf{u}_s$, current density $\mathbf{j}_s(\mathbf{x},t) = q_s \mathbf{u}_s$, and scalar pressure (the isotropic portion of the pressure) $p_s(\mathbf{x},t) = \frac{1}{3} Tr(\underline{\Pi}_s)$ where the individual particle mass m_s and charge q_s are used. In the following section we will drop the index s for a more compact representation but remind the reader that there is a separate set of fluid equations for each particle species.

The fluid equations are determined by the moments of the Boltzmann equation, i.e.,

$$\int_{-\infty}^{\infty} d^3v\,\mathbf{v^i}(Boltzmann\,Equ.)$$

To account for the collision term in (1.23) we define

$$Q^\rho(\mathbf{x},t) = m\int_{-\infty}^{\infty} d^3v\,\left.\frac{\partial f}{\partial t}\right|_c \tag{3.4}$$

$$\mathbf{Q}^p(\mathbf{x},t) = m\int_{-\infty}^{\infty} d^3v(\mathbf{v}-\mathbf{u})\,\left.\frac{\partial f}{\partial t}\right|_c \tag{3.5}$$

$$Q^E(\mathbf{x},t) = \frac{1}{2}m\int_{-\infty}^{\infty} d^3v(\mathbf{v}-\mathbf{u})^2\,\left.\frac{\partial f}{\partial t}\right|_c \tag{3.6}$$

The precise form of these terms depends on the particular properties of the systems and will not be specified at this point.

3.1.2 Fluid Moments

To provide an example for the evaluation of the moments of the Boltzmann equations let us evaluate the 0th moment of the integral. The first term of the equation becomes

$$\int_{-\infty}^{\infty} d^3v\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} d^3v\,f(\mathbf{x},\mathbf{v},t) = \frac{\partial n(\mathbf{x},t)}{\partial t}.$$

The second term is

$$\int_{-\infty}^{\infty} d^3v\mathbf{v}\cdot\nabla f = \int_{-\infty}^{\infty} d^3v\nabla\cdot(\mathbf{v}f(\mathbf{x},\mathbf{v},t)) = \nabla\cdot\int_{-\infty}^{\infty} d^3v\mathbf{v}f(\mathbf{x},\mathbf{v},t) = \nabla\cdot(\mathbf{u}(\mathbf{x},t)n(\mathbf{x},t))$$

and the third term is

$$\int_{-\infty}^{\infty} d^3v\frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}}f = \sum_i\int_{-\infty}^{\infty} d^3v\frac{F_i}{m}\frac{\partial f}{\partial v_i} = \sum_i\int_{-\infty}^{\infty} d^2v\left[\frac{F_i}{m}f\right]_{v_i=-\infty}^{v_i=\infty} - \sum_i\int_{-\infty}^{\infty} d^3v\frac{f}{m}\frac{\partial F_i}{\partial v_i}$$

The terms on the rhs. in the above equation are 0 because the $f = 0$ for $v_i = \begin{cases} +\infty \\ -\infty \end{cases}$ for each component v_i and in because $\partial F_i/\partial v_i = 0$ (see homework for the Lorentz force). Therefore

$$\int_{-\infty}^{\infty} d^3v\frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}}f = 0$$

The collision term of the Boltzmann equation reduces to Q^ρ (see equation 3.4) such that in summary the 0th moment of the Boltzmann equation reduces to

$$\frac{\partial n(\mathbf{x},t)}{\partial t} + \nabla \cdot (n\mathbf{u}) = \frac{1}{m} Q^\rho . \tag{3.7}$$

This is the usual continuity equation for the particle number density with a source term on the right side. Multiplying (3.7) with the particle mass yields the continuity equation for mass density

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = Q^\rho \tag{3.8}$$

The source term describes production or annihilation of mass for instance through chemical reactions or ionization or recombination. It is noted that (3.8) is for one species only. In the case of several neutral constituents or ion species a corresponding continuity equation is obtained for each species. The total production rate of mass has to be zero.

Similar to the 0th moment the 1st moment of the Boltzmann equation $[m \int_{-\infty}^{\infty} d^3v\, \mathbf{v}\, (Boltzmann\, equation)]$ and 2nd moment $[\frac{3}{2} m \int_{-\infty}^{\infty} d^3v\, v^2\, (Boltzmann\, equation)]$ yield the equations for the fluid momentum (or velocity) and energy

$$\begin{aligned} \frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla \cdot \underline{\Pi} + qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{u} Q^\rho + \mathbf{Q}^p && (3.9) \\ \frac{\partial}{\partial t}\left(\frac{1}{\gamma-1} p + \frac{1}{2}\rho u^2\right) &= -\nabla \cdot \left(\frac{1}{2}\rho u^2 \mathbf{u} + \frac{1}{\gamma-1} p\mathbf{u} + \mathbf{u} \cdot \underline{\Pi} + \mathbf{L}\right) \\ &\quad + qn\mathbf{u} \cdot \mathbf{E} + \frac{1}{2} u^2 Q^\rho + \mathbf{u} \cdot \mathbf{Q}^p + Q^E && (3.10) \end{aligned}$$

with the heat flux $\mathbf{L}(\mathbf{x},t) = \frac{1}{2} m \int_{\infty}^{\infty} d^3v (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u})^2 f(\mathbf{x}, \mathbf{v}, t)$ and γ is the ratio of specific heats, i.e., $\gamma = 5/3$ if a gas has 3 degrees of freedom for motion.

Elimination of $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right)$ in the energy equation (with the aid of (3.8) and (3.9)) yields

$$\frac{1}{\gamma - 1}\left(\frac{\partial}{\partial t} p + \nabla \cdot p\mathbf{u}\right) = -(\underline{\Pi} \cdot \nabla) \cdot \mathbf{u} - \nabla \cdot \mathbf{L} + Q^E \tag{3.11}$$

Notes:

- As before the "fluid" Lorentz force $\mathbf{E} + \mathbf{u} \times \mathbf{B}$ requires to solve the corresponding field equations (1.15) - (1.18) .
- The Lorentz force (or any velocity dependent force now depends on the bulk velocity and velocity is a dependent variable in the fluid equations rather than an independent variable as in the Boltzmann equation.
- The source terms for mass Q^ρ, momentum $\mathbf{Q}^p$, and energy Q^E depend on system properties and need to be specified through these or through a systematic collision operator and the corresponding velocity integrals. The terms reflect mass generation and annihilation Q^ρ, momentum exchange through friction $\mathbf{Q}^p$, and energy exchange collisions Q^E. The collisions term $\mathbf{Q}^p$ can be expressed in terms of an effective collision frequency $\mathbf{Q}^p =$

- The pressure tensor is often split into a scalar pressure and a viscous tensor $\underline{\Pi} = p\underline{1} + \underline{\mathbf{w}}$, with $p = \frac{1}{3}Tr(\Pi)$ and the viscous tensor $\underline{\mathbf{w}}$.
- Often a kinematic viscosity $\sigma_{ik} = a\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + b\frac{\partial u_l}{\partial x_l}\delta_{ik}$ is used on the right of the momentum equation yielding a term $\nabla \cdot \underline{\sigma} = \eta\Delta\mathbf{u} + \left(\zeta + \frac{\eta}{3}\right)\nabla(\nabla \cdot \mathbf{u})$.

The fluid equations without the source terms imply the conservation of the corresponding property (mass, momentum, and energy). Consider the mass in a given volume defined by

$$M_V = \int_V \rho d^3x$$

The change of mass in the volume is

$$\frac{dM_V}{dt} = \int_V \frac{\partial\rho}{\partial t} d^3x = -\int_V \nabla \cdot \mathbf{v}\rho d^3x = -\oint_{S_V} \rho\mathbf{v} \cdot d\mathbf{s}$$

where S_V is the surface of V. In other words the mass in the volume V changes only if there is a nonzero density flux (velocity) across the surface of the volume. Similarly momentum and energy are conserved. However, for the energy one has to include momentum and energy which is contained in the fields as well.

Exercise: Determine the integral of the 1st order moment for the first two terms in the Boltzmann equation.

Exercise: Derive the 1st order moment force term for a gravitational force and the Lorentz force (velocity dependent).

Exercise: Do the same for the energy equation (i.e., multiply the Boltzmann equation (1.23) with $\frac{1}{2}mv^2$ and integrate).

3.1.3 Typical Fluid Approximations

Equations (3.8) - (3.10) establish the typical set of fluid equations which are used in many simulations of fluids and gases like weather simulations, air flow around aircraft or cars, water flow in pipes or round boats, and many other research and technical applications. Using the set of equations (3.8), (3.9), and (3.11) we can derive most equations commonly used in fluid simulations:

- For a known velocity profile $\mathbf{u}$ and no sources $Q^\rho = 0$ it is sufficient to model the continuity equation for instance to derive the evolution of density of a gas or the concentration of dust, aerosols, etc. in any medium like air water etc.:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0$$

- If the velocity profile is incompressible $\nabla \cdot \mathbf{u} = 0$ the equation reduces to the common advection equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0$$

- With the total derivative along the fluid path defined as $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ the advection equation is

$$\frac{d\rho}{dt} = 0$$

- In the hydrodynamic case)no electric and magnetic forces):
  - For no sources Q^ρ, $\mathbf{Q}^p = 0$, no viscosity $\underline{\sigma} = 0$ (scalar pressure) and gravitational acceleration for the force term one obtains Euler's equation :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \ (+\, \mathbf{g})$$

  - With a kinematic viscosity included the momentum equation is known as the Navier-Stokes equation:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta \Delta \mathbf{u} + \left(\zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{u}) \ (+\, \rho \mathbf{g} +)$$

- Neglecting pressure and external force terms and assuming a simplified viscosity one obtains Burger's equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} = 0$$

- Diffusion and heat conduction: A diffusion equation can be obtained from the continuity equation and a redefinition of the bulk velocity in the presence of several particle species. However, the more straightforward equation for diffusion is obtained for heat conduction (i.e., diffusion of temperature). Equation (3.11) can be re-written in different forms. Defining $\frac{p}{\gamma - 1} = \rho \varepsilon$ yields an equation for the internal energy ε of a gas. More commonly used is the ideal gas law $p = nkT$ to re-write the energy equation into an equation for temperature. Assuming scalar pressure, constant density, and a heat flux driven by a temperature gradient $\mathbf{L} = -\kappa \nabla T$ on obtains

$$\frac{\partial T}{\partial t} = -\mu \Delta T$$

- Steady state equation are generated from the above sets by assuming $\partial/\partial t = 0$. For steady condition the velocity is often determined from a potential which can be scalar if the flow is assumed irrotational ($\mathbf{u} = \nabla\Phi$) or incompressible flow is modeled sometimes by a vector potential ($\mathbf{u} = \nabla \times \mathbf{V}$).

Exercise: Using the continuity equation and the stated assumptions derive Euler's equation.

Exercise: Derive the equation for heat conduction with the stated assumptions.

Exercise: Derive the heat conduction equation for nonzero velocity $\mathbf{u}$.

Exercise: Derive the continuity equation and momentum equation for irrotational flow.

Exercise: Assume a scalar pressure, $\mathbf{L} = 0$, and $Q^E = 0$ in the pressure equation (3.11). Consider a function $g = p^a\rho^b$ and determine a and b such that the resulting equation for g assumes a conservative form, i.e., $\partial g/\partial t + \nabla \cdot g\mathbf{u} = 0$.

Exercise: Assume a scalar pressure, $\mathbf{L} = 0$, and $Q^E = 0$ in the pressure equation (3.11). Consider a function $h = p^a\rho^b$ and determine a and b such that the resulting equation for h assumes a total derivative, i.e., $\partial h/\partial t + \mathbf{u} \cdot \nabla h = 0$. For $\gamma = 5/3$ this equation becomes a measure for entropy because entropy is conserved for adiabatic changes.

3.2 Two Fluid Plasma Equations

In the absence of ionization and energy exchange collisions and considering a simple two component (ion and electron) plasma, one obtains the so-called two fluid equations. With $\rho_s = m_s n_s$ one obtains the following continuity, momentum, and energy equation for species s

$$\begin{aligned}
\frac{\partial \rho_s}{\partial t} &= -\nabla \cdot (\rho_s \mathbf{u}_s) && (3.12)\\
\frac{\partial \rho_s \mathbf{u}_s}{\partial t} &= -\nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s) - \nabla \cdot \underline{\Pi}_s + q_s n_s(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + \mathbf{Q}_s^p && (3.13)\\
\frac{1}{\gamma - 1}\frac{\partial p_s}{\partial t} &= -\frac{1}{\gamma - 1}\nabla \cdot p_s \mathbf{u}_s - (\underline{\Pi}_s \cdot \nabla) \cdot \mathbf{u}_s - \nabla \cdot \mathbf{L}_s + Q_s^E && (3.14)
\end{aligned}$$

where $\mathbf{Q}_s^p$ describes friction between the two components and effectively describes an electric resistance. Effectively one can express the friction terms as $\mathbf{Q}_s^p = \nu_{st} m_s n_s (\mathbf{v}_t - \mathbf{v}_s)$ where ν_{st} is the collision frequency for particles of species s to collide with particles of species t.

Specifically we have for electrons

$$\mathbf{Q}_e^p = \nu_{ei} m_e n_e (\mathbf{u}_i - \mathbf{u}_e) \tag{3.15}$$

with the condition:

$$\mathbf{Q}_e^p + \mathbf{Q}_i^p = 0 \tag{3.16}$$

and the Coulomb collision frequency ν_{ei} as derived in section 1.4. The condition (3.16) is required because of the conservation of total momentum. For a two-component plasma with single charged ions the energy conservation is

$$\mathbf{u}_i \cdot \mathbf{Q}_i^p + \mathbf{u}_e \cdot \mathbf{Q}_e^p + Q_i^E + Q_e^E = 0$$

or

$$Q_i^E + Q_e^E = (\mathbf{u}_i - \mathbf{u}_e)\, \mathbf{Q}_e^p$$

Note that for many applications quasi-neutrality, i.e., $n_e = n_i = n$ is a good assumption. The corresponding equations are sometimes also called two fluid equations. Often it is also assumed that the pressure is scalar or gyrotropic, i.e., is different parallel and perpendicular to the magnetic field. The heat conduction term is often neglected. Further use of these equations will be made in derivation of the MHD equations in the following section and in chapter 4 on two fluid properties such as waves and instabilities.

3.3 Single Fluid or MHD Equations

3.3.1 Derivation of the MHD equations:

While considerably much simpler the two fluid equation contain still considerable complexity which is not needed for many plasma systems. Thus it is desirable to formulate a more appropriate set of equations which is applicable for large scale systems. This set of equations are the so-called MHD equations. Assuming electrons and single charged ions with $q_i = -q_e = e$ and a charge neutral plasma $n_e = n_i$ the total current density is

$$\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e)$$

We can also define the total mass density as ρ, effective mass M, and bulk velocity or total mass density flux $\rho\mathbf{u}$ as

$$\begin{aligned} \rho &= n(m_i + m_e) \\ M &= m_i + m_e \\ \rho\mathbf{u} &= n(m_i\mathbf{u}_i + m_e\mathbf{u}_e) \end{aligned}$$

with these definitions one obtains

$$\begin{aligned} \nabla \cdot j &= 0 \\ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{u} &= 0 \end{aligned} \tag{3.17}$$

Exercise: Derive the above equations.

With the above definitions we can uniquely express $\mathbf{u}_i$ and $\mathbf{u}_e$ in terms of $\mathbf{u}$ and $\mathbf{j}$ the goal being to derive equations which substitute the two-fluid equations for momentum (3.13) and energy (3.14) density.

$$\begin{aligned} \mathbf{u}_i &= \mathbf{u} + \frac{m_e}{m_i}\frac{\mathbf{j}}{ne} \simeq \mathbf{u} \\ \mathbf{u}_e &= \mathbf{u} - \frac{\mathbf{j}}{ne} \end{aligned}$$

It is also assumed that the pressure is scalar for both the electron and the ion components even though this is not necessary. By taking the sum of the momentum equations and substituting $\mathbf{u}_i$ and $\mathbf{u}_e$ one obtains:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) + \frac{m_e m_i}{e^2} \nabla \cdot (\frac{1}{\rho}\mathbf{j}\mathbf{j}) = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{3.18}$$

with $p = p_i + p_e$. Note that the total momentum has to be conserved such that $\mathbf{Q}_i^p + \mathbf{Q}_e^p = 0$.

Exercise: Derive the momentum equations.

A second equation is required for uniqueness (there are two momentum equations for the two fluids). This is obtained by multiplying the ion equation with q_i/m_i and the electron equation with q_e/m_e and the sum of the modified equations:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{m_e m_i}{e^2 \rho} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u}) \right] - \frac{M}{e\rho} \nabla p_e + \frac{m_i}{e\rho} \mathbf{j} \times \mathbf{B} + \eta \mathbf{j} \tag{3.19}$$

with the resistivity $\eta = m_e \nu_c / ne^2$ where ν_c is the collision frequency between electrons and ions (or neutrals). This equation is usually termed **generalized Ohm's law**. In the above equation the first term on the rhs is often called the inertia term because it represents the electron inertia in this equation.

Exercise: Derive Ohm's law from the two fluid approximation.

The second term is the electron pressure force and the third term is the Hall term.

Note that thus far there has been no approximation in our derivation (except for the pressure isotropy which is not really required) such that the above form of Ohm's law is fully equivalent to the two-fluid equations.

Finally one can take the sum of the electron and ion pressure equations and keep the electron pressure equation unmodified to obtain

$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p+\nabla\cdot p\mathbf{u}\right) = -p\nabla\cdot\mathbf{u}-\nabla\cdot\mathbf{L}+\eta\mathbf{j}^2+... \quad (3.20)$$
$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p_e+\nabla\cdot p_e\mathbf{u}_e\right) = -p_e\nabla\cdot\mathbf{u}_e-\nabla\cdot\mathbf{L}_e+\eta\mathbf{j}^2$$

Note that the ohmic heating term is present both in the sum of the pressure equations as well as in the electron equation. Here it is assumed that For the MHD equations we will now neglect the terms associated with the electron pressure, and the first, second, and third term on the rhs of generalized Ohm's law. Summarizing the equations and complementing them with the non-relativistic Maxwell equations one obtains the following set of equations termed the resistive MHD equations:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{u} = 0 \quad (3.21)$$
$$\frac{\partial\rho\mathbf{u}}{\partial t}+\nabla\cdot(\rho\mathbf{u}\mathbf{u}) = -\nabla p+\mathbf{j}\times\mathbf{B} \quad (3.22)$$
$$\mathbf{E}+\mathbf{u}\times\mathbf{B} = \eta\mathbf{j} \quad (3.23)$$
$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p+\nabla\cdot p\mathbf{u}\right) = -p\nabla\cdot\mathbf{u}+\eta\mathbf{j}^2 \quad (3.24)$$
$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} \quad (3.25)$$
$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} = 0 \quad (3.26)$$

The above equations do not contain $\nabla\cdot\mathbf{B}=0$. This equation enters actually as an initial condition. If $\nabla\cdot\mathbf{B}=0$ is satisfied initially then the induction equation implies $\nabla\cdot\mathbf{B}=0$ at all times.

Exercise: Use Ampere's law and $\nabla\cdot\mathbf{B}=0$ to show that the momentum equation can also be written as

$$\frac{\partial\rho\mathbf{u}}{\partial t}=-\nabla\cdot\left[\rho\mathbf{u}\mathbf{u}+\left(p+\frac{B^2}{2\mu_0}\right)\underline{\underline{1}}-\frac{1}{\mu_0}\mathbf{B}\mathbf{B}\right]$$

In the above equation the term $B^2/(2\mu_0)$ is referred to as a magnetic pressure. This terminology makes sense as will be shown in simple magnetic equilibrium situations.

3.3.2 Approximations used in the MHD equations:

a) First it should be noted that the MHD equations are derived from the two fluid equations and thus are subject to the general limitations of fluid equations, i.e., they cannot address discrete or single particle effects. such as gyro motion. Note, however, that it is not necessary to have just two charged particle species to derive the MHD equations.

b) In terms of the full set of two fluid equations we have assumed exact charge neutrality

$$n_e = n_i$$

This assumption basically removes plasma phenomena on the plasma frequency because plasma oscillations are an electrostatic process which requires small space charges. Thus MHD equations are valid for much lower frequencies than the plasma frequency.

c) In Maxwell's equations the displacement current has been neglected by assuming that there are no electromagnetic waves propagating at the speed of light. Divergence $\mathbf{B}$ is satisfied through the initial condition (show this) and quasi-neutrality satisfies Coulomb's equation.

d) In the momentum equation we assumed an isotropic pressure. In general the pressure is a tensor, however in particular in a collisional environment, collision will keep a distribution function close to local thermal equilibrium. In the absence of collisions a strongly anisotropic pressure can cause kinetic instabilities which will also tend to isotropise the pressure. A later chapter will discuss corresponding processes and effects from a gyrotropic pressure, i.e„ a pressure tensor with different pressure parallel and perpendicular to the magnetic field. Nonisotropic pressure components are also sometimes included in kinematic viscosity (or gyroviscosity). The momentum equation also does not include the $\frac{m_e m_i}{e^2}\nabla \cdot \left(\frac{1}{\rho}\mathbf{jj}\right)$ term. this term is small for various reasons one of which has to do with a scaling argument which will be addressed in connection to Ohm's law. In addition the magnitude of electric current is often smaller than the magnitude of the bulk flow in the $\nabla \cdot (\rho \mathbf{uu})$ term.

e) The form of Ohm's law in equation (3.23) is called resistive Ohm's law and the resulting set of MHD equation is called resistive MHD. If $\eta = 0$ we use the term ideal MHD. The main assumptions (simplifications) in Ohm's law can be better understood by examining general Ohm's law (3.19) through a normalization or dimensional analysis.

Measuring all quantities in typical units, i.e., the magnetic induction $\mathbf{B}$ in units of a typical magnetic field B_0 such that $\mathbf{B} = B_0\widehat{\mathbf{B}}$ where $\widehat{\mathbf{B}}$ is now of order unity, we can examine the coefficients of the different terms in Ohm's law. Note that velocities should be measured in units of the Alfvén speed $u_0 = B_0/\sqrt{\mu_0\rho_0}$, time in units of Alfvén travel time $t_0 = L_0/u_0$, and length scales in a typical scale for gradients in the system L_0. Applying this scaling yields

$$\widehat{\mathbf{E}} + \widehat{\mathbf{u}} \times \widehat{\mathbf{B}} = \frac{c^2}{\omega_{pe}^2 L_0^2}\frac{1}{\widehat{\rho}}\left[\frac{\partial \widehat{\mathbf{j}}}{\partial t} + \nabla \cdot \left(\widehat{\mathbf{u}}\widehat{\mathbf{j}} + \widehat{\mathbf{j}}\widehat{\mathbf{u}}\right)\right] - \frac{c}{\omega_{pi} L_0}\frac{1}{\widehat{\rho}}\nabla \widehat{p}_e + \frac{c}{\omega_{pi} L_0}\frac{1}{\widehat{\rho}}\widehat{\mathbf{j}} \times \widehat{\mathbf{B}} + \eta'\widehat{\mathbf{j}}$$

The terms

$$c/\omega_{pe} = \left(\epsilon_0 m_e c^2/ne^2\right)^{1/2}$$

and

$$c/\omega_{pi} = (m_i/m_e)^{1/2} c/\omega_{pe} \gg c/\omega_{pe}$$

are called electron and ion inertia scales (or electron and ion skin depth because of the extinction length of waves in a medium). Note that all MHD quantities are measured in typical units such the the typical values of $\widehat{\mathbf{B}}$, $\widehat{\rho}$, etc is of order unity. Thus the inertia term (first term on the right) is important only if the typical length scales of a system L is comparable to or smaller than the electron inertia scale. Similarly, the electron pressure term and the hall term (2nd and 3rd term on the right) are important only if typical gradient scales are comparable to or smaller than the ion inertia scale c/ω_{pi}. Thus it is justified to neglect these terms if the gradients in the system are on a much larger length scale then these inertia scales.

Other notes on Ohm's law

- The $\mathbf{j} \times \mathbf{B}$ term is usually addressed as the Hall term. With $\mathbf{u} \times \mathbf{B} - \frac{m_i}{e\rho}\mathbf{j} \times \mathbf{B} = \mathbf{u}_e \times \mathbf{B}$ Ohm's law can be written as

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = \frac{m_e m_i}{e^2 \rho}\left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u})\right] - \frac{M}{e\rho}\nabla p_e + \eta \mathbf{j}$$

- If the electron inertia terms (c/ω_{pe}) are neglected the corresponding set of equation are often addressed as Hall MHD.
- If both electron an ion inertia terms are neglected Ohm's law is called resistive Ohm's law $\mathbf{E} + \mathbf{u} \times \mathbf{B} = +\eta \mathbf{j}$ and the MHD equations are addressed as resistive MHD.
- If in addition resistivity $\eta = 0$ we address Ohm's law and the MHD as ideal Ohm's law and Ideal MHD.

Exercise: Assume a plasma density of 1 cm^{-3}, temperature equivalent to 1 keV, and a magnetic field of 20 nT which are typical for the near Earth magnetotail. Determine electron and ion inertia scales. Assume that quasi-neutrality is violated in a sphere with the radius of the electron inertia length by 1 % (e.g. 1% of the ion charge is not compensated by electrons. If outside were a vacuum what is the electric field outside the sphere? What velocity perpendicular to the magnetic field is required by Ohm's law to generate an electric field magnitude equal to that on the surface of the sphere?

Exercise: For the plasma in the prior exercise, determine the temperature in degrees Kelvin. Determine the energy density in kW hours/m^3 and kW hours /R_e^3 (1 R_E = 6370 km). For the sake of simplicity assume that the plasma sheet is represented by a cylinder with 10 R_E radius and 100 R_E length. How long could a power plant with an output of 1000 MW operate on the energy stored in the plasma?

3.4 Properties of the MHD equations:

3.4.1 Frozen-in Condition

The MHD equations are a very commonly used plasma approximation. They conserve mass, momentum, and energy. As mentioned above they are valid on scales larger than the ion inertia scale. It is important to note that the ideal MHD equation do not have any intrinsic physical length scale. This

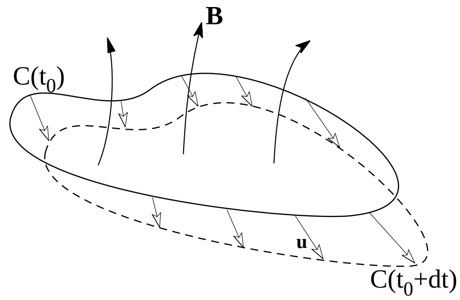


Figure 3.1: Illustration of the frozen-in condition.

implies for instance a self-similarity in the sense that the dynamics on small physical length scales is exactly the same as for large scale systems with the only difference that the larger system evolve slower. This can be illustrated by normalizing the equations to a particular length L_0 which implies that the typical time scale is $\tau_0 = L_0/u_A$ (with $u_A = B_0/\sqrt{\mu_0\rho_0}$). For a system which is identical except that is is 10 times larger the length scale is $10L_0$ and the time scale is $10\tau_0$. Thus a simple re-normalization yields exactly the same dynamics.

Ideal MHD assumes that the plasma is an ideal conductor (resistivity $\eta = 0$) and that terms on the ion and electron inertia scales can be neglected. Thus Ohm's law becomes

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

which implies that the magnetic flux is frozen into the plasma motion. This can be seen from the following arguments. The magnetic flux through the surface C is the surface integral

$$\Phi_C = \int_C \mathbf{B} \cdot d\mathbf{s}$$

with $d\mathbf{s}_C$ being the surface element of the contour C. The contour elements move with the fluid velocity $\mathbf{u}$. The change of the magnetic flux from time t_0 do $t_0 + dt$ is

$$\begin{aligned}
\Phi_C(t_0 + dt) - \Phi_C(t_0) &= \int_{C(t_0)}^{C(t_0+dt)} \mathbf{B} \cdot d\mathbf{s} + dt \int_{C(t_0)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \\
&= dt \oint_{\partial C(t_0)} \mathbf{B} \cdot (\mathbf{u}_C \times d\mathbf{l}) - dt \int_{C(t_0)} (\nabla \times \mathbf{E}) \cdot d\mathbf{s} \\
&= dt \oint_{\partial C(t_0)} (\mathbf{B} \times \mathbf{u}_C) \cdot d\mathbf{l} - dt \oint_{\partial C(t_0)} (-\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \\
&= dt \oint_{\partial C(t_0)} [(\mathbf{u} - \mathbf{u}_C) \times \mathbf{B}] \cdot d\mathbf{l}
\end{aligned}$$

where the first term on the rhs represents the contribution from the change of the shape of C and the second term the contribution from the change of $\mathbf{B}$. It follows that

$$\frac{d\Phi_C}{dt} = 0$$

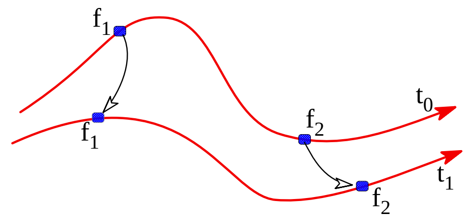


Figure 3.2: Illustration of line conservation

if the surface is moving with the fluid $\mathbf{u}_C = \mathbf{u}$. In other words the amount of magnetic flux through any given cross sectional area of the MHD fluid does not change in time if this area which moves with the fluid. The frozen-in condition can also be understood in the following way. Two fluid elements are always connected by a magnetic field line if they were connected at one time by a field line (a line defined by the direction of the magnetic field at any moment in time). In other words a field line can be identified by fluid plasma elements. This property is sometimes called line conservation. This requires that ideal Ohm's law applies, i.e., electrical resistivity is zero.

A more complete form of Ohm's law should be considered if gradients on smaller scales exist in a plasma. Since the ion inertia scale is by a factor of $\sqrt{m_i/m_e}$ larger than electron inertia effects the first terms to consider are the Hall term and the electron pressure term (note that the electron pressure is typically an order of magnitude smaller than the ion pressure such that contributions from this term are small. It is interesting to note that

$$\mathbf{u}_e = \mathbf{u} - \frac{m_i}{e\rho}\mathbf{j} \tag{3.27}$$

Comparing this with generalized Ohm's law we can re-write this neglecting the electron inertia scale as

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = -\frac{M}{e\rho}\nabla p_e \tag{3.28}$$

In other words the addition of the Hall term transforms Ohm's law from $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ in MHD into $\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = 0$. With this form it is clear that the frozen-in condition for the magnetic flux applies now to the electron fluid (note that one can also include a scalar electron pressure term in Ohm's law if the density does not vary strongly).

There will be various applications using the fluid and the kinetic equations. Typical applications consider waves, discontinuities and shocks, instabilities, steady state solutions, and equilibrium solutions. Particularly for the last topic it is important to note the following terminology.

Steady state assumes time stationary solutions with nonzero velocity, $\partial/\partial t = 0$ and $\mathbf{u} \neq 0$.

Equilibrium solutions assume $\partial/\partial t = 0$ and $\mathbf{u} = 0$. Note that for kinetic systems the velocity in phase space is always nonzero for physical systems. Also the electron velocity is nonzero in current regions.

Electrostatic solutions assume $\partial\mathbf{B}/\partial t = 0$. This implies $\nabla \times \mathbf{E} = 0$ or $\mathbf{E} = -\nabla\phi$. In this case Ohm's law must be replaced by the Coulomb equation.

3.4.2 Entropy and Adiabatic Convection

In an ordinary gas or fluid the change of heat due to pressure or temperature changes is

$$\Delta Q = C_V dT + pdV$$

Assuming an ideal gas (with 3 degrees of freedom)

$$T = pV/R$$

and the specific heat and coefficients of specific heat

$$\begin{aligned} C_V = \frac{3}{2}R \qquad C_p = \frac{5}{2}R \\ c_V = \frac{3}{2} \qquad c_p = \frac{5}{2} \end{aligned}$$

and the ratio $\gamma = c_p/c_V = 5/3$.
Thus the change in heat becomes

$$\begin{aligned} \Delta Q &= c_V V dp + c_p p dV \\ &= c_V pV \left(\frac{dp}{p} + \gamma \frac{dV}{V} \right) \end{aligned}$$

Entropy changes are defined as $dS = \Delta Q/T$ or

$$S \propto C_V \ln \left(pV^\gamma \right)$$

For adiabatic changes of the state of a system we have $dS = 0$ which can also be expressed as

$$\frac{d\left(p\rho^{-\gamma}\right)}{dt} = \frac{\partial}{\partial t} p\rho^{-\gamma} + \mathbf{u} \cdot \nabla \left(p\rho^{-\gamma} \right) = 0 \qquad (3.29)$$

with the specific volume $V \propto 1/\rho$. This equation implies that the entropy of a plasma element does not change (moving with this element). The same relation can be derived from the total pressure equation

$$\frac{1}{\gamma - 1} \left(\frac{\partial}{\partial t} p + \nabla \cdot p\mathbf{u} \right) = -p \nabla \cdot \mathbf{u}$$

with the help of the continuity equation for density.

Exercise: Derive equation (3.29) from the pressure equation.

Properties of magnetic flux tubes:

The volume of a closed magnetic flux tube (magnetosphere or magnetic mirror) is given by

$$V_f = \int_{l_1}^{L_2} \left(\int \int_{A_c(l)} ds \right) dl$$

which is taken along the entire length of the flux tube with a cross section that is varying along the flux tube. For a ‘sufficiently thin flux tube the cross section of the tube varies as

$$A_c \propto \frac{1}{B}$$

This leads us to the definition of the differential flux tube volume

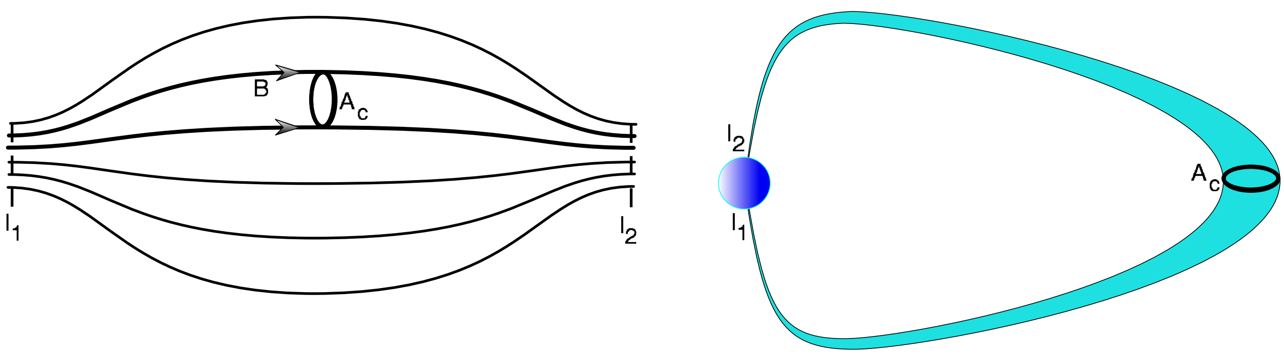
$$V = \int_{l_1}^{L_2} \frac{dl}{B}$$

Examine properties of such a magnetic flux tube:

The total number of particles on the flux tube is

$$N_f = \int_{l_1}^{l_2} \left(\int \int_{A_c(l)} n ds \right) dl$$

which is the plasma density integrated over the volume of a magnetic flux tube.



The change of the total number of particles over a time period from t to $t + \Delta t$ is given by the change in length of the flux tube, changes in the cross section boundary, and a change in the local number density

$$\begin{aligned} \Delta N &= \int_{l_{1,t+\Delta t}}^{l_{2,t+\Delta t}} \left(\int \int_{A_c,t} n d\mathbf{s} \right) \cdot d\mathbf{l} - \int_{l_{1,t}}^{l_{2,t}} \left(\int \int_{A_c,t} n d\mathbf{s} \right) \cdot d\mathbf{l} \\ &\quad + \int_{l_{1,t}}^{l_{2,t}} \left(\int \int_{A_c,t+\Delta t} n d\mathbf{s} - \left(\int \int_{A_c,t} n d\mathbf{s} \right) \right) \cdot d\mathbf{l} \end{aligned}$$

$$
\begin{aligned}
& + \int_{l_1}^{l_2} \left(\int \int_{A_c(l)} \left(n\left(t+\Delta t\right) - n\left(t\right)\right) d\mathbf{s} \right) \cdot d\mathbf{l} \\
= & \int_{l_{2,t}}^{l_{2,t+\Delta t}} \left(\int \int_{A_c,t} n d\mathbf{s} \right) \cdot d\mathbf{l} - \int_{l_{1,t}}^{l_{1,t+\Delta t}} \left(\int \int_{A_c,t} n d\mathbf{s} \right) \cdot d\mathbf{l} \\
& + \int_{l_{1,t}}^{l_{2,t}} \left(\oint_{A_c,t+\Delta} n \mathbf{u}_f \Delta t \times d\mathbf{c} \right) \cdot d\mathbf{l} + \int_{l_1}^{l_2} \left(\int \int_{A_c(l)} \left(n\left(t+\Delta t\right) - n\left(t\right)\right) d\mathbf{s} \right) \cdot d\mathbf{l}
\end{aligned}
$$

where A_c is the cross section of the flux, $\mathbf{u}_f$ is the velocity of the boundary of the flux tube, and $d\mathbf{c}$ is the line element along the flux tube boundary. The temporal change of the total number of particles in the flux tube is

$$
\begin{aligned}
\frac{\Delta N}{\Delta t} = & \ \mathbf{u}_{f2} \cdot \int \int_{A_{c2}} n d\mathbf{s} + \mathbf{u}_{f1} \cdot \int \int_{A_{c1}} n d\mathbf{s} \\
& + \int_{l_{1,t}}^{l_{2,t}} \oint_{A_c,t} n \mathbf{u}_f \cdot d\mathbf{s} + \int_{l_1}^{l_2} \left(\int \int_{A_c,t} \frac{n\left(t+\Delta t\right) - n\left(t\right)}{\Delta t} d\mathbf{s} \right) \cdot d\mathbf{l}
\end{aligned}
$$

here $\mathbf{u}_{f1}$ and $\mathbf{u}_{f2}$ is the velocity of the flux tube boundary at the points l_1 and l_2. And in the limit $\Delta t \to 0$

$$
\begin{aligned}
\frac{dN}{dt} = & \ \mathbf{u}_{f2} \cdot \int \int_{A_{c2},t} n d\mathbf{s} + \mathbf{u}_{f1} \cdot \int \int_{A_{c1},t} n d\mathbf{s} \\
& + \int_{l_{1,t}}^{l_{2,t}} \oint_{A_c,t} n \mathbf{u}_f \cdot d\mathbf{s} - \int_{l_1}^{l_2} \int \int_{A_c,t} \nabla \cdot \left(n \mathbf{u} \right) d^3 r \\
= & \int \int_{surface(V_f)} \left(n (\mathbf{u}_f - \mathbf{u}) \right) \cdot d\mathbf{s}
\end{aligned}
$$

In other words the number of particle changes only as a result of a motion of the flux tube boundary different from the plasma motion. In particular this implies that

- flow through the ends of the flux tube (into the ionosphere or out of a magnetic mirror configuration) can change the total number of particles on a flux tube
- the number of particle does not change for closed ends (i.e., no flow through the ends) and ideal MHD because the boundary of the flux tube are magnetic field lines and plasma elements move with the magnetic field (= boundary of the flux tube).
- using the definition of a differential flux tube

  $$N = \int_{l_1}^{L_2} \frac{n dl}{B}$$

  is conserved in ideal MHD in the absence of flow at the ends of the flux tube

The above discussion considered the the total number of particles which is the integral over the number density. For the derivation of the constancy of total number of particles on flux tubes we only required that the number density satisfies a continuity equation. This implies that any property which satisfies a continuity equation is conserved. In particular it can be shown that

$$\frac{\partial p^{1/\gamma}}{\partial t} + \nabla \cdot p^{1/\gamma} = 0$$

if heat flux and energy sources are negligible. For slow (adiabatic) changes the pressure is constant on magnetic field lines such that we can define

$$H = p^{1/\gamma} V = \int_{l_1}^{L_2} \frac{p^{1/\gamma} dl}{B}$$

Since $p^{1/\gamma}$ satisfies a continuity equation the derivation for dH/dt follows exactly that for density such that $dH/dt = 0$ for ideal MHD and no flux through the ends of a flux tube. Consider the specific entropy of a flux tube as $S = H^{\gamma}$. This implies that the specific entropy satisfies

$$\frac{dS}{dt} = \frac{dpV^{\gamma}}{dt} = 0$$

This is a generalization of the local conservation of entropy

$$\frac{dp\rho^{-\gamma}}{dt} = 0$$

The number of particle on flux tubes and the flux tube entropy are important to achieve insight into convection. Considering for instance the magnetosphere these quantities can be evaluated in the equatorial plane thus providing a two-dimensional map of flux tube entropy. Steady convection implies $\partial/\partial t = 0$, i.e., the magnetic and plasma configuration does not change. Thus steady convection can occur only along contours of constant flux tube entropy.

Other equations of state:

The rigorous derivation of the MHD equations from the Boltzmann implies $\gamma = 5/3$. However there are situations where it is more appropriate to assume other values for gamma to emulate a particular equation of state. We illustrate these using the equation for pressure.

$$\frac{1}{\gamma - 1}\left(\frac{\partial}{\partial t} p + \nabla \cdot p\mathbf{u}\right) = -p\nabla \cdot \mathbf{u}$$

a) Adiabatic changes: The value of $\gamma = 5/3$ corresponds to adiabatic change and we can transform the pressure equation to an equation for adiabatic changes.

b) Isothermal changes: With $p = nk_bT$ we can re-write the pressure equation as

$$\partial T/\partial T + \mathbf{u}\cdot\nabla T = -\left(\gamma-1\right)T\nabla\cdot\mathbf{u}$$

Thus a value of $\gamma = 1$ leads to $dT/dt = \partial T/\partial T + \mathbf{u}\cdot\nabla T = 0$ and implies isothermal changes.

c) Incompressible changes can be modeled in the limit of $\gamma = \infty$ because this limit implies $\nabla\cdot\mathbf{u} = 0$ such that $d\rho/dt = 0$.

3.4.3 Conservation laws:

The MHD equation satisfy mass, momentum, and energy conservation.

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= -\nabla\cdot\rho\mathbf{u} \\
\frac{\partial\rho\mathbf{u}}{\partial t} &= -\nabla\cdot\left[\rho\mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu_0}\right)\underline{\underline{1}} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}\right] \\
\frac{\partial w_{tot}}{\partial t} &= -\nabla\cdot\left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1} + \frac{1}{\mu_0}B^2\right)\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{B}}{\mu_0}\mathbf{B} + \frac{\eta}{\mu_0}\mathbf{j}\times\mathbf{B}\right]
\end{aligned}$$

with the total energy density

$$w_{tot} = \frac{1}{2}\rho u^2 + \frac{p}{\gamma-1} + \frac{1}{2\mu_0}B^2 \tag{3.30}$$

Exercise: Demonstrate the validity of the energy conservation equation.

The various terms in (3.30) are the energy density of the

Bulk flow: $\frac{1}{2}\rho u^2$

3.5 MHD Equilibria

Resistive Diffusion

Before discussing equilibrium properties let us first consider effects of electric resistivity. Using the resistive form of Ohm's law (with constant resistivity η) and the induction equation

$$\begin{aligned}
\frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E} \\
&= \nabla\times\left[\mathbf{u}\times\mathbf{B} - \frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right] \\
&= \nabla\times\left[\mathbf{u}\times\mathbf{B}\right] + \frac{\eta}{\mu_0}\Delta\mathbf{B}
\end{aligned}$$

In cases where the velocity is negligible one obtains

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0}\Delta \mathbf{B}$$

This equation indicates that the magnetic field evolves in the presence of a finite resistivity even in the absence of any plasma flow. This process is called resistive diffusion. Dimensional analysis yields for the typical time scale

$$\tau_{diff} \sim \frac{\mu_0 L^2}{\eta}$$

i.e., diffusion is fast for large values of η or small typical length scales. For the prior discussion of the frozen-in condition we require that the resistivity is 0 or at least so small that the diffusion time is long compared to the time scale for convection where the frozen-in condition is applied.

Basic equilibrium equations and properties:

Equilibrium requires $\partial/\partial t = 0$ and $\mathbf{u} = 0$. Thus the MHD equations lead to

$$\begin{aligned} -\nabla p + \mathbf{j} \times \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{3.31}$$

Taking the scalar product of the momentum equation with $\mathbf{B}$ and $\mathbf{j}$ yields

$$\begin{aligned} \mathbf{B} \cdot \nabla p &= 0 \\ \mathbf{j} \cdot \nabla p &= 0 \end{aligned}$$

In other words the pressure is constant on magnetic field lines and on current lines.

Note: There is no equation for the plasma density. Only the pressure needs to be determined. From

$$p = n k_B T$$

it follows that only the product nT is fixed and either density or temperature can be chosen arbitrarily.

The momentum equation can also be expressed as

$$\nabla \cdot \left[\left(p + \frac{B^2}{2\mu_0} \right) \underline{\underline{1}} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B} \right] = 0$$

or

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)-\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\mathbf{B}=0$$

This implies particularly simple equilibria for $\mathbf{B}\cdot\nabla\mathbf{B}=0$ (a simple case of this condition is a magnetic field with straight magnetic field lines). In this case the equilibrium requires total pressure balance where the sum of thermal and magnetic pressure are 0, i.e.,

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)=0$$

or

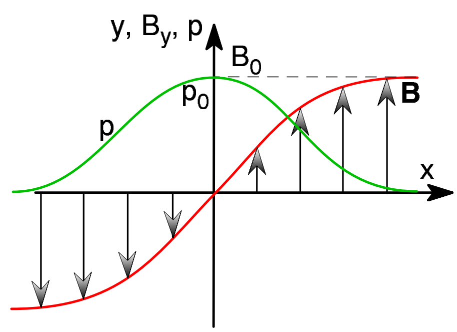
$$p+\frac{B^2}{2\mu_0}=\text{const} \tag{3.32}$$

A particular example of this class of equilibria is the plain current sheet with the specific example of a so-called Harris sheet with

$$\mathbf{B}=B_0\tanh\frac{x}{L}\,\mathbf{e}_y$$

with

$$\begin{aligned} p &= p_0\cosh^{-2}\frac{x}{L} \\ \mathbf{j} &= \frac{B_0}{\mu_0 L}\cosh^{-2}\frac{x}{L}\,\mathbf{e}_z \\ \rho &= \rho_0\cosh^{-2}\frac{x}{L} \end{aligned}$$



with $p_0=B_0^2/(2\mu_0)$.

Note that one can add any constant to the pressure. It is also straightforward to modify the magnetic field, for instance to an asymmetric configuration. the pressure is computed from (3.32) and only subject to the condition that it must be larger than 0 everywhere.

The magnetic and plasma configuration also determines how important pressure relative to magnetic forces are. It is common to use the so-called plasma β as a measure of the thermal pressure to magnetic field pressure (this is also a measure of the corresponding thermal and magnetic energy densities.). The plasma β is defined as

$$\beta=\frac{2\mu_0 p}{B^2}.$$

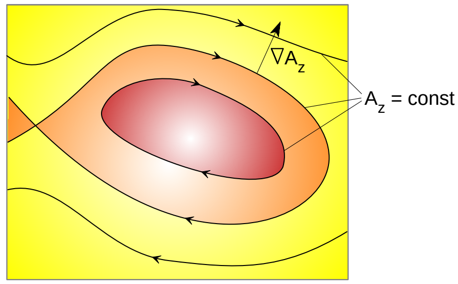


Figure 3.3: Representation of field lines by the the vector potential

Two-Dimensional Equilibria

A systematic approach to solve the equilibrium equations usually requires to represent the magnetic field through the vector potential.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Consider 2d system with $\partial/\partial z = 0$ such that $\mathbf{B} = \mathbf{B}(x, y)$. In this case the magnetic field is uniquely represented through the z component of the vector potential $A_z(x, y)$ and the $B_z(x, y)$ component:

$$\mathbf{B} = \nabla \times (A_z \mathbf{e}_z) + B_z \mathbf{e}_z = \nabla A_z \times \mathbf{e}_z + B_z \mathbf{e}_z$$

We only need two dependent variables because of $\nabla \cdot \mathbf{B} = 0$. Note also that this form of $\mathbf{B}$ always satisfies $\nabla \cdot \mathbf{B} = 0$. Denoting the field in the x, y plane as

$$\mathbf{B}_\perp = \nabla \times A_z(x, y)\mathbf{e}_z = \nabla A_z(x, y) \times \mathbf{e}_z \tag{3.33}$$

it follows that $\mathbf{B}_\perp$ is perpendicular to ∇A_z and $\mathbf{e}_z$.

Therefore lines of constant A_z (contour lines of A_z) are magnetic field lines projected into the x, y plane. The difference of the vector potential between two field lines is a direct measure of the magnetic flux bound by these field lines. The vector potential can be obtained by integrating

$B_x = \partial A_z/\partial y$ and $B_y = -\partial A_z/\partial x$.

In spherical coordinates the use of a component of the vector potential is similar but slightly modified. Consider two-dimensionality with $\partial/\partial \varphi = 0$. In this case the magnetic field is expressed as

$$\mathbf{B}(r, \theta) = \nabla \times A_\varphi(r, \theta)\mathbf{e}_\varphi + B_\varphi(r, \theta)\mathbf{e}_\varphi \tag{3.34}$$

Explicitly the magnetic field components are

$$\begin{aligned} B_r &= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (r \sin\theta \, A_\varphi) \\ B_\theta &= -\frac{1}{r \sin\theta} \frac{\partial}{\partial r} (r \sin\theta \, A_\varphi) \end{aligned}$$

Remembering that the gradient in the r, θ plane is defined as

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}e_\theta$$

one can choose $f = r\sin\theta\, A_\varphi$ such that $\mathbf{B}\cdot\nabla f = 0$. It follows that magnetic field lines are determined by

$$r\sin\theta\, A_\varphi = const \tag{3.35}$$

Exercise: Consider the magnetic field $B_x = B_0 y/L$, $B_y = \epsilon B_0$. Compute the equations for magnetic field lines.

Exercise: Repeat the above formulation for cylindrical coordinates with $\partial/\partial\varphi = 0$.

With $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$ the current density becomes

$$\begin{aligned}\mathbf{j} &= \frac{1}{\mu_0}\nabla\times[\nabla A_z\times\mathbf{e}_z + B_z\mathbf{e}_z]\\ &= -\frac{1}{\mu_0}\Delta A_z\mathbf{e}_z + \frac{1}{\mu_0}\nabla B_z\times\mathbf{e}_z\end{aligned}$$

such that the z component of the current density is

$$j_z = -\frac{1}{\mu_0}\Delta A_z$$

Substituting the current density in the force balance equation and using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

$$\begin{aligned}0 &= -\nabla p + \mathbf{j}\times\mathbf{B}\\ &= -\nabla p + \frac{1}{\mu_0}[-\Delta A_z\mathbf{e}_z + \nabla B_z\times\mathbf{e}_z]\times[\nabla A_z\times\mathbf{e}_z + B_z\mathbf{e}_z]\\ &= -\nabla p + \frac{1}{\mu_0}\{-\Delta A_z\nabla A_z - ((\nabla B_z\times\mathbf{e}_z)\cdot\nabla A_z)\mathbf{e}_z - B_z\nabla B_z\}\\ &= -\nabla p + \frac{1}{\mu_0}\{-\Delta A_z\nabla A_z - B_z\nabla B_z + \nabla B_z\times\nabla A_z\}\end{aligned}$$

Here the term $\nabla B_z\times\nabla A_z$ has only a z component because ∇B_z and ∇A_z are both in the x, y plane. Since it is the only term in the z it follows that $\nabla B_z\times\nabla A_z = 0$ or $\nabla B_z \| \nabla A_z$. Thus we can in general express $B_z = B_z(A_z)$. This can be used to express

$$B_z \nabla B_z = \frac{1}{2} \nabla B_z^2 = \frac{1}{2} \frac{dB_z^2}{dA_z} \nabla A_z$$

In summary the force balance condition leads to

$$\nabla p = \left(j_z - \frac{1}{2\mu_0} \frac{dB_z^2}{dA_z} \right) \nabla A_z$$

Since the pressure gradient is along the gradient of A_z the pressure must be a function of A_z. It follows that

$$\Delta A_z = -\mu_0 \frac{d}{dA_z} \left(p(A_z) + \frac{B_z(A_z)^2}{2\mu_0} \right)$$

In this equation p represents the thermal pressure and $B_z^2/2\mu_0$ is the magnetic pressure due to the z component of the magnetic field. In other words in two dimensions the magnetic field along the invariant direction acts mostly as a pressure to maintain an equilibrium. Defining

$$\widetilde{p}(A_z) = p + \frac{B_z^2}{2\mu_0}$$

we have to seek the solution to

$$\Delta A_z = -\mu_0 \frac{d}{dA_z} \widetilde{p}(A_z). \tag{3.36}$$

Usually $\widetilde{p}(A_z)$ is defined as a relatively simple form. Most convenient for traditional solution methods is to define $\widetilde{p}(A_z)$ as a linear function of A_z. More realistic is a definition which requires a kinetic background. We will later show that local thermodynamic equilibrium implies $p(A_z) \propto \exp\left(-2A_z/A_c\right)$ where A_c is a constant which follows from the later kinetic treatment of the equilibrium.

Exercise: Consider $p(A_z) = p_0 \exp\left(-2A_z/A_c\right)$ and $B_z = 0$. Show that a one-dimensional ($\partial/\partial y = 0$ and $\partial/\partial z = 0$) solution of equation (3.36) has the form $A_z = A_c \ln\cosh\left(x/L\right)$ and that the magnetic field in this case is the Harris sheet field.

Exercise: Consider one-dimensional solutions with $\partial/\partial x \neq 0$. Obtain the first integral of the equation (3.36) by multiplying the equation with dA_z/dx and integration. The resulting equation is the equation of total pressure balance. Interpret the term in the first integral in this manner.

Exercise: Consider a two-dimensional plasma $\partial/\partial z = 0$ with $B_z = 0$ and use the representation of the magnetic field through the vector potential. Replace the magnetic field in resistive Ohm's law $\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\eta}{\mu_0} \nabla \times \mathbf{B}$ through the vector potential and show for $\eta = \text{const}$ that this yields for the z component $\partial A_z/\partial z + \mathbf{u} \cdot \nabla A_z = \frac{\eta}{\mu_0} \Delta A_z$. For $\eta = 0$ this equation directly demonstrates the frozen-in condition. Explain why.

Note that the above discussion can be generalized to coordinate systems other than Cartesian. For Laboratory plasmas with an azimuthal invariance it is often convenient to use cylindrical coordinates with $\partial/\partial\phi = 0$. If there is invariance along a cylinder axis one can also use $\partial/\partial z = 0$.

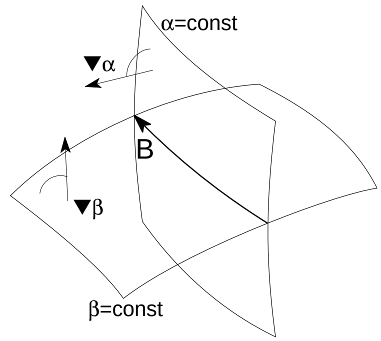


Figure 3.4: Sketch of the field interpretation of Euler potentials.

Three-Dimensional Equilibria:

In three dimensions we cannot make the simplification of using a single component of the vector potential. However there is a formulation which lends itself to a similar treatment.

Introducing so-called Euler potentials α and β: $\mathbf{A} = \alpha\nabla\beta + \nabla\Xi$ =>

$$\begin{aligned} \mathbf{B} = \nabla\times\mathbf{A} &= \nabla\times[\alpha\nabla\beta + \nabla\Xi] \\ &= \nabla\alpha\times\nabla\beta \end{aligned} \tag{3.37}$$

Note that Euler potentials imply $\mathbf{A}\cdot\mathbf{B} = 0$ which is not generally satisfied but it is always possible to find a gauge such that $\mathbf{B}$ is perpendicular to $\mathbf{A}$.

Using Euler potentials the magnetic field is perpendicular to $\nabla\alpha$ and $\nabla\beta$ or - in other words - magnetic field lines are the lines where isosurfaces of α and β intersect. The prior two-dimensional treatment is actually a special case of Euler coordinates with $\alpha = A_z$ and $\beta = z$.

The current density is now given by

$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

$$\begin{aligned} \mathbf{j} &= \frac{1}{\mu_0}\nabla\times[\nabla\alpha\times\nabla\beta] \\ &= \frac{1}{\mu_0}(\Delta\beta\nabla\alpha - \Delta\alpha\nabla\beta + \nabla\beta\cdot\nabla(\nabla\alpha) - \nabla\alpha\cdot\nabla(\nabla\beta)) \end{aligned}$$

Force balance: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

$$\begin{aligned} 0 &= -\nabla p + \mathbf{j}\times\mathbf{B} \\ &= -\nabla p + \frac{1}{\mu_0}[\nabla\times(\nabla\alpha\times\nabla\beta)]\times[\nabla\alpha\times\nabla\beta] \\ &= -\nabla p + \frac{1}{\mu_0}\{[\nabla\beta\cdot\nabla\times(\nabla\alpha\times\nabla\beta)]\nabla\alpha \\ &\qquad - [\nabla\alpha\cdot\nabla\times(\nabla\alpha\times\nabla\beta)]\nabla\beta\} \end{aligned}$$

This yields the basic dependencies of $p = p(\alpha\,\beta)$ and the equilibrium equations

$$\nabla\beta \cdot \nabla \times (\nabla\alpha \times \nabla\beta) = \mu_0 \frac{\partial p\,(\alpha, \beta)}{\partial \alpha} \tag{3.38}$$

$$\nabla\alpha \cdot \nabla \times (\nabla\alpha \times \nabla\beta) = -\mu_0 \frac{\partial p\,(\alpha, \beta)}{\partial \beta} \tag{3.39}$$

Note that in cases with a gravitational force the force balance equation is modified to

$$-\nabla p + \mathbf{j} \times \mathbf{B} - \rho \nabla \Phi = 0$$

where Φ is the gravitational potential. This is for instance important for solar applications of the equilibrium theory. In this case the pressure has to be also a function of Ψ and the equilibrium equations become

$$\begin{aligned} \nabla\beta \cdot \nabla \times (\nabla\alpha \times \nabla\beta) &= \mu_0 \frac{\partial p\,(\alpha, \beta, \Phi)}{\partial \alpha} \\ \nabla\alpha \cdot \nabla \times (\nabla\alpha \times \nabla\beta) &= -\mu_0 \frac{\partial p\,(\alpha, \beta, \Phi)}{\partial \beta} \\ \rho &= -\frac{\partial p\,(\alpha, \beta, \Phi)}{\partial \Phi} \end{aligned}$$

These equation are often referred to as the Grad Shafranov equations and they are commonly used to compute three-dimensional equilibrium configurations. While there are some analytic solutions these equations are mostly used with computational techniques. A numerical procedure to find solution usually starts with a straightforward initial solution. For instance, a vacuum magnetic field is always a solution to equations (3.38) and (3.39) for constant pressure (vacuum field refers to a magnetic field for which the current density is 0, i.e., there are no current carriers). Examples are constant magnetic fields, dipole or higher multipole fields or any magnetic field which is derived through a scalar potential $\mathbf{B} = -\nabla\Psi$. Note that Ψ has to satisfy $\Delta\Psi = 0$ otherwise $\nabla \cdot \mathbf{B} = 0$ is violated.

Magnetic dipole in spherical coordinates (r, θ, φ):

$$\Psi = -\frac{\mu_0 M_D}{4\pi} \frac{\cos\theta}{r^2} \tag{3.40}$$

Noting that

$$\nabla\Psi = \frac{\partial\Psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\varphi}\mathbf{e}_\varphi$$

the dipole magnetic field components become

$$B_r = -\frac{\mu_0 M_D}{2\pi}\frac{\cos\theta}{r^3}$$
$$B_\theta = -\frac{\mu_0 M_D}{4\pi}\frac{\sin\theta}{r^3}$$

The formalism derived for spherical coordinates in two dimensions can then be used to derive Euler potentials for the start solution. In the course of the numerical solution of (3.38) and (3.39) the pressure is increased as a specified function of α and β. Also any changes in terms of boundary conditions etc. are applied in the iterative solution of the equations. Note that a nontrivial point of the system (3.38) and (3.39) is the existence of solution or possible multiple branches of solutions for the same boundary conditions.

Exercise: Determine the vector potential component A_ϕ for the dipole field and derive the equations for the magnetic field lines in spherical coordinates.

3.6 MHD Stability

The stability of plasmas both in laboratory and in the natural environment is of central importance to understand plasma systems. It is worth noting, however that the interest of laboratory plasma research is usually the stability of a configuration (for instance the stable confinement of plasma in a fusion device) while the interest in space plasma is clearly more in the instability of such systems. Either way plasma instability is a central issue in both communities.

A priory it is clear that a homogeneous plasma with all particle species moving a the same bulk velocity and all species having equal temperatures with Maxwell particle distributions is the ultimate stable system because it is in global thermal equilibrium. However, any deviation from this state has the potential to cause an instability. In this section we are interested in the stability of fluid plasma such that detail of the distribution function such as non-Maxwellian anisotropic distributions cannot be addressed here and will be left for later discussion.

In the case of fluid plasma the main driver for instability is spatial inhomogeneity, such as spatially vary magnetic field, current, and bulk velocity distributions. Most of the following discussion will assume that the bulk velocity is actually zero and the plasma is in an equilibrium state as discussed in the previous section.

There are several methods to address plasma stability/instability. First, modern numerical methods allow to carry out computer simulation if necessary on massive scales. Numerical studies have advantages and disadvantages. For instance, a computer simulation can study not only small perturbations but also nonlinear perturbations of the equilibrium and the basic formulation is rather straightforward. However, a simulation can test only one configuration, one set of boundary conditions, and one type of perturbation at a time. thus it may be cumbersome to obtain a good physical understanding of how stability properties change when system parameter change.

There are two basic analytic methods to study stability and instability of a plasma. The first methods uses a small perturbation and computes the characteristic evolution of the system. If all modes are have constant or damped amplitudes in time the plasma system is stable, if there are (exponentially) growing modes the system is unstable. This analysis is called the normal mode analysis. The second approach also uses small perturbations but considers a variational or energy principle. This is similar to a simple mechanical system where the potential has a local maximum or minimum. Thus this method attempts to formulate a potential for the plasma system which can be examined for stability. It will soon be clear that although the basic idea is the same, plasma systems are considerably more complex than simple mechanical examples.

3.6.1 Small oscillations near an equilibrium

Consider a configuration which satisfies the equilibrium force balance condition. For conditions $-\nabla p + \mathbf{j} \times \mathbf{B} = 0$. Equilibrium quantities are denoted by an index 0, i.e., p_0, $\mathbf{B}_0$, and $\mathbf{j}_0 = \frac{1}{\mu_0} \nabla \times \mathbf{B}_0$. All equilibrium plasma properties are a function of space only.

In the following we assume small perturbations from the equilibrium state. For these perturbations we use the index 1. The coordinate of a fluid element is $\mathbf{x}_0(t)$. Considering a small displacement of the plasma coordinate in the frame moving with the plasma (Lagrangian displacement) such that the new coordinate is

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \boldsymbol{\xi}(\mathbf{x}_0, t)$$

which satisfies $\xi(\mathbf{x}_0, 0) = 0$. A Taylor expansion of the fluid velocity yields

$$\mathbf{u}_1(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}_0, t) + \left(\dot{\boldsymbol{\xi}} \cdot \nabla\right) \mathbf{u}_1(\mathbf{x}_0, t) + ...$$

with $\dot{\boldsymbol{\xi}} = \partial \boldsymbol{\xi} / \partial t$. However, since both $\boldsymbol{\xi}$ and $\mathbf{u}_1$ are perturbation quantities (and hence small) we can neglect the terms of the Taylor expansion except for the 0th order implying:

$$\mathbf{u}_1(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}_0, t)$$

Here $\mathbf{u}_1(\mathbf{x}, t)$ represent the Eulerian velocity at the location $\mathbf{x}$ and $\mathbf{u}_1(\mathbf{x}_0, t)$ the Lagrangian description in the co-moving frame. Since $\mathbf{u}_1(\mathbf{x}_0, t) = \partial \boldsymbol{\xi}(\mathbf{x}_0, t)/\partial t$. Therefore we can replace replace the Eulerian velocity in the plasma equations by the Lagrangian by $\partial \boldsymbol{\xi}(\mathbf{x}_0, t)/\partial t$.

We now linearize the ideal MHD equations around the equilibrium state and substitute $\mathbf{u}_1$ with $\partial \boldsymbol{\xi} / \partial t$.

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\nabla \cdot \left(\rho_0 \dot{\boldsymbol{\xi}}\right) \\
\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1 \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times \left(\dot{\boldsymbol{\xi}} \times \mathbf{B}_0\right) \\
\frac{\partial p_1}{\partial t} &= -\dot{\boldsymbol{\xi}} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \dot{\boldsymbol{\xi}}
\end{aligned}$$

We can now integrate the continuity equation, the induction equation, and the pressure equation in time (assuming that the initial perturbation is zero) which yields

$$\begin{aligned}
\rho_1 &= -\nabla \cdot (\rho_0 \boldsymbol{\xi}) \\
\mathbf{B}_1 &= \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \equiv \mathbf{Q}_\xi \\
p_1 &= -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}
\end{aligned}$$

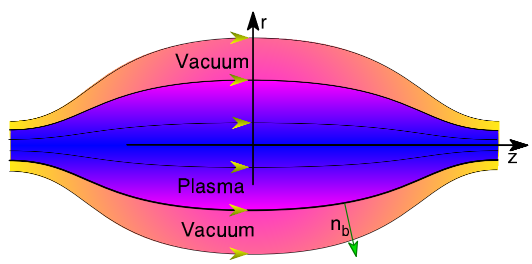
and substitute $\mathbf{u}_1$ with $\dot{\boldsymbol{\xi}}$. Take the time derivative of the momentum equation and replace the perturbed density, magnetic field, and pressure in the momentum equation.

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \nabla \left(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi} \right) + \frac{1}{\mu_0} \left[(\nabla \times \mathbf{Q}_\xi) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{Q}_\xi \right]$$

To solve this equation it is necessary to provide initial conditions for $\boldsymbol{\xi}$ and $\dot{\boldsymbol{\xi}} = \mathbf{u}_1$, and boundary conditions for $\boldsymbol{\xi}$.

Boundary Conditions

Frequently used conditions for a laboratory system are that the plasma is confined to a region which is embedded in a vacuum which in turn is bounded by an ideal conducting wall. This is an assumption seldom realized but suitable for the mathematical treatment.



In the case of space plasma systems boundary conditions are even more difficult to formulate because they are not confined to a particular region. However, if the volume is taken sufficiently large surface contributions to the interior are small such that for instance an assumption that the perturbations are zero at the boundary is a suitable choice. In the case of solar prominences the field is anchored in the solar photosphere which is an almost ideal conductor which implies that the foot points of prominences are moving with the photospheric gas.

i) Conducting wall

If the boundary to the plasma system is an ideal conducting wall the boundary condition simplifies considerably. In this case it is necessary that the tangential electric field is zero because the wall has a constant potential:

$$\mathbf{n}_w \times \mathbf{E}_1 = 0$$

where $\mathbf{n}_w$ is the unit vector of the outward normal direction to the wall. With Ohm's law $\mathbf{E}_1 = \dot{\boldsymbol{\xi}} \times \mathbf{B}_0$ we obtain

$$\mathbf{n}_w \times \left(\dot{\boldsymbol{\xi}} \times \mathbf{B}_0 \right) = (\mathbf{n}_w \cdot \mathbf{B}_0)\, \dot{\boldsymbol{\xi}} - \left(\mathbf{n}_w \cdot \dot{\boldsymbol{\xi}} \right) \mathbf{B}_0 = 0$$

If the magnetic field is tangential to the wall which is usually the case for fusion devices the first term vanishes and the boundary condition is

$$\mathbf{n}_w \cdot \dot{\boldsymbol{\xi}} = 0$$

which is satisfied if the perturbation is tangential to the boundary.

ii) Vacuum boundary:

Considering a very small section of the boundary one usually neglects curvature terms $\mathbf{B} \cdot \nabla \mathbf{B}$ in the force balance equation. In this case the local equilibrium is determined by total pressure balance at the plasma - vacuum boundary

$$p_0 (\mathbf{x}_0) + \frac{B_0^2 (\mathbf{x}_0)}{2\mu_0} = \frac{B_{v0}^2 (\mathbf{x}_0)}{2\mu_0}$$

where $\mathbf{x}_0$ denotes a point on the unperturbed plasma - vacuum boundary and the index v denotes variables in the vacuum region. Note that this is an idealization because it implies a jump of the tangential magnetic field (which also implies a surface current). Note that the transition to vacuum implies that the boundary is a pressure boundary and from $\mathbf{B}_0 \cdot \nabla p_0 = 0$ we no that the magnetic field must be tangential to the boundary! If there were a magnetic field threading through the boundary it is obvious that the pressure is not constant on a field line and therefore that the equilibrium condition of $p_0 = \mathrm{const}$ on magnetic field lines is violated. A point on the perturbed boundary has the coordinate

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{n}_b \xi_n$$

where $\mathbf{n}_b$ is the outward normal unit vector and ξ_n is the component of the displacement normal to the boundary. At the perturbed boundary the total pressure must also be continuous

$$p_0 (\mathbf{x}) + p_1 (\mathbf{x}) + \frac{(B_0 (\mathbf{x}) + B_1 (\mathbf{x}))^2}{2\mu_0} = \frac{(B_{v0} (\mathbf{x}) + B_{v1} (\mathbf{x}))^2}{2\mu_0}$$

Now we have to express this condition in terms of $\boldsymbol{\xi}$ which can be done by expanding equilibrium quantities in a Taylor expansion around $\mathbf{x}_0$ such as

$$\begin{aligned}
p_0 (\mathbf{x}) &= p_0 (\mathbf{x}_0) + \xi_n \mathbf{n}_b \cdot \nabla p_0 (\mathbf{x}_0) + ... \\
p_1 (\mathbf{x}) &= -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi} \\
B_0^2 (\mathbf{x}) &= B_0^2 (\mathbf{x}_0) + \xi_n \mathbf{n}_b \cdot \nabla B_0^2 (\mathbf{x}_0) + ... \\
\mathbf{B}_0 (\mathbf{x}) \cdot \mathbf{B}_1 (\mathbf{x}) &= \mathbf{B}_0 (\mathbf{x}_0) \cdot \mathbf{B}_1 (\mathbf{x}_0) + ...
\end{aligned}$$

Substitution into the pressure balance equation yields

$$-\gamma p_0 \nabla \cdot \boldsymbol{\xi} + \frac{\mathbf{B}_0 (\mathbf{x}_0) \cdot \mathbf{B}_1 (\mathbf{x}_0)}{\mu_0} + \frac{\xi_n}{2\mu_0} \frac{\partial B_0^2 (\mathbf{x}_0)}{\partial n} = \frac{\mathbf{B}_{v0} (\mathbf{x}_0) \cdot \mathbf{B}_{v1} (\mathbf{x}_0)}{\mu_0} + \frac{\xi_n}{2\mu_0} \frac{\partial B_{v0}^2 (\mathbf{x}_0)}{\partial n}$$

where we have made use of $\xi_n \mathbf{n}_b$ being along ∇p_0.

A second boundary condition can be obtained from the fact that the electric field in a frame moving with the plasma velocity is zero $\mathbf{E}_1 + \mathbf{u}_1 \times \mathbf{B}_0 = 0$ and the tangential component must be continuous into the vacuum region, i.e.

$$\mathbf{n}_b \times (\mathbf{E}_{v1} + \mathbf{u}_1 \times \mathbf{B}_{v0}) = 0$$

This condition can be expressed as

$$\mathbf{n}_b \times \mathbf{E}_{v1} = (\mathbf{n}_b \cdot \mathbf{u}_1)\, \mathbf{B}_{v0} = u_n \mathbf{B}_{v0}$$

Introducing the vector potential for the perturbation in the vacuum region $\mathbf{A}$ the electric and magnetic fields are

$$\mathbf{E}_{v1} = -\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B}_1 = \nabla \times \mathbf{A}$$

which yields

$$\mathbf{n}_b \times \mathbf{A} = -\xi_n \mathbf{B}_{v0}$$

on the conducting wall the boundary condition for $\mathbf{A}$ is

$$\mathbf{n}_w \times \mathbf{A} = 0$$

i.e., the vector potential has only a component along the normal direction of the wall

3.6.2 Energy principle

We have derived an partial differential equation of the form

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}) = -\underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi}$$

where $\underline{\underline{\mathbf{K}}}$ is the differential operator

$$\begin{aligned} \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} &= -\nabla \left(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi} \right) - \frac{1}{\mu_0} \left[(\nabla \times \mathbf{Q}_\xi) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{Q}_\xi \right] \\ \text{with} & \quad \mathbf{Q}_\xi = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \end{aligned}$$

Assuming

$$\boldsymbol{\xi}(\mathbf{x},t) = \boldsymbol{\xi}(\mathbf{x}) \exp{(i\omega t)}$$

the PDE becomes an Eigenvalue equation for ω^2:

$$\omega^2 \rho_0 \boldsymbol{\xi} = \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi}$$

and stability depends on the sign of ω^2. In the MHD case the operator $\underline{\underline{\mathbf{K}}}$ is self-adjoint, i.e.,

$$\int_V \boldsymbol{\eta} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} d\mathbf{x} = \int_V \boldsymbol{\xi} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\eta} d\mathbf{x}$$

where the integrals are carried out over the plasma volume. The eigenvalues of a self-adjoint operator are always real that means either positive or negative. Explicitly the Eigenvalues are given by

$$\omega^2 = \frac{\int_V \boldsymbol{\xi} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} d\mathbf{x}}{\int_V \rho_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi} d\mathbf{x}}$$

For $\omega^2 > 0$ the values for ω are real and the solution is oscillating but not growing in time, i.e., the solution is stable. However if there are negative eigenvalues $\omega^2 < 0$ there are solutions which are growing exponentially in time and are therefore unstable!

In the following it is demonstrated that $\underline{\underline{\mathbf{K}}}$ is self-adjoint:

$$\begin{aligned} U_{\xi\eta} = \int_V \boldsymbol{\eta} \cdot \underline{\underline{\mathbf{K}}} \cdot \boldsymbol{\xi} d\mathbf{x} &= -\int_V \boldsymbol{\eta} \cdot \nabla \left(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi} \right) d\mathbf{x} \\ &\quad - \frac{1}{\mu_0} \int_V \boldsymbol{\eta} \cdot \left[(\nabla \times \mathbf{Q}_\xi) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \mathbf{Q}_\xi \right] d\mathbf{x} \end{aligned}$$

Identities to be used:

$$\begin{aligned} \nabla \cdot (\boldsymbol{\eta}\phi) &= \boldsymbol{\eta} \cdot \nabla \phi + \phi \nabla \cdot \boldsymbol{\eta} \\ \nabla \cdot [\mathbf{A} \times \mathbf{B}] &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\ \nabla \cdot [(\boldsymbol{\eta} \times \mathbf{B}_0) \times \mathbf{Q}_\xi] &= \boldsymbol{\eta} \cdot [(\nabla \times \mathbf{Q}_\xi) \times \mathbf{B}_0] + \mathbf{Q}_\xi \cdot [\nabla \times (\boldsymbol{\eta} \times \mathbf{B}_0)] \\ &= \boldsymbol{\eta} \cdot [(\nabla \times \mathbf{Q}_\xi) \times \mathbf{B}_0] + \mathbf{Q}_\xi \cdot \mathbf{Q}_\eta \\ \text{with} \quad & \mathbf{Q}_\eta = \nabla \times (\boldsymbol{\eta} \times \mathbf{B}_0) \end{aligned}$$

which yields

$$U_{\xi\eta} = \int_V (\nabla \cdot \boldsymbol{\eta}) \left(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi} \right) d\mathbf{x}$$

$$
\begin{aligned}
&+\frac{1}{\mu_0}\int_V \mathbf{Q}_\xi\cdot\mathbf{Q}_\eta d\mathbf{x} - \frac{1}{\mu_0}\int_V \boldsymbol{\eta}\cdot\left[(\nabla\times\mathbf{B}_0)\times\mathbf{Q}_\xi\right]d\mathbf{x}\\
&-\int_V \nabla\cdot\left[\boldsymbol{\eta}\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}\right)-\frac{1}{\mu_0}\left(\boldsymbol{\eta}\times\mathbf{B}_0\right)\times\mathbf{Q}_\xi\right]d\mathbf{x}\\
= &\int_V\left(\gamma p_0\nabla\cdot\boldsymbol{\eta}\nabla\cdot\boldsymbol{\xi}+\frac{1}{\mu_0}\mathbf{Q}_\xi\cdot\mathbf{Q}_\eta\right)d\mathbf{x}\\
&+\int_V\left(\boldsymbol{\xi}\cdot\nabla p_0\nabla\cdot\boldsymbol{\eta}-\frac{1}{\mu_0}\boldsymbol{\eta}\cdot\left[(\nabla\times\mathbf{B}_0)\times\mathbf{Q}_\xi\right]\right)d\mathbf{x}\\
&-\int_{S_V}\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}-\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{Q}_\xi\right)\eta_n ds
\end{aligned}
$$

where we have used Gauss theorem for the surface integrals with $B_n = 0$ on the plasma boundary. With the boundary conditions derived in the prior section

$$
\begin{aligned}
-\gamma p_0\nabla\cdot\boldsymbol{\xi}+\frac{\mathbf{B}_0(\mathbf{x}_0)\cdot\mathbf{B}_1(\mathbf{x}_0)}{\mu_0}+\xi_n\frac{\partial B_0^2(\mathbf{x}_0)}{\partial n} &= \frac{\mathbf{B}_{v0}(\mathbf{x}_0)\cdot\mathbf{B}_{v1}(\mathbf{x}_0)}{\mu_0}+\xi_n\frac{\partial B_{v0}^2(\mathbf{x}_0)}{\partial n}\\
\mathbf{B}_0 &= \frac{1}{\xi_n}\mathbf{n}_b\times\mathbf{A}_\xi\\
\mathbf{Q}_\xi=\mathbf{B}_1 &= \nabla\times\mathbf{A}_\xi
\end{aligned}
$$

For the surface terms we can use the boundary conditions

$$
\begin{aligned}
U_{\mathrm{S},\xi\eta} &= -\int_{S_V}\left(\boldsymbol{\xi}\cdot\nabla p_0+\gamma p_0\nabla\cdot\boldsymbol{\xi}-\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{Q}_\xi\right)\eta_n ds\\
&= -\int_{S_V}\left(\boldsymbol{\xi}\cdot\nabla p_0+\frac{\mathbf{B}_0\cdot\mathbf{Q}_\xi}{\mu_0}-\frac{\mathbf{B}_{v0}\cdot\mathbf{Q}_{v\xi}}{\mu_0}+\frac{\xi_n}{2\mu_0}\frac{\partial\left(B_0^2(\mathbf{x}_0)-B_{v0}^2(\mathbf{x}_0)\right)}{\partial n}\right)\eta_n ds\\
&\quad -\int_{S_V}\left(-\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{Q}_\xi\right)\eta_n ds\\
&= -\int_{S_V}\left(\xi_n\frac{\partial p_0}{\partial n}-\frac{\mathbf{B}_{v0}\cdot\mathbf{Q}_{v\xi}}{\mu_0}+\frac{\xi_n}{2\mu_0}\frac{\partial\left(B_0^2(\mathbf{x}_0)-B_{v0}^2(\mathbf{x}_0)\right)}{\partial n}\right)\eta_n ds
\end{aligned}
$$

and the field terms in the vacuum region can be treated as follows

$$
\begin{aligned}
\mu_0 U_{\mathbf{B},\xi\eta} &= \int_{S_V}\mathbf{B}_{v0}\cdot\mathbf{Q}_\xi\eta_n ds=\int_{S_V}\left(\mathbf{n}_b\times\mathbf{A}_\eta\right)\cdot\left(\nabla\times\mathbf{A}_\xi\right)ds\\
&= \int_{S_V}\left(\mathbf{A}_\eta\times\nabla\times\mathbf{A}_\xi\right)d\mathbf{s}=\int_{V_{vacuum}}\nabla\cdot\left(\mathbf{A}_\eta\times\nabla\times\mathbf{A}_\xi\right)d\mathbf{r}\\
&= \int_{V_{vacuum}}\left(\nabla\times\mathbf{A}_\eta\right)\cdot\left(\nabla\times\mathbf{A}_\xi\right)d\mathbf{r}
\end{aligned}
$$

such that the sum of the surface terms can be written as

$$U_{\mathrm{S},\xi\eta} = -\int_{S_V} \left(\frac{\partial p_0}{\partial n} + \frac{1}{2\mu_0} \frac{\partial \left(B_0^2 \left(\mathbf{x}_0 \right) - B_{v0}^2 \left(\mathbf{x}_0 \right) \right)}{\partial n} \right) \xi_n \eta_n ds + \frac{1}{\mu_0} \int_{V_{vacuum}} \left(\nabla \times \mathbf{A}_\eta \right) \cdot \left(\nabla \times \mathbf{A}_\xi \right) d\mathbf{r}$$

Here it is clear that the surface contributions are symmetric in ξ and η.

Finally we have to demonstrate the symmetry of the remaining non-symmetric terms

$$\widetilde{U}_{\xi\eta} = \int \boldsymbol{\xi} \cdot \nabla p_0 \nabla \cdot \boldsymbol{\eta} - \frac{1}{\mu_0} \boldsymbol{\eta} \cdot \left[\left(\nabla \times \mathbf{B}_0 \right) \times \mathbf{Q}_\xi \right] d\mathbf{x}$$

Consider decomposition of

$$\begin{aligned} \boldsymbol{\xi} = \boldsymbol{\xi}_{||} + \boldsymbol{\xi}_\perp \quad & \text{with} \quad \boldsymbol{\xi}_{||} = \alpha \mathbf{B}_0 \\ \boldsymbol{\eta} = \boldsymbol{\eta}_{||} + \boldsymbol{\eta}_\perp \quad & \text{with} \quad \boldsymbol{\eta}_{||} = \beta \mathbf{B}_0 \end{aligned}$$

Considering that $\boldsymbol{\xi}_{||} \cdot \nabla p_0 = \alpha \mathbf{B}_0 \cdot \nabla p_0 = 0$ and $\mathbf{Q}_{\xi_{||}} = \nabla \times \left(\boldsymbol{\xi}_{||} \times \mathbf{B}_0 \right) = 0$ the non-symmetric parts of the integral reduce to

$$\widetilde{U}_{\xi\eta} = \int \boldsymbol{\xi}_\perp \cdot \nabla p_0 \nabla \cdot \boldsymbol{\eta} - \frac{1}{\mu_0} \boldsymbol{\eta} \cdot \left[\left(\nabla \times \mathbf{B}_0 \right) \times \mathbf{Q}_{\xi_\perp} \right] d\mathbf{x}$$

The contributions from $\boldsymbol{\eta}_{||} = \beta \mathbf{B}_0$ can also be symmetrized. The algebra is somewhat more tedious and we note that the integrand for the $\boldsymbol{\eta}_{||}$ may be reduced to a form $\nabla \cdot (\boldsymbol{\xi} \cdot \nabla p_0 \boldsymbol{\eta})$. This give a contribution of $\boldsymbol{\eta}_{||,boundary} \propto B_{0n} = 0$ but for typical boundary conditions it is assumed that the normal magnetic field is 0 (i.e. the magnetic field is aligned with the boundary).

To show the symmetry of the perpendicular components it is necessary to decompose the perpendicular displacement into components along the equilibrium current and along the equilibrium pressure gradient:

$$\begin{aligned} \boldsymbol{\xi}_\perp &= \xi_1 \mu_0 \mathbf{j}_0 + \xi_2 \mathbf{e} \\ \boldsymbol{\eta}_\perp &= \eta_1 \mu_0 \mathbf{j}_0 + \eta_2 \mathbf{e} \\ \text{with} \qquad & \mathbf{e} = \frac{\nabla p_0}{|\nabla p_0|} \end{aligned}$$

With these definitions consider the term

$$\begin{aligned} -\boldsymbol{\eta}_\perp \cdot \left[\mathbf{j}_0 \times \mathbf{Q}_{\xi_\perp} \right] &= -\eta_2 \mathbf{e} \cdot \left[\mathbf{j}_0 \times \left(\nabla \times \left(\left(\xi_1 \mu_0 \mathbf{j}_0 + \xi_2 \mathbf{e} \right) \times \mathbf{B}_0 \right) \right) \right] \\ &= -\eta_2 \mathbf{e} \cdot \left[\mathbf{j}_0 \times \left(\nabla \times \left(\xi_1 \mu_0 \nabla p_0 + \xi_2 \mathbf{e} \times \mathbf{B}_0 \right) \right) \right] \\ &= -\eta_2 \mathbf{e} \cdot \left[\mathbf{j}_0 \times \left(\mu_0 \nabla \xi_1 \times \nabla p_0 + \xi_1 \mu_0 \nabla \times \nabla p_0 \right) \right] \\ & \quad -\eta_2 \mathbf{e} \cdot \left[\mathbf{j}_0 \times \left(\nabla \xi_2 \times \left(\mathbf{e} \times \mathbf{B}_0 \right) + \xi_2 \nabla \times \left(\mathbf{e} \times \mathbf{B}_0 \right) \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= -\mu_0\eta_2\mathbf{e}\cdot\left[(\mathbf{j}_0\cdot\nabla p_0)\nabla\xi_1-(\mathbf{j}_0\cdot\nabla\xi_1)\nabla p_0\right] \\
&\quad -\eta_2\mathbf{e}\cdot\left[\mathbf{j}_0\cdot(\mathbf{e}\times\mathbf{B}_0)\nabla\xi_2-(\mathbf{j}_0\cdot\nabla\xi_2)(\mathbf{e}\times\mathbf{B}_0)+\xi_2\mathbf{j}_0\times\nabla\times(\mathbf{e}\times\mathbf{B}_0)\right] \\
&= \mu_0\eta_2(\mathbf{e}\cdot\nabla p_0)(\mathbf{j}_0\cdot\nabla\xi_1) \\
&\quad \eta_2(\mathbf{e}\cdot\nabla p_0)(\mathbf{e}\cdot\nabla\xi_2)-\eta_2\xi_2\mathbf{e}\cdot\mathbf{j}_0\times\nabla\times(\mathbf{e}\times\mathbf{B}_0) \\
&= \boldsymbol{\eta}_\perp\cdot\nabla p_0\nabla\cdot\boldsymbol{\xi}_\perp-\eta_2\xi_2\mathbf{e}\cdot\mathbf{j}_0\times\nabla\times(\mathbf{e}\times\mathbf{B}_0)
\end{aligned}
$$

Thus the sum of the non-symmetric integrands becomes

$$
\begin{aligned}
\tilde{u}_{\xi\eta} &= \boldsymbol{\xi}\cdot\nabla p_0\nabla\cdot\boldsymbol{\eta}-\frac{1}{\mu_0}\boldsymbol{\eta}\cdot\left[(\nabla\times\mathbf{B}_0)\times\mathbf{Q}_\xi\right] \\
&= \boldsymbol{\xi}\cdot\nabla p_0\nabla\cdot\boldsymbol{\eta}+\boldsymbol{\eta}\cdot\nabla p_0\nabla\cdot\boldsymbol{\xi}-\eta_2\xi_2\mathbf{e}\cdot\mathbf{j}_0\times\nabla\times(\mathbf{e}\times\mathbf{B}_0)
\end{aligned}
$$

Obviously this form is symmetric in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and therefore the operator $\underline{\underline{\mathbf{K}}}$ is self-adjoint implying that all Eigenvalues are real.

With this property we can now formulate a Lagrangian function with the kinetic energy

$$T=\frac{1}{2}\int_V d\mathbf{r}\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\right)^2$$

and the generalized potential energy

$$
\begin{aligned}
U &= \frac{1}{2}\int_V\left[\gamma p_0(\nabla\cdot\boldsymbol{\xi})^2+\frac{1}{\mu_0}(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0))^2\right. \\
&\qquad \left.+\boldsymbol{\xi}\cdot\nabla p_0\nabla\cdot\boldsymbol{\xi}-\frac{1}{\mu_0}(\boldsymbol{\xi}\times(\nabla\times\mathbf{B}_0))\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\right]d\mathbf{x} \\
&\quad -\int_{S_V}\left(\frac{\partial p_0}{\partial n}+\frac{1}{2\mu_0}\frac{\partial\left(B_0^2(\mathbf{x}_0)-B_{v0}^2(\mathbf{x}_0)\right)}{\partial n}\right)\xi_n^2 ds+\frac{1}{\mu_0}\int_{V_{vacuum}}(\nabla\times\mathbf{A})^2 d\mathbf{r}
\end{aligned}
$$

Here the operator $\underline{\underline{\mathbf{K}}}$ (and integration of volume play the role of a potential and the potential energy is the above expression for the small displacement ξ. This is analogous to the formulation in classical mechanics. Considering a kinetic energy T and a potential U one obtains the Lagrangian $L(q,\dot{q},t)=T-U$ for the generalized coordinate q and velocities $\dot{q}$. The Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}-\frac{\partial L}{\partial q}=0$$

An equilibrium point i.e., a point where the acceleration is zero is given by

$$\dot{P}=\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}=0$$

which implies

$$\frac{\partial L}{\partial q} = - \left.\frac{\partial U}{\partial q}\right|_{q=q_0} = 0$$

We now define the coordinates as $\xi = q - q_0$ and assume the ξ to be small displacements from the equilibrium point. The potential can be expanded in terms of the displacement at the equilibrium point i.e.

$$U(\xi) = U(q_0) + \frac{\partial U(q_0)}{\partial q}\xi + \frac{1}{2}\frac{\partial^2 U(q_0)}{\partial q^2}\xi^2 + ..$$

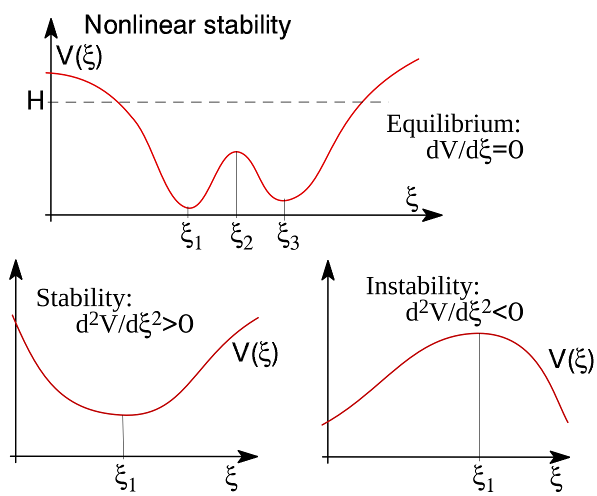
Substituting ξ and in the Lagrangian with the kinetic energy $\frac{m}{2}\dot{\xi}^2$ yields the equation of motion

$$m\ddot{\xi} + \frac{\partial^2 U(q_0)}{\partial q^2}\xi = 0$$

with the solutions

$$\xi = \exp(\pm i\omega t) \quad \text{with} \quad \omega^2 = \frac{1}{m}\frac{\partial^2 U(q_0)}{\partial q^2} \quad for \quad \frac{\partial^2 U(q_0)}{\partial q^2} > 0$$
$$\xi = \exp(\pm \omega t) \quad \text{with} \quad \omega^2 = -\frac{1}{m}\frac{\partial^2 U(q_0)}{\partial q^2} \quad for \quad \frac{\partial^2 U(q_0)}{\partial q^2} < 0$$

Thus the solution is oscillatory and therefore stable if the potential has a local minimum and the solution is exponentially growing if the potential has a local maximum.

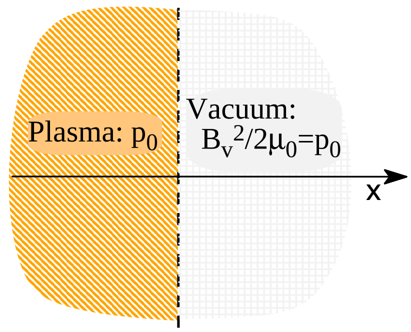


The energy principle for the MHD equations has to be interpreted in a similar way. An equilibrium is stable if the potential energy is positive for all small perturbations ξ and it is unstable if there are perturbations for which the potential energy can become negative (Note that the equilibrium value of the potential is 0.

3.6.3 Applications of the energy principle

Stability of a plane plasma magnetic field interface

Consider the plane boundary at $x = 0$ between a homogeneous plasma (for $x < 0$) with constant pressure and density and a vacuum region at $x > 0$. Inside the plasm region the magnetic field is assumed to be 0. The plasma pressure is balanced by the magnetic field in the vacuum region. In this case the potential energy becomes

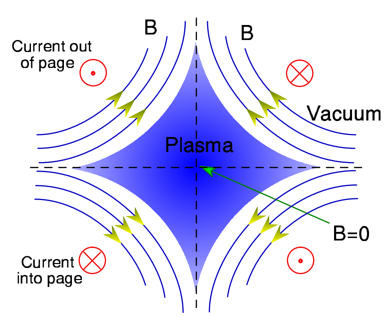


$$U = \frac{1}{2}\int_V \gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 d\mathbf{r} + \frac{1}{2\mu_0}\int_{V_{vacuum}} \mathbf{B}_{v1}^2 d\mathbf{r} + \frac{1}{2\mu_0}\int_{x=0} \frac{\partial B_{v0}^2\left(\mathbf{x}_0\right)}{\partial n}\xi_n^2 ds$$

Properties:

- the $\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2$ term requires compressible perturbations
- $\gamma p_0 \left(\nabla\cdot\boldsymbol{\xi}\right)^2 > 0$ always => stabilizing contribution for compressible perturbations!
- incompressible perturbations have a lower stability threshold
- the $\mathbf{B}_{v1}^2$ term is due to the perturbation of the vacuum field
- $\mathbf{B}_{v1}^2 > 0$ => always stabilizing
- $\partial B_{v0}^2\left(\mathbf{x}_0\right)/\partial n < 0$ can cause instability.
- all simple boundaries with $\partial B_{v0}^2\left(\mathbf{x}_0\right)/\partial n > 0$ are stable.

Magnetic cusp: This configuration is generated by two coils one in the upper half and one in the lower half of the system and the configuration is azimuthally symmetric with respect to the the vertical axis. Locally the plasma magnetic interface can be treated as a plane surface. On the larger system scale the magnetic field is curved into the plasma (as shown) such that $\partial B_{v0}^2\left(\mathbf{x}_0\right)/\partial n > 0$. The cause for this curvature is that the magnetic field increases closer to the coils. Therefore this configuration is always stable.



A similar result is obtained for the interchange instability. This is an instability where a flux tube from the plasma region is bulging out into the vacuum region and vice versa vacuum flux tube enters the plasma region. This instability also requires a magnetic field line curvature opposite to that shown for the cusp configuration.

The plasma can be unstable if $\partial B_{v0}^2\left(\mathbf{x}_0\right)/\partial n < 0$. Consider the same simple plane plasma vacuum boundary as before with $\mathbf{n}_b = \mathbf{e}_x$, and $\mathbf{B}_{v0} = B_{v0}\mathbf{e}_z$.

Consider a test function for the displacement and the vacuum perturbation field of the form

$$\begin{aligned} \boldsymbol{\xi} &= \widetilde{\boldsymbol{\xi}} \exp i\left(k_x x + k_y y + k_z z\right) \\ \mathbf{B} &= \widetilde{\mathbf{b}} \exp i\left(k_x x + k_y y + k_z z\right) \end{aligned}$$

where we dropped the indices v because we discuss exclusively the perturbed field in the vacuum region. The normal component of the perturbation field is given by

$$B_{1x} = \nabla \times \left(\boldsymbol{\xi} \times \mathbf{B}_0\right)\big|_x = ik_z\xi_x B_{v0}$$

The perturbed field in the vacuum region satisfies

$$\begin{aligned} \nabla \times \nabla \times \mathbf{B}_1 &= \mathbf{k} \times \mathbf{k} \times \mathbf{B}_1 = 0 \\ \nabla \cdot \mathbf{B}_1 &= \mathbf{k} \cdot \mathbf{B}_1 = 0 \end{aligned}$$

which yields

$$\mathbf{k}^2 = 0$$

or $k_x = i\sqrt{k_y^2 + k_z^2}$. Since the current density in the vacuum region is zero we can use

$$j_{1x} = \frac{i}{\mu_0}(k_y B_{1z} - k_z B_{1y}) = 0$$

or $B_{1z} = \left(k_z/k_y\right) B_{1y}$ in $\nabla \cdot \mathbf{B}_1 = 0$ to obtain the y and z components of the perturbed field

$$\begin{aligned} B_{1y} &= -\frac{k_x k_y}{k_y^2 + k_z^2} B_{1x} \\ B_{1z} &= -\frac{k_x k_y}{k_y^2 + k_z^2} B_{1x} \end{aligned}$$

Thus we obtain

$$\begin{aligned} \mathbf{B}_1^2 &= \left(\left|B_{1x}\right|^2 + \left|B_{1y}\right|^2 + \left|B_{1z}\right|^2\right) \\ &= \left(1 + \frac{\left|k_x^2\right|}{k_y^2 + k_z^2}\right)\left|B_{1x}\right|^2 = 2\left|B_{1x}\right|^2 = 2\widetilde{b}_x^2 \exp\left(-2\sqrt{k_y^2 + k_z^2}x\right) \end{aligned}$$

Since the Magnetic field is exponetially decreasing the perturbation ξ_x has to take the also form

$$\xi_x = \tilde{\xi}_x \exp\left(-\sqrt{k_y^2 + k_z^2}x\right) \exp i\left(k_y y + k_z z\right)$$

such that

$$\mathbf{B}_1^2 = 2k_z^2 \tilde{\xi}_x^2 B_{v0}^2 \exp\left(-2\sqrt{k_y^2 + k_z^2}x\right)$$

and integration over the entire vacuum region from $x = 0$ to ∞ yields

$$\int_{V_{vacuum}} \mathbf{B}_{v1}^2 d\mathbf{r} = \int_{x=0} \frac{k_z^2 \tilde{\xi}_x^2 B_{v0}^2}{\sqrt{k_y^2 + k_z^2}} ds$$

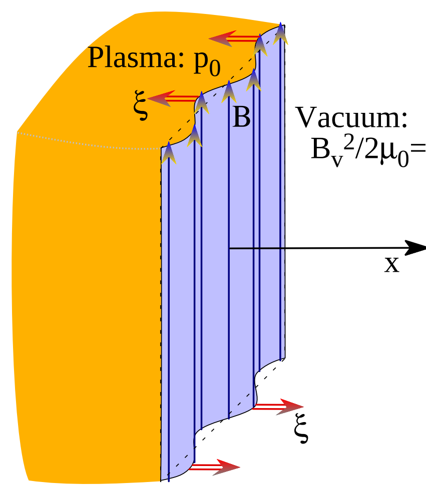
Assuming the most unstable, i.e., incompressible $\nabla \cdot \boldsymbol{\xi}$ perturbations we obtain for the potential energy

$$U = \frac{1}{2\mu_0} \int_{x=0} \left(\frac{2k_z^2}{\sqrt{k_y^2 + k_z^2}} B_{v0}^2 + \frac{\partial B_{v0}^2\left(\mathbf{x}_0\right)}{\partial n} \right) \tilde{\xi}_x^2 ds$$

This result implies stability for

$$\frac{2k_z^2}{\sqrt{k_y^2 + k_z^2}} > -\frac{1}{B_{v0}^2} \frac{\partial B_{v0}^2\left(\mathbf{x}_0\right)}{\partial n}$$

Therefore configurations with $\partial B_{v0}^2\left(\mathbf{x}_0\right)/\partial n < 0$ are unstable. Instability occurs preferably for small values of k_z and the most unstable modes have $k_z = 0$. The perturbation moves magnetic field into a region where the plasma pressure is greater than the magnetic pressure



Stability of the Theta pinch

In cylindrical geometry with a magnetic field in the θ direction and only radial dependence the force ballance equation can be written as

$$\frac{dp_0}{dr} = -\frac{B_0}{\mu_0 r} \frac{d}{dr}\left(rB_0\right)$$

where B_0 is the θ component of the magnetic field. Note that this equation has straightforward solutions. We can for instance specify B_0 then integrate the equation to obtain p_0 or vice versa specify p_0

to integrate and obtain B_0. This is similar to the case of straight field lines in cartesian coordinates, however, for the θ pinch field lines are curved and $\mathbf{B} \cdot \nabla \mathbf{B} \neq 0$. A straightfarward solution for this case can be found by assuming the current density (along z) to be constant up to a radius a for which the pressure drops to 0. for $r > a$ the current density has to be 0 otherwise the pressure would be reuired negative which is unphysical.

Exercise: Assume a constant current j_0 along the z direction in a cylindrical coordinate system. Compute the magnetic field $B_\theta(r)$ and integrate the force balance equation to obtain $p(r)$. The pressure at $r = 0$ is p_0. Determine the critical radius for which the pressure decreases to 0.

The resulting configuration is a column or cylinder in wich the current is flowing along the cylinder axis in the z direction and the magnetic field is in the azimuthal θ direction. An equilibrium configuration which has somesimilarity with the θ pinch is the Z pinch in which the magnetic field is along the z direction and the perturbations of this configuration have the form

$$\boldsymbol{\xi}(r,\theta,z) = \boldsymbol{\xi}_0(r,z)\exp(im\theta)$$

where m is the wave number in the θ direction.

In cylindrical coordinates the perturbations contributing to the potential energy are

$$\begin{aligned}
\nabla \cdot \boldsymbol{\xi} &= \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \\
\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) &= -B_0\left[\frac{1}{r}\frac{\partial \xi_r}{\partial \theta}\mathbf{e}_r - \left(\frac{1}{B_0}\frac{\partial}{\partial r}(B_0\xi_r) + \frac{\partial \xi_z}{\partial z}\right)\mathbf{e}_\theta + \frac{1}{r}\frac{\partial \xi_z}{\partial \theta}\mathbf{e}_z\right] \\
\boldsymbol{\xi} \times (\nabla \times \mathbf{B}_0) &= \frac{1}{r}\frac{\partial}{\partial r}(rB_0)(\xi_\theta \mathbf{e}_r - \xi_r \mathbf{e}_\theta)
\end{aligned}$$

We will assume that the boundary is a conducting wall at which the displacement is 0. With these relations the potential becomes

$$\begin{aligned}
U = \frac{1}{2}\int_V &\left[\gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + \frac{\partial p}{\partial r}\xi_r \nabla \cdot \boldsymbol{\xi}\right. \\
&+ \frac{B_0^2}{\mu_0}\left(\left(\frac{1}{B_0}\frac{\partial}{\partial r}(B_0\xi_r) + \frac{\partial \xi_z}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right) \\
&\left.+ \frac{B_0}{\mu_0 r}\frac{\partial}{\partial r}(rB_0)\left(\frac{\xi_\theta}{r}\frac{\partial \xi_r}{\partial \theta} + \xi_r\left(\frac{1}{B_0}\frac{\partial}{\partial r}(B_0\xi_r) + \frac{\partial \xi_z}{\partial z}\right)\right)\right] d\mathbf{x}
\end{aligned}$$

Azimuthally symmetric perturbations

In this case m is zero and all θ derivatives are 0. To evaluate the stability we bring the potential into the quadratic form

$$U = \frac{1}{2}\int_V \left[a_{11}\left(\nabla\cdot\boldsymbol{\xi}\right)^2 + 2a_{12}\frac{\xi_r}{r}\nabla\cdot\boldsymbol{\xi} + a_{22}\frac{\xi_r^2}{r^2}\right]d\mathbf{x}$$

where the coefficients are

$$a_{11} = \gamma\beta + 2 \qquad \beta = \frac{2\mu_0 p_0}{B_0^2}$$

$$a_{12} = \frac{d\ln B_0}{d\ln r} + \frac{\beta}{2}\frac{d\ln p_0}{d\ln r} - 1$$

$$a_{22} = \left(\frac{d\ln B_0}{d\ln r} - 1\right)a_{12}$$

and stability requires

$$a_{11}a_{22} - a_{12}^2 > 0$$

Using the force balance equation one can simplify the stability equation to

$$-\frac{d\ln p_0}{d\ln r} = -\frac{r}{p_0}\frac{dp_0}{dr} < \frac{4\gamma}{2+\beta\gamma}$$

For the θ pinch the plasma pressure decreases with radial distance from the pinch axis such that at large distances $\beta \ll 1$. Here the rhs assumes a maximum such that stability requires

$$\frac{dp}{p} > -2\gamma\frac{dr}{r}$$

or

$$p >\sim r^{-2\gamma}$$

However to avoid plasma contact with the wall the density and pressure must be close to 0. Thus the θ pinch is unstable with respect to symmetric perturbations which lead to a periodic pinching of the plasma column.

Azimuthally asymmetric perturbations

In this case the θ dependence is nonzero and on can assume a displacement of the following form

$$\begin{aligned} \xi_r &= \xi_r(r,z)\sin m\theta \\ \xi_\theta &= \xi_\theta(r,z)\cos m\theta \\ \xi_z &= \xi_z(r,z)\sin m\theta \end{aligned}$$

To simplify the analysis one usually assumes $\nabla \cdot \boldsymbol{\xi} = 0$. With this assumption the only change in the potential energy is that the term $\gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2$ is replaced by

$$\frac{B_0^2}{\mu_0}\frac{m^2}{r^2}\left(\xi_r^2 + \xi_z^2\right)$$

Stability is obtained for

$$-\frac{d\ln p_0}{d\ln r} = -\frac{r}{p_0}\frac{dp_0}{dr} < \frac{m^2}{\beta}$$

One can compare the stability limit for non-symmetric perturbation with that for symmetric perturbations. For small values of β the symmetric condition is always more unstable. For values with $\beta > 1$ the $m = 1$ or even $m = 2$ mode can be unstable even though the symmetric condition may imply stability.

The $m = 1$ mode is called kink instability (or corkscrew instability because of the form of the bending of the plasma collumn.

Stability and magnetic flux tube volume

In a plasma with a small plasma β the evolution is strongly dominated by the magnetic field such that magnetic flux tubes carry much of the energy. we had defined the magnetic flux tube volume as

$$V = \int \frac{dl}{B}$$

The thermal energy contained in a flux tube is pU and since the flux tube volume changes during convection the energy associated with the flux tube changes. Using the pressure equation we have

$$p + \Delta p = p - \gamma p \nabla \cdot \mathbf{u} = p + \gamma p \frac{\Delta V}{V}$$

Note that $dp/dt = -\gamma p \nabla \cdot \mathbf{u}$ and $\nabla \cdot \mathbf{u} = -\frac{1}{n}\frac{dn}{dt} = -\frac{1}{V}\frac{dV}{dt}$ which is obtained from

$$\frac{dS}{dt} = \frac{dpV^\gamma}{dt} = 0$$

The pressure change in the vicinity of the flux tube is

$$p(V + \Delta V) = p + \frac{dp}{dV}\Delta V$$

Considering a small displacement from the equilibrium: If the pressure change in the flux tube is greater than the pressure in the surrounding tubes then lower energy state is the equilibrium and energy has to be brought into the system to achieve the change. In other words the configuration is stable for

$$\gamma p \frac{\Delta V}{V} + \frac{dp}{dV}\Delta V > 0$$

or

$$-\frac{V}{p}\frac{dp}{dV} < \gamma$$

or

$$-\frac{d\ln p}{d\ln V} < \gamma$$

Note that adiabatic convection yields

$$-\frac{d\ln p}{d\ln V} = \gamma$$

For the sheet pinch this yields

$$-\frac{d\ln p}{d\ln r} < 2\gamma$$

which is the small β approximation of our prior result.

Note, however that this is somewhat heuristic or intuitive and lacks the rigour or the prior discussion.

3.7 Magnetohydrodynamic Waves

In the following we want to examine typical waves in the single fluid plasma. The treatment is similar to for instance sound waves but clearly one expects that the magnetofluid has more degrees of freedom and wave propagation should depend on teh orientation of the magnetic field, i.e. it is not isotropic as in the case of sound waves. Note that in the cause of waves in gravitationally stratified atmosphere wave propagation is also nonistropic and gravity allows for a new wave mode the so-called gravity wave.

In the case of MHD we start from the full set of linearized ideal MHD equations.

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) \\
\frac{\partial p}{\partial t} &= -\mathbf{u} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{u}
\end{aligned}
$$

where we have already combined the induction equation and Ohm's law. Now we linearize the equation as in the previous section, however, we do not introduce a small displace but rather keep $\mathbf{u}_1$ as a variable. Specifically the MHD variables are expressed as

$$
\begin{aligned}
\rho &= \rho_0 + \rho_1 \\
\mathbf{u} &= \mathbf{u}_1 \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1 \\
p &= p_0 + p_1
\end{aligned}
$$

For simplicity we assume all equilibrium quantities to be constant, i.e., we do not consider and inhomogeneous plasma. Substitution of the perturbations into the MHD equations yields similar to the small displacement treatment

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\rho_0 \nabla \cdot \mathbf{u}_1 \\
\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} &= -\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \\
\frac{\partial p_1}{\partial t} &= -\gamma p_0 \nabla \cdot \mathbf{u}_1
\end{aligned}
$$

Taking the second derivative in time of the momentum equation yields

$$
\begin{aligned}
\rho_0 \frac{\partial^2 \mathbf{u}_1}{\partial t^2} &= -\nabla \frac{\partial p_1}{\partial t} + \frac{1}{\mu_0} \left(\nabla \times \frac{\partial \mathbf{B}_1}{\partial t} \right) \times \mathbf{B}_0 \\
&= \gamma p_0 \nabla (\nabla \cdot \mathbf{u}_1) + \frac{B_0^2}{\mu_0} (\nabla \times (\nabla \times (\mathbf{u}_1 \times \mathbf{e}_B))) \times \mathbf{e}_B
\end{aligned}
$$

where we used $\mathbf{B}_0 = B_0 \mathbf{e}_B$. Now dividing by ρ_0 and with the definitions for the speed of sound c_s and the Alfven speed v_A

$$
\begin{aligned}
c_s^2 &= \frac{\gamma p_0}{\rho} \\
v_A^2 &= \frac{B_0^2}{\mu_0 \rho_0}
\end{aligned}
$$

we obtain

$$
\frac{\partial^2 \mathbf{u}_1}{\partial t^2} = c_s^2 \nabla \left(\nabla \cdot \mathbf{u}_1\right) + v_A^2 \left(\nabla \times \left(\nabla \times \left(\mathbf{u}_1 \times \mathbf{e}_B\right)\right)\right) \times \mathbf{e}_B
$$

We will now try to find the solution as plane waves by assuming

$$
\mathbf{u}_1 = \hat{\mathbf{u}}_1 \exp\left(i\mathbf{k} \cdot \mathbf{x} - i\omega t\right)
$$

which leads us to

$$
\omega^2 \mathbf{u}_1 = c_s^2 \mathbf{k} \left(\mathbf{k} \cdot \mathbf{u}_1\right) - v_A^2 \mathbf{e}_B \times \left(\mathbf{k} \times \left(\mathbf{k} \times \left(\mathbf{u}_1 \times \mathbf{e}_B\right)\right)\right)
$$

We now choose a coordinate system where the z axis points along the equilibrium (constant) magnetic field and the $\mathbf{k}$ vector is in the x, z plane and given by

$$
\mathbf{k} = k_\perp \mathbf{e}_x + k_{||} \mathbf{e}_z
$$

This leads to the following three components for the equation:

$$
\begin{aligned}
\left(\omega^2 - v_A^2 k_{||}^2 - \left(c_s^2 + v_A^2\right) k_\perp^2\right) u_{1x} - c_s^2 k_{||} k_\perp u_{1z} &= 0 \\
\left(\omega^2 - v_A^2 k_{||}^2\right) u_{1y} &= 0 \\
\left(\omega^2 - c_s^2 k_{||}^2\right) u_{1z} - c_s^2 k_{||} k_\perp u_{1x} &= 0
\end{aligned}
$$

or in matrix form $\underline{\underline{\mathbf{W}}} \cdot \mathbf{u}_1 = 0$

$$
\left[\begin{array}{ccc}
\omega^2 - v_A^2 k_{||}^2 - \left(c_s^2 + v_A^2\right) k_\perp^2 & 0 & -c_s^2 k_{||} k_\perp \\
0 & \omega^2 - v_A^2 k_{||}^2 & 0 \\
-c_s^2 k_{||} k_\perp & 0 & \omega^2 - c_s^2 k_{||}^2
\end{array}\right]
\left(\begin{array}{c} u_{1x} \\ u_{1y} \\ u_{1z} \end{array}\right) = 0
$$

Exercise: Derive the above matrix.

These equations are linear independent and provide nontrivial solutions if the determinant is equal to zero.

a) Alfvén wave:

The first nontrivial solution is determined by the y component of the above equations. This component separates from the other two because u_{1y} does not appear in the first and third equation. In the determinant $\omega^2 - v_A^2 k_{||}^2$ appears a a common factor. The solution is

$$\omega^2 = v_A^2 k_{||}^2 = \frac{k_{||}^2 B_0^2}{\mu_0 \rho_0}$$

or

$$\omega = \pm \frac{\mathbf{k} \cdot \mathbf{B}_0}{(\mu_0 \rho_0)^{1/2}} = \pm k v_A \cos\theta$$

The group velocity $\mathbf{v}_g = d\omega/d\mathbf{k}$ is

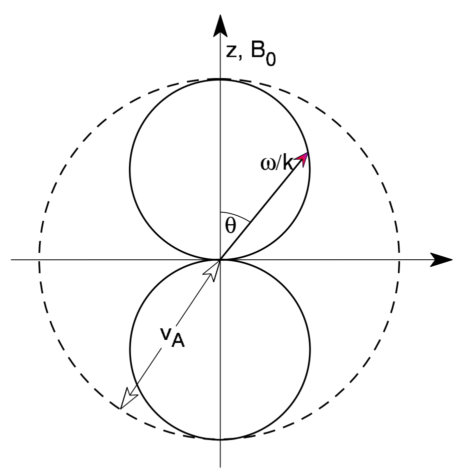
$$\mathbf{v}_g = \pm \frac{\mathbf{B}_0}{(\mu_0 \rho_0)^{1/2}} = \pm v_A \mathbf{e}_B$$

which is independent of $\mathbf{k}$ and always along the magnetic field. The phase velocity is

$$v_{ph} = \frac{\omega}{k} = \pm v_A \cos\theta$$

where theta is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. This wave is the Alf\ven wave.

A common representation of phase and group velocities is the Clemmov-Mullaly-Allis diagram which represents phase and group velocities in a polar plot where the radius vector represents the magnitude of the velocity and the angle is the propagation angle theta. Note that the direction of the group velocity for Alfv\'en waves (which is the direction in which energy and momentum is carried) is always along the magnetic field! Note also that the case for $\theta = 90°$ is singular in that wave does not propagate and the points with $\theta = 90°$ should be excluded from the group velocity plot.

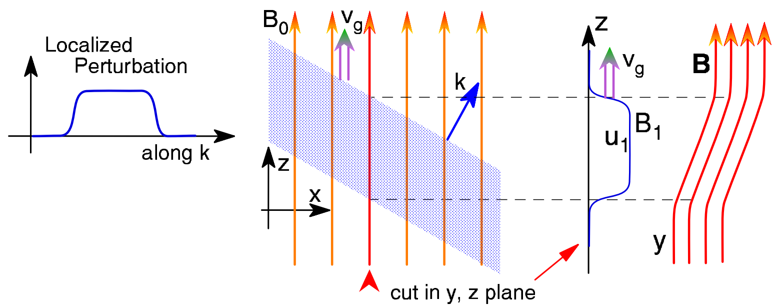


Further properties of Alfvén waves:

- Velocity perturbation $\mathbf{u}_1 \perp \mathbf{B}_0$
- Wave is incompressible: $\nabla \cdot \mathbf{u}_1 = i\mathbf{k} \cdot \mathbf{u}_1 = 0$ or $\mathbf{u}_1 \perp \mathbf{k}$

- Magnetic field perturbation: $\mathbf{B}_1 = -\mathbf{k} \times (\mathbf{u}_1 \times \mathbf{B}_0) = -k_{\|}\mathbf{u}_1$, i.e., the magnetic field perturbation is aligned with the velocity perturbation (and both are perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$.
- Current density $\mathbf{j}_1 \sim \mathbf{k} \times \mathbf{B}_0$, i.e., the current perturbation is perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$.
- Alfvén wave also exist without steepening with nonlinear amplitudes.

Considering a localized perturbation for $\mathbf{B}_1$ along the $\mathbf{k}$ vector propagation of an Alfven wave and the deformation of the magnetic field is presented in the following figure.



The Alfvén wave is of large importance in many plasma systems. It is very effective in carrying energy and momentum along the magnetic field. Among many space plasma applications the Alfvén wave is particulalry important for the coupling between the magnetosphere and the ionosphere. Here perturbations (convection) from the magnetosphere are carried into the ionosphere and depending on ionospheric conditions cause convection in the high latitude ionosphere.

A coomon diagnostic to identify Alfvén waves in space physics is the use of the so-called the Walen relation.

$$\Delta \mathbf{u} = \pm \Delta \mathbf{v}_A = \pm \Delta \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}}$$

This relation is valid for linear and nonlinear Alfvén waves, and can also be used to identify the corresponding (nonlinear) discontinuities.

b) **Fast and slow mode:**

The other nontrivial solution is given by $\det \underline{\underline{\mathbf{W}}} = 0$ or

$$\left(\omega^2 - v_A^2 k_{\|}^2 - \left(c_s^2 + v_A^2\right) k_{\perp}^2\right)\left(\omega^2 - c_s^2 k_{\|}^2\right) - c_s^4 k_{\|}^2 k_{\perp}^2 = 0$$

This is the dispersion relation for 4 additional solutions which are given by the roots of the equation. Since the equation is double quadratic, i.e., depends only on ω^2 there are two different solutions for ω^2. Abbreviating $c_f^2 = c_s^2 + v_A^2$ (where the f stands for fast mode - the reason for this will be clear in a moment) one can re-write the equation as

$$\omega^4 - c_f^2 k^2 \omega^2 + v_A^2 c_s^2 k_{\|}^2 k^2 = 0 \tag{3.41}$$

with the solutions

$$\begin{aligned}\omega^2 &= \frac{k^2}{2}\left\{c_f^2 \pm \left[c_f^4 - 4v_A^2 c_s^2 \frac{k_{\|}^2}{k^2}\right]^{1/2}\right\} \\ &= \frac{k^2}{2}\left\{v_A^2 + c_s^2 \pm \left[\left(v_A^2 + c_s^2\right)^2 - 4v_A^2 c_s^2 \cos^2\theta\right]^{1/2}\right\}\end{aligned}$$

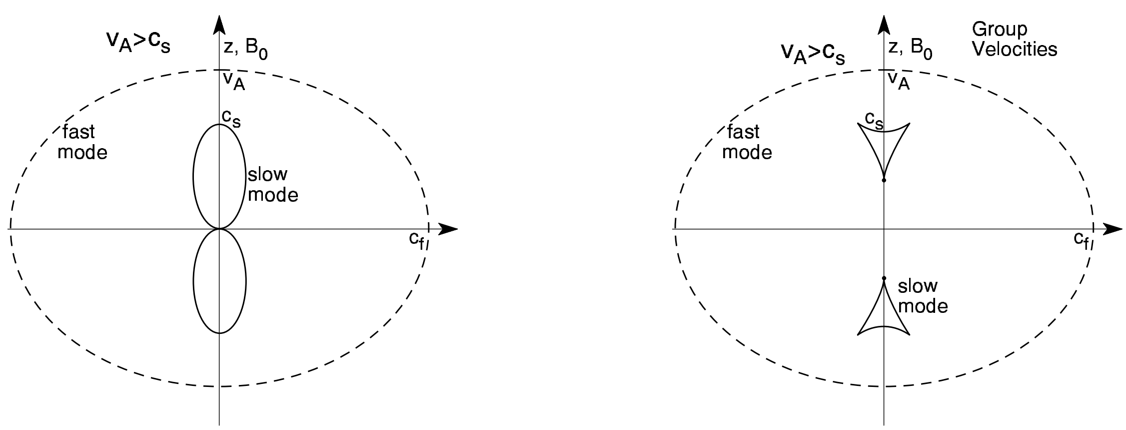
Here the solution with the $+$ sign is called the fast mode and the solution with the $-$ sign the slow mode.

Fast wave:

This is the MHD wave with the fastest propagation speed, i.e., information in an MHD plasma cannot travel faster thant this velocity.

- Phase speed along the magnetic field: $\omega^2/k^2 = \max\left[v_A^2,\ c_s^2\right]$
- Phase speed perpendicular to the magnetic field $\omega^2/k^2 = c_f^2 \equiv v_A^2 + c_s^2$
- Limit for $c_s^2 \ll v_A^2 : \omega^2 - v_A^2 k^2 = 0$ => the fast mode becomes the so-called compressional Afvén wave. Note that the compressional Alfven waves has a group velocity equal to its phase velocity (different from the regular or shear Alfven wave).
- Limit for $v_A^2 \ll c_s^2 : \omega^2 - c_s^2 k^2 = 0$ => the fast mode becomes a sound wave.

The most prominant example of a fast wave in space physics is the Earth's bow shock. This shock forms as a result of the solar wind velocity which is faster than the fast mode speed. The fast wave is also important in terms of transporting energy perpendicular to the magnetic field. The slow mode group velocity is as for the Alfven wave along the magnetic feild and it is comparably small. Thus the fast wave is the only MHD wave able to carry energy perpendicular to the magnetic field.



Slow wave:

The slow wave is the third of the basic MHD waves. The phase velocity is always smaller than or equal to $\min[v_A^2, c_s^2]$. Basic properties:

Phase speed along the magnetic field: $\omega^2/k^2 = \min[v_A^2, c_s^2]$.

Phase speed perpendicular to the magnetic field $\omega^2/k^2 = 0$.

In the limits of $c_s^2 \ll v_A^2$ and $v_A^2 \ll c_s^2$ the slow wave disappears.

Applications of the slow wave are not as eye catching than those for the other MHD wave but the wave is important to explain plasma structure in the region between the bow shock and the magnetosphere, and well know for applications such as magnetic reconnection.

Concluding remarks:

There is an additional wave which in MHD which is the sound wave. This is obtained by using $\mathbf{k}_\perp = 0$ and $\mathbf{u}_\perp = 0$ such that the velocity perturbation is along the magnetic field and thus the magnetic field perturbation is 0. However, it is singular in that it exists only for propagation exactly along the magnetic field, i.e. the velocity is exactly parallel to $\mathbf{B}_0$ and the $\mathbf{k}$ vector is also exactly parallel to $\mathbf{B}_0$. In fact the sound wave can be either part of the slow wave or part of the fast wave solution depending on whether $c_s^2 < v_A^2$ or $c_s^2 > v_A^2$.

Exercise: Derive the dispersion relations for $\mathbf{k}$ parallel to $\mathbf{B}_0$. What are the three wave solutions.

It should also be remarked that similar to a sonic shock wave the MHD waves are associated with respective shocks or discontinuities, I.e., there is a fast shock for the transition from plasma flow that is super-fast (faster than the fast mode speed) to sub-fast (slower than the fast mode speed), a slow shock, and an intermediate shock corresponding the Alfvén wave.