Chapter 4

Two Fluid Equations and Waves

The two fluid equations have been derived in the previous chapter where they were simplified to the set of MHD equations. Here we want to return to the set of two fluid equations and examine properties which are beyond the usual MHD plasma description. Specifically this chapter focusses on waves and instabilities in the two fluid approximation. Many of these wave also exesit in the kinetic plasma descript. Many of these waves exist only in the cause of not exactly charge neutral plasmas. This is important only on relatively small spatial scales comparable with the Debye length. Coorepondingly thes wase have much higher frequencies then the typical MHD waves. However before discussing these wave phenomena we first want to address pressure anisotropy and fluid plasma drifts.

4.1 Pressure Anisotropy

In many cases the pressure in a collisionless plasma is not isotropic. However the plasma is usualy well gyrotropic meaning that it is well described by a parallel and perpendicular pressure. The reason for the gyrotropic pressure is the rapid gyromotion which isotropizes the plasma motion efficiently in the plane perpendicular to the magnetic field. In this case the pressure tensor can be expressed as

$$\underline{\underline{\mathbf{p}}} = p_\perp \underline{\underline{1}} + (p_{||} - p_\perp)\frac{\mathbf{BB}}{B^2}$$

Here as in the remainder of this subsection we have dropped an index s to indicate that the equations are valid for each particle species in a plasma. For both pressures the ideal gas equation is a good approximation

$$\begin{aligned} p_{||} &= nk_B T_{||} \\ p_\perp &= nk_B T_\perp \end{aligned}$$

If the adiabatic approximation is satisfiedit is tempting to use the concept of a parallel and perpendicular entropy $s_{||} = p_{||} n^{-\gamma_{||}} = \text{const}$ and $s_\perp = p_\perp n^{-\gamma_\perp} = \text{const}$ to obtain equations for the parallel and perpendicular pressure. Here the adiabatic index can be obtained using the general definition

$$\gamma = (d+2)/d$$

with d being the degree of freedom. For the parallel motion we have $d = 1$ and for perpendincular motion $d = 2$ which yields the adiabatic equations of state

$$\begin{aligned} p_{||} &= p_{||0} \left(\frac{n}{n_0}\right)^3 \\ p_{\perp} &= p_{\perp 0} \left(\frac{n}{n_0}\right)^2 \end{aligned}$$

However, the adiabatic equations do not consider the coupling between the parallel and perpendicular pressures. For instance the perpendicular pressure should increase as a particle distribution move into a region of larger magnetic field strength which is not the case for the adiabatic equations. Also the pressures are combined a measure for the internal energy such that the equations should satisfy energy conservation which is also not the case.

A better approximation can be found by considering the adiabatic invariants of single particle motion. Avering the magnetic moment for a particle distribution function yields

$$\langle \mu \rangle = \frac{k_B T_\perp}{B} = \frac{p_\perp}{nB}$$

Since the average magnetic moment must be conserved (if the partcle gyro-motion is faster than other temporal changes and the gyro radius is smaller than length scales of gradients in the plasma) the perpendicular adiabatic law is

$$\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = 0$$

The parallel adiabatic equation is more complicated and basically requires to consider energy conservations, i.e., it requires to integrate the collisionless Boltzmann equation for the parallel energy and for the perpendicular energy separately. The resulting equations can be combined to

$$p_\perp \frac{dp_{||}}{dt} + 2p_{||}\frac{dp_\perp}{dt} + 5p_\perp p_{||} \nabla \cdot \mathbf{u} = 0 \qquad (4.1)$$

With the continuity equation

$$\nabla \cdot \mathbf{u} = \frac{1}{n}\left(\frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n\right) = \frac{1}{n}\frac{dn}{dt}$$

the pressure equation becomes

$$\frac{d}{dt}\left(\frac{p_{||}p_{\perp}^{2}}{n^{5}}\right)=\frac{d}{dt}\left(\frac{p_{||}B^{2}}{n^{3}}\right)=0$$

Using the ideal gas laws the equations of state can be re-written as

$$\begin{aligned}\frac{d}{dt}\left(\frac{T_{\perp}}{B}\right) &= 0\\ \frac{d}{dt}\left(\frac{B^{2}T_{||}}{n^{2}}\right) &= 0\end{aligned}$$

Thus a plasma which moves into a region of higher magnetic field strength will have an increasing perpendicular and a decreasing parallel temperature. This is for instance the case for the magnetosheath plasma as it gets closer to the dayside magnetopause with the result of an increasing temperature anisotropy.

Exercise: Show that the equation for the perpendicular kinetic energy is

$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_{\perp}^{2}+p_{\perp}\right)=-\nabla\cdot\left(\frac{1}{2}\rho u_{\perp}^{2}+p_{\perp}\right)\mathbf{u}-\nabla\cdot p_{\perp}\mathbf{u}_{\perp}-\nabla\cdot\mathbf{L}_{\perp}-qn\mathbf{u}_{\perp}\cdot\mathbf{E}$

Exercise: Show that the equation for the parallel kinetic energy is

$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_{||}^{2}+p_{||}\right)=-\nabla\cdot\left(\frac{1}{2}\rho u_{||}^{2}+p_{||}\right)\mathbf{u}-\nabla\cdot p_{||}\mathbf{u}_{||}-\nabla\cdot\mathbf{L}_{||}-qn\mathbf{u}_{||}\cdot\mathbf{E}$

Exercise: Show that by combining the above two equations and the momentum equations for parallel and perpendicular pressure one can derive eqution 4.1.

4.2 Fluid Plasma Drifts

The fluid equation of motion for such a plasma is

$$m_{s}n_{s}\left(\frac{\partial\mathbf{u}_{s}}{\partial t}+\mathbf{u}_{s}\cdot\nabla\mathbf{u}_{s}\right)=-\nabla p_{s\perp}-\nabla\cdot\left[(p_{s||}-p_{s\perp})\frac{\mathbf{BB}}{B^{2}}\right]+q_{s}n(\mathbf{E}+\mathbf{u}_{s}\times\mathbf{B}) \tag{4.2}$$

For a stationary plasma with sufficiently small velocities (such that the $\mathbf{u}_s\cdot\nabla\mathbf{u}_s$ can be neglected) the force balance equation is

$$q_{s}n(\mathbf{E}+\mathbf{u}_{s}\times\mathbf{B})=\nabla p_{s\perp}+\nabla\cdot\left[(p_{s||}-p_{s\perp})\frac{\mathbf{BB}}{B^{2}}\right]$$

Taking the cross-product of this equation with $\mathbf{B}/B^2$ and dividing by $q_s n$ yields

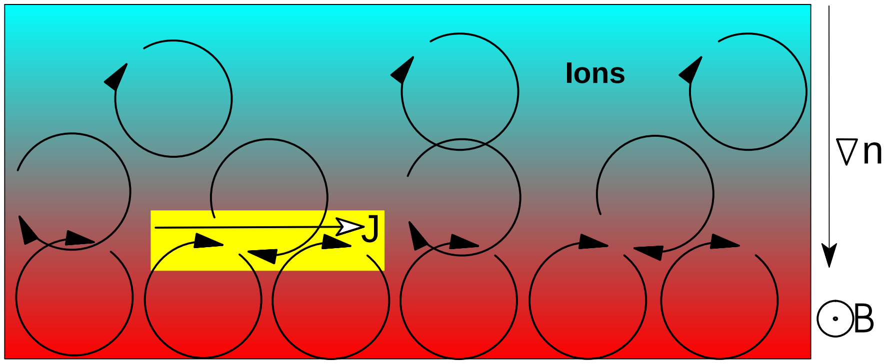


Figure 4.1: Illustration of the diamagnetic drift

$$\mathbf{u}_s = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{1}{q_s n B^2}\mathbf{B}\times\nabla p_{s\perp} + \frac{1}{q_s n B^2}\mathbf{B}\times\nabla\cdot\left[(p_{s||}-p_{s\perp})\frac{\mathbf{B}\mathbf{B}}{B^2}\right] \tag{4.3}$$

which defines the fluid drifts in a stationary plasma configuration similar to the single particle drifts discussed earlier. The first term is the familiar $\mathbf{E}\times\mathbf{B}$ drift which has to be present as a result of the Lorentz transformation. The second and third terms are new and not present in this form in the single particle drifts. The new drifts arise due to the collective particle interactions.

The second term decribes a particle drift perpendicular to the magnetic field and perpendicular to the gradient of the perpendicular pressure. This drift is called the diamagnetic drift. and is present if either a gradient in the density or a gradient in the temperature of the plasma exist (or both).

Let us consider a gradient in the plasma number density. Particles gyrate in the magnetic field all in the same direction for the same charge. However, in the presence of a density gradient there are more particle in the direction of the gradient than in the opposite direction. Thus an observer at a fixed location would see more particle going in one direction (due to gyromotion and due to the larger number of particle in the density gradient direction) then in the opposite direction. Thus at a given location a net bulk velocity arises due to the density gradient. Note that this does not require for the center of gyromotion to move. Similarly a gradient in the temperature results in different gyroradii in the direction of the gradient with the same net result for the bulk motion.

Since the diamagnetic drift velocity

$$\mathbf{v}_{dia,s1} = \frac{1}{q_s n B^2}\mathbf{B}\times\nabla p_{s\perp}$$

depends on the charge electrons and ions move in opposite directions giving rise to a diamagnetic current

$$\mathbf{j}_{dia} = \mathbf{B}\times\nabla p_\perp$$

with $p_\perp = p_{e\perp} + p_{i\perp}$. If the plasma is isotropic the diamagnetic drift is the only plasma drift because $p_{s||} = p_{s\perp}$.

For a nonisotropic plasma we can re-write the last term on the rhs of (4.3) using the radius of curvature definition as

$$\mathbf{v}_{dia,s2} = \frac{p_{s||} - p_{s\perp}}{q_s n B^2 R_c}\mathbf{B} \times \mathbf{n}$$

where $\mathbf{n}$ is the outer normal of the field line curvature and R_c is the radius of curvature. Thus this drift exists only for curved magnetic fields similar to the single particle curvature drift but it depends on parallel and perpendicular pressure and it can be positive or negative depending on the ratio of these pressures. The corresponding current density is given by

$$\mathbf{j}_{dia} = \frac{p_{||} - p_{\perp}}{B^2 R_c}\mathbf{B} \times \mathbf{n}$$

where $p_{||} = p_{e||} + p_{i||}$.

Applications of these drifts to various plasma environments are obvious. In the case of the Harris sheet there is a maximum pressure in the center of the current sheet. The corresponding perpendicular pressure gradient drives a diamagnetic current which in turn accounts selfconsistently for the increase in magnetic field strength away from the center of the current sheet.

Exercise: Compute the diamagnetic drift for electrons and iosn for the Harris sheet. Is it consistent with the Harris sheet current?

Polarization Drifts

Thus far we have considered a stationary configuration. If there are slow changes in the configuration we can compute the additional drifts by including the inertia term in the equation for the electric field

$$q_s n(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \nabla p_{s\perp} + \nabla \cdot \left[(p_{s||} - p_{s\perp})\frac{\mathbf{BB}}{B^2} \right] + \rho_s \frac{\partial \mathbf{u}_s}{\partial t}$$

and by taking the cross-product with $\mathbf{B}/B^2$ and dividing by $q_s n$ obtain the additional term

$$\mathbf{v}_s = \frac{m_s}{q_s B^2}\mathbf{B} \times \frac{\partial \mathbf{u}_s}{\partial t}$$

where we can substitute

$$\frac{\partial \mathbf{u}_s}{\partial t} = \frac{\partial}{\partial t}\left\{ \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{1}{q_s n B^2}\mathbf{B} \times \nabla p_{s\perp} + \frac{1}{q_s n B^2}\mathbf{B} \times \nabla \cdot \left[(p_{s||} - p_{s\perp})\frac{\mathbf{BB}}{B^2} \right] \right\}$$

For the first term this results in the guiding center polarization drift

$$\mathbf{v}_{p,s} = \frac{m_s}{q_s B^2} \frac{d\mathbf{E}}{dt}$$

known from the single particle drifts. The second term yields a new polarization drift which in the case of constant magnetic field and temperature results in

$$\mathbf{v}_{pn,s} = \frac{m_s k_B T_s}{q_s^2 B^2} \nabla_\perp \frac{d \ln n}{dt}$$

Finally from generalized Ohm's law one obtains a drift similar to the ones above from the current inertia term:

$$\mathbf{v}_{pc,s} = \frac{m_e}{n e_s^2 B^2} \mathbf{B} \times \frac{\partial \mathbf{j}}{\partial t}$$

This drift is a motion of the bulk of the entire plasma such that it does not cause any current.

4.3 Basic Two Fluid Wave Equations

For most two fluid waves the following set of equations is fully sufficient.

$$\begin{aligned} \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) &= 0 \\ m_s n_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) &= -\nabla p_s + q_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) \\ \frac{\partial p_s}{\partial t} + \mathbf{u}_s \cdot \nabla p_s &= -\gamma p_s \nabla \cdot \mathbf{u}_s \end{aligned}$$

where the index s denotes electrons (index e) and ions (index i). The equations are completed with Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{e}{\epsilon_0} (n_i - n_e) \\ \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 e (n_i \mathbf{u}_i - n_e \mathbf{u}_e) \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}$$

In the following we will discuss various examples of two fluid waves. The discussion will start with waves in a nonmagnetized plasma, meaing that the equilibrium or background magnetic field is zero. Thereafter we discuss the magnetized waves. Another basic distinction tocategorize waves are electrostatic and electromagnetic waves. In the electrostatic case the magnetic field perturbation is zero which implies with $\partial \mathbf{B}/\partial t = 0$ also $\nabla \times \mathbf{E} = 0$. Thus the electric field can be represented by a potential. Electromagnetic waves are all waves which carry a magnetic field perturbation.

4.4 Nonmagnetized Plasma Waves

4.4.1 Langmuir wawes

The most basic plasma wave is the Langmuir wave which is basically the same as the space charge oscillation which we used to derive the plasma frequency. Considering only high frequencies such that the ion dynamics can be neglected and a dependance only on the x coordinate the basic equations are

$$\begin{aligned} \partial_t n_e + \partial_x (n_e u_{ex}) &= 0 \\ m_e n_e (\partial_t u_{ex} + u_{ex} \partial_x u_{ex}) &= -\partial_x p_e - e n_e E_x \\ \partial_x E_x &= \frac{e}{\epsilon_0} (n_i - n_e) \end{aligned}$$

Next we linearize the equations whereequilibrium properties have an index 0 and perturbed quantities an index 1. Assuming

$$\begin{aligned} n_e &= n_0 + n_1 \\ E_x &= E_1 \\ u_{ex} &= u_1 \end{aligned}$$

and assuming cold electrons $p_e = 0$ the first order equations are

$$\begin{aligned} \partial_t n_1 + n_0 \partial_x u_1 &= 0 \\ m_e n_0 \partial_t u_1 &= -e n_0 E_1 \\ \partial_x E_1 &= -\frac{e}{\epsilon_0} n_1 \end{aligned}$$

Assuming plane wave solutions of the form $n_1 = \tilde{n}_1 \exp(-i\omega t + ikx)$ for n_1, E_1, and u_1 we can rewrite the equations in algebraic form

$$\begin{aligned} -\omega n_1 + k n_0 u_1 &= 0 \\ -i\omega m_e n_0 u_1 &= -e n_0 E_1 \\ ikE_1 &= -\frac{e}{\epsilon_0} n_1 \end{aligned}$$

Note that the different variables may actuall have a different phase in the plane wave solution. However, we can absorb the pahse into the amplitude $\tilde{n}_1$ etc by allowing these ampplitude to be complex.

With $n_1 = k n_0 u_1 / \omega$ we obtain from the third equation

$$ikE_1 = -\frac{e k n_0}{\epsilon_0 \omega} u_1$$

and multiplication of this equation with the second equation in the set above yields

$$k\omega m_e n_0 = e n_0 \frac{e k n_0}{\epsilon_0 \omega}$$

or

$$\omega^2 = \frac{e^2 n_0}{\epsilon_0 m_e} = \omega_{pe}^2$$

These are the well known electron plasma oscillations now as a result of the most basic electron plasma wave. For a warm plasma this wave is known as the Langmuir wave (Langmuir, 1926).

The electron plasma oscillations do not depend on the wave vector. Therefore the group velocity $\mathbf{v}_g = d\omega/d\mathbf{k}$ is zero and the oscillations do not carry energy.

Exercise: Determine the phase shift between E_1, n_1, and u_1. Assuming a solution of $u_1 = \tilde{u}_1 \cos(\omega t - kx)$ sketch the solutions for E_1, n_1, and u_1.

For a warm plasma the actual dispersion relation require stricly a kinetic treatment. However, using intuition we can guess the the particle motion in this case is one-dimensional. Therefore $\gamma = d + 2/d = 3$ such that we can include the linearized pressure equation

$$\partial_t p_1 = -\gamma p_0 \partial_x u_1$$

with the result

$$p_1 = \frac{k}{\omega} \gamma p_0 u_1$$

into our set of equations for the dispersion

$$\begin{aligned}
-\omega n_1 + k n_0 u_1 &= 0 \\
-i\omega m_e n_0 u_1 &= -ikp_1 - e n_0 E_1 \\
ikE_1 &= -\frac{e}{\epsilon_0} n_1
\end{aligned}$$

which yields

$$\begin{aligned}
-i\omega m_e n_0 \left(1 - \frac{k^2}{\omega^2} \frac{\gamma p_0}{m_e}\right) u_1 &= -e n_0 E_1 \\
ikE_1 &= -\frac{e k n_0}{\epsilon_0 \omega} u_1
\end{aligned}$$

or

$$\omega^2 = \frac{e^2 n_0}{\epsilon_0 m_e} - k^2 \frac{\gamma p_0}{m_e n_0} = \omega_{pe}^2 + k^2 \gamma v_{te}^2$$

with the thermal speed

$$v_{te}^2 = \frac{p_0}{m_e n_0} = \frac{k_B T_e}{m_e}$$

This is now the Langmuir wave. Usually $\gamma p_0 / m_e n_0$ is the sound speed in a medium. However, since we have already identified that $\gamma = 3$ it is the sound speed in a one-dimensional medium. Note also that the assumption of the pressure equation assumes that the electron compression is adiabatic, i.e., electrons travel only a short distance through the wave over a wave period $v_{te}/\omega \ll \lambda$ or

$$v_{te} \ll \frac{\omega}{k} \approx \frac{\omega_{pe}}{k}$$

which is equivalent to $k\lambda_{de} \ll 1$, i.e., wavelength's much larger then the Debye length. With this the solution can be expressed as

$$\omega \cong \pm \left(\omega_{pe}^2 + k^2 \gamma v_{te}^2 \right)^{1/2}$$

with the group velocity

$$v_g = \left| \frac{d\omega}{d\mathbf{k}} \right| = 3k\lambda_{de} v_{te}$$

Exercise: For a solution of $u_1 = \tilde{u}_1 \cos(\omega t - kx)$ determine the solutions for E_1 and n_1.

4.4.2 Dielectric Function

When we compare a wave propagation in a plasma to an ordinary medium one can re-formulate Poisson's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}$$

with the displacement in the form

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0 \\ \mathbf{D} &= \underline{\underline{\epsilon}} \cdot \mathbf{E} \end{aligned}$$

where $\underline{\underline{\epsilon}}$ is the dielectric tensor which incorporates the properties of the medium into the displacement $\mathbf{D}$. For the one-dimensional plasma case we can re-write

$$ikE = \frac{\rho_c}{\epsilon_0}$$

as

$$ik\left(E - \frac{\rho_c}{ik\epsilon_0}\right) \equiv ik\epsilon E$$

with the dielectric function ϵ. Using the derivation for the electron plasma oscillations one obtains

$$ik\left(1 - \frac{\omega^2}{\omega_{pe}^2}\right) E = 0$$

such tha the dielectric function becomes

$$\epsilon = 1 - \frac{\omega^2}{\omega_{pe}^2}$$

Note that the dispersion relation implies $\epsilon = 0$

For the Langmuir wave Poisson's equation becomes

$$ik\left(1 - \frac{\omega^2}{\omega_{pe}^2 - 3k^2 v_{te}^2}\right) E = 0$$

or for the dielectric function

$$\epsilon = 1 - \frac{\omega^2}{\omega_{pe}^2 - 3k^2 v_{te}^2}$$

with the dispersion relation from $\epsilon = 0$

4.4.3 Ion plasma waves

Langmuir waves are a typical high frequency phenomenon for which the ion dynamics can be neglected. Here we will consider the ion motion for the corresponding waves. The discussion on the plasma freuency demsontrates that a corresponding ion oscillation ocurrs on the ion plasma frequency ω_{pi}. Thus the term high frequency refers to $\omega \geq O(\omega_{pe})$ whereas low frequency refers to $\omega \leq O(\omega_{pi}) \ll \omega_{pe}$. Note that in a magnetized plasma one has to consider the gyro frequencies as well with the typical ordering $\omega_{gi} \ll \omega_{pi} \ll \omega_{pe} < \omega_{ge}$.

Exercise: Why is $\omega_{gi} \ll \omega_{pi} \ll \omega_{pe} < \omega_{ge}$ the typical ordering for a plasma?

The linearized pressure equations can be combined with the continuity equations to yield

$$\frac{\partial p_{s1}}{\partial t} = -\gamma_s p_{s0} \nabla \cdot \mathbf{u}_{s1} = \frac{\gamma_s p_{s0}}{n_{s0}} \frac{\partial n_{s1}}{\partial t}$$

or

$$\frac{\partial p_{s1}}{\partial t} = \gamma_s k_B T_{s0} \frac{\partial n_{s1}}{\partial t}$$

which can be integrated in time to yield $p_{s1} = \gamma_s k_B T_{s0} n_{s1}$.

The full set of electron and ion equations is

$$\begin{aligned}
\partial_t n_e + \partial_x (n_e u_{ex}) &= 0 \\
m_e n_e (\partial_t u_{ex} + u_{ex} \partial_x u_{ex}) &= -\gamma_e k_B T_{e0} \partial_x n_e - e n_e E_x \\
\partial_t n_i + \partial_x (n_i u_{ix}) &= 0 \\
m_i n_i (\partial_t u_{ix} + u_{ix} \partial_x u_{ix}) &= -\gamma_i k_B T_{i0} \partial_x n_i + e n_i E_x \\
\partial_x E_x &= \frac{e}{\epsilon_0} (n_i - n_e)
\end{aligned}$$

Linearizing these equations as we did before for the Langmuir waves with

$$\begin{aligned}
n_e &= n_0 + n_{e1} \\
n_i &= n_0 + n_{i1} \\
u_{ex} &= u_e \\
u_{ix} &= u_i \\
T_{e0} &= T_e \\
T_{i0} &= T_i \\
E_x &= E
\end{aligned}$$

yields

$$\begin{aligned}
\partial_t n_{e1} + n_0 \partial_x u_e &= 0 \\
m_e n_0 \partial_t u_e &= -\gamma_e k_B T_{e0} \partial_x n_e - e n_0 E \\
\partial_t n_{i1} + n_0 \partial_x u_i &= 0 \\
m_i n_0 \partial_t u_i &= -\gamma_i k_B T_{i0} \partial_x n_i + e n_0 E \\
\partial_x E_1 &= \frac{e}{\epsilon_0} (n_i - n_e)
\end{aligned}$$

For low frequency phenomena the electrons always tend to neutralize the ion charges. Thus a first attempt to solve the linearized equation can assume $n_{e1} \approx n_{i1} = n_1$. This eliminates Poisson's equation. Taking now the sum of the momentum equations yields

$$n_0 \partial_t (m_e u_e + m_i u_i) = -(\gamma_e k_B T_e + \gamma_i k_B T_i) \partial_x n_1$$

Note that T_e and T_i are the equilibrium temperatures. Multiplying the continuity equations with m_e and m_i respectively and taking the time derivative yields

$$(m_e + m_i)\frac{\partial^2 n_1}{\partial t^2} = -n_0 \partial_x \partial_t (m_e u_e + m_i u_i) = (\gamma_e k_B T_e + \gamma_i k_B T_i)\frac{\partial^2 n_1}{\partial x^2}$$

or

$$\frac{\partial^2 n_1}{\partial t^2} = c_s^2 \frac{\partial^2 n_1}{\partial x^2}$$

with

$$c_s \equiv \left(\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}\right)^{1/2}$$

where we neglected terms of $O(m_e/m_i)$. Using the plane approach leads to the ion-acoustic dispersion relation

$$\omega^2 = c_s^2 k^2$$

The name indicates that this wave is really a sound wave, however with the sound speed not only determined by the ion temperature but also by the electron temperature (or pressure). The coupling to the electrons occurs through the electric field. Note that the electric field was eliminated by adding the two momentum equations which introduced the electron pressure gradient instead of the electric field. The inertia term in the total momentum equation is dominated by the ions. Note that the coefficients γ_e and γ_i depend much on the detailed kinetic physics. However, since the electron thermal velocity $(k_B T_e/m_e)^{1/2} \gg c_s$ the electrons typically remain isothermal imlying $\gamma_e = 1$. There are also various plasma environments in which the electron temperature actually dominates such that the ion soundspeed becomes $c_s = (k_B T_e/m_i)$.

To solve the full set of electron and ion equations we take the time derivatives of the continuity equations and the x derivaitves of the momentum equations

$$\begin{aligned}
\partial_t^2 n_{e1} &= -n_0 \partial_t \partial_x u_e \\
n_0 \partial_x \partial_t u_e &= -\frac{\gamma_e k_B T_e}{m_e}\partial_x^2 n_{e1} - \frac{e n_0}{m_e}\partial_x E \\
\partial_t^2 n_{i1} &= -n_0 \partial_t \partial_x u_i \\
n_0 \partial_x \partial_t u_i &= -\frac{\gamma_i k_B T_i}{m_i}\partial_x^2 n_{i1} + \frac{e n_0}{m_i}\partial_x E
\end{aligned}$$

Defining

$$\begin{aligned}
c_{se}^2 &= \frac{\gamma_e k_B T_e}{m_e} \\
c_{si}^2 &= \frac{\gamma_i k_B T_i}{m_i}
\end{aligned}$$

and substituting the momentum equations in the corresponding continuity equations

$$\begin{aligned}\partial_t^2 n_{e1} &= c_{se}^2 \partial_x^2 n_{e1} + \frac{en_0}{m_e}\partial_x E \\ \partial_t^2 n_{i1} &= c_{si}^2 \partial_x^2 n_{i1} - \frac{en_0}{m_i}\partial_x E\end{aligned}$$

Using the plane wave approach $n_1 = \tilde{n}_1 \exp\left(-i\omega t + ikx\right)$ the basic equations are

$$\begin{aligned}\left(\omega^2 - k^2 c_{se}^2\right) n_{e1} &= -\frac{en_0}{m_e} ikE \\ \left(\omega^2 - k^2 c_{si}^2\right) n_{i1} &= \frac{en_0}{m_i} ikE \\ &\text{and} \\ ikE &= \frac{e}{\epsilon_0}\left(n_{i1} - n_{e1}\right)\end{aligned}$$

and substitution of n_{i1} and n_{e1} in Poisson's equation yields

$$\begin{aligned}ikE &= \frac{e}{\epsilon_0}\left(\frac{en_0}{m_i}\frac{1}{\omega^2 - k^2 c_{si}^2} + \frac{en_0}{m_e}\frac{1}{\omega^2 - k^2 c_{se}^2}\right) ikE \\ &= \left(\frac{\omega_{pi}^2}{\omega^2 - k^2 c_{si}^2} + \frac{\omega_{pe}^2}{\omega^2 - k^2 c_{se}^2}\right) ikE\end{aligned}$$

where we have used

$$\omega_{ps} = \left(\frac{n_0 e^2}{m_s \epsilon_0}\right)^{1/2}.$$

Thus the dispersion relation becomes

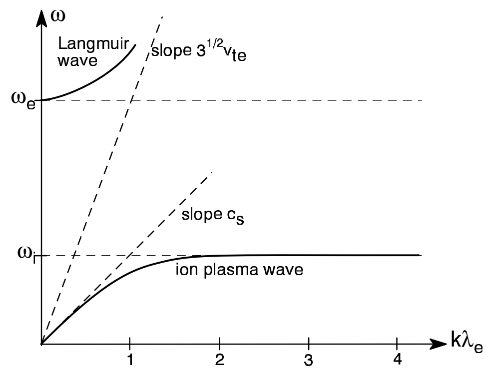
$$1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 c_{si}^2} - \frac{\omega_{pe}^2}{\omega^2 - k^2 c_{se}^2} = 0$$

This dispersion relation contains both the ion wave as well as the Langmuir wave solution. In the case of $\omega^2 = k^2 c_{se}^2$ we obtain the Langmuir wave. For $\omega^2 \ll k^2 c_{se}^2$ the solution is obtained from

$$1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 c_{si}^2} + \frac{\omega_{pe}^2}{k^2 c_{se}^2} = 0$$

$$\omega^2 = k^2 c_{si}^2 + \frac{k^2 c_{se}^2 \omega_{pi}^2}{k^2 c_{se}^2 + \omega_{pe}^2} = k^2 c_{si}^2 + \frac{k^2 \gamma_e k_B T_e / m_i}{k^2 \lambda_{de}^2 + 1}$$

Note that we could have arrived at this result also by neglecting the elctron inertia term in the above equations. For values of $k^2\lambda_{de}^2 \ll 1$ we obtain the prior dispersion relation for ion-acoustic waves. In the case of $k^2\lambda_{de}^2 \gg 1$ the second term dominates. In this limit the wave frequency approaches the ion plasma frequency.



The Langmuir and ion wave dispersion is illustrated in the above figure. The asymptotic values are the electron and the ion plasma frequency. The ion wave has a slope of c_s and the limiting slope for the langmuir wave was $\sqrt{k_B T_e}$.

4.4.4 Electromagnetic waves

The only other class of waves in an umagnetized plasma are electromagnetic waves. These are usually high frequency wave such that the ion dynamics can be neglected. We will further assume $\nabla \cdot \mathbf{u}_e = 0$ such that $dn_e/dt = 0$ and Poisson's equation becomes $\nabla \cdot \mathbf{E} = 0$. Together with $\nabla \cdot \mathbf{B} = 0$ we have the conditions $\mathbf{k} \cdot \mathbf{B} = 0$ and $\mathbf{k} \cdot \mathbf{E} = 0$. A wave satisfying the last condition is called transverse. Thus the set of basic equations for these waves is

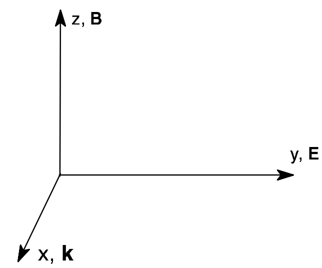
$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ \nabla \times \mathbf{B} &= -\mu_0 e n_e \mathbf{u}_e + \frac{1}{c^2}\partial_t \mathbf{E} \\ m_e n_e \left(\partial_t \mathbf{u}_e + \mathbf{u}_e \cdot \nabla \mathbf{u}_e\right) &= -\nabla p_e - e n_e \left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) \end{aligned}$$

Linearizing these equation where we assume $\mathbf{u}_{e0} = 0$. Further with $n_e = 0 \quad => \; p_{e1} = 0$ and we can neglect $\mathbf{u}_e \times \mathbf{B}$ because we consider only linear terms in the perturbation

$$\begin{aligned} \nabla \times \mathbf{E} &= -\partial_t \mathbf{B} \\ \nabla \times \mathbf{B} &= -\mu_0 e n_0 \mathbf{u}_e + \frac{1}{c^2}\partial_t \mathbf{E} \\ m_e n_0 \partial_t \mathbf{u}_e &= -e n_0 \mathbf{E} \end{aligned}$$

Taking the curl of the induction equation one obtains

$$
\begin{aligned}
\nabla \times \nabla \times \mathbf{E} &= -\partial_t \nabla \times \mathbf{B} \\
&= \mu_0 e n_0 \partial_t \mathbf{u}_e - \frac{1}{c^2}\partial_t^2 \mathbf{E} \\
\text{or} & \\
\nabla\left(\nabla \cdot \mathbf{E}\right) &= \frac{\mu_0 e^2 n_0}{m_e}\mathbf{E} + \frac{1}{c^2}\partial_t^2 \mathbf{E}
\end{aligned}
$$



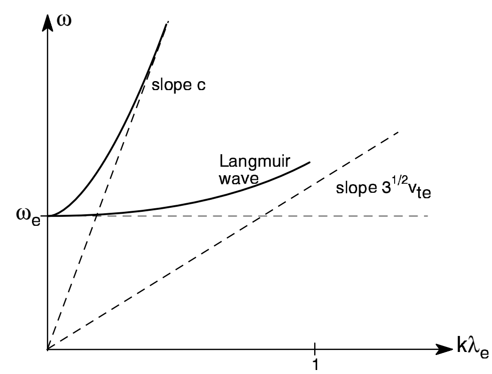
Assuming a plane wave with the $\mathbf{k}$ vector along x one can choose $\mathbf{E}$ along the y direction ($\mathbf{k} \cdot \mathbf{E} = 0$) such that $\mathbf{B}$ is along the z direction. This yields

$$-k^2 - \frac{\omega_{pe}^2}{c^2} + \frac{\omega^2}{c^2} = 0$$

or

$$\omega^2 = \omega_{pe}^2 + k^2 c^2$$

In the limit of zero density we have $\omega_{pe} \rightarrow 0$ and thus recover the dispersion relation for free space light waves: $\omega = \pm kc$. The dispersion relation is sketched in the figure to the right.



In the limit of zero density we have $\omega_{pe} \rightarrow 0$ and thus recover the dispersion relation for free space light waves: $\omega = \pm kc$.

It is instructive to recall the index of refraction in optical media as

$$n \equiv \frac{ck}{\omega}$$

For the electromagnetic waves we have

$$n \equiv \frac{\sqrt{\omega^2 - \omega_{pe}^2}}{\omega} = \sqrt{1 - \omega_e^2/\omega^2}$$

Thus the index of refraction and therefore the wavevector k become imaginary when $\omega < \omega_{pe}$. This coresponds to evanescence of the wave in this medium. In a medium with a positive density gradient the plasma frequency increase in the direction of the density gradient. An electromagnetic wave propagating in this medium reflects at the point $\omega = \omega_{pe}$ which the point of critical density gradient. This effect is important in laser fusion and in the interaction of radio wave with the ionosphere.

Exercise: Sketch the group velocity and the phase velocity for electromagnetic waves in an umagnetized plasma.

4.5 Magnetized Plasma Waves

So far we considered unmagnetized plasma waves. In the next section we consider plasma waves in a plasma with a homogeneous magnetic field background. This provides an additional complication or degree of freedom. The unmagnetized plasma waves have been isotropic in the sense that the wave dispersion does not depend on the direction of the wave propagation. In the presence of a magnetic field the direction of the wave vector relative to the magnetic field will become important. In classifying the magnetized plasma waves there are the following important properties to consider

- If $\mathbf{k}$ is along the magnetic field the wave is called parallel while for $\mathbf{k} \cdot \mathbf{B}_0 = 0$ the wave is perpendicular.
- If the wave vektor is parallel to the perturbed electric field the wave is called longitudinal. For $\mathbf{k} \cdot \mathbf{E}_1 = 0$ the wave is transverse.
- If the perturbed magnetic field $\mathbf{B}_1 = 0$ the wave is called electrostatic and for $\mathbf{B}_1 \neq 0$ the wave is electromagnetic. We have seen already that the MHD wave are all electromagnetic.

Not all wave can be simply classified in these categories. For instance a wave with a wave vector that has a 45 degree angle with the magnetic field is neither parallel nor perpendicular. These classifications are als not independent. Using faraday's law (induction equation) $\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$ we obtain $\mathbf{k} \times \mathbf{E}_1 = \omega \mathbf{B}$. For longitudinal waves $\mathbf{k} \times \mathbf{E}_1 = 0$ such that longitudinal waves are electrostatic. Vice versa transverse waves are electromagnetic. We will start the following discussion with electrostatic waves and discuss electromagnetic wave in the second part of this section.

4.5.1 Electrostatic magnetized waves

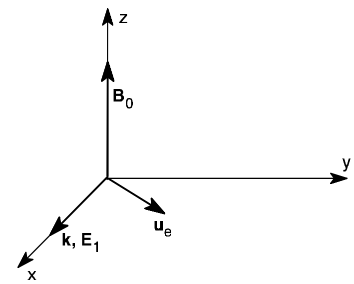
Upper hybrid waves

Let us first consider the high frequency electrostatic waves. As before we can neglect the ion dynamics in this case (they form a background with charge n_0) and we consider a cold plasma $T_e = 0$. We also assume the wave to be longitudinal i.e. the wave vector is along the electric feild perturbation which according to our prior comments implies an electrostatic wave. The resulting basic equations are

$$
\begin{aligned}
\partial_t n_e + \nabla \cdot (n_e \mathbf{u}_e) &= 0 \\
m_e n_e (\partial_t \mathbf{u}_e + \mathbf{u}_e \cdot \nabla \mathbf{u}_e) &= -e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) \\
\nabla \cdot \mathbf{E} &= \frac{e}{\epsilon_0} (n_0 - n_e)
\end{aligned}
$$

Exercise: Why are the other Maxwell equations ignored?

We choose now as the base coordinate system $\mathbf{E}_1 = E_1\mathbf{e}_x$, $\mathbf{k} = k\mathbf{e}_x$, $\mathbf{B}_0 = B_0\mathbf{e}_z$. Note that the $\mathbf{u}_e \times \mathbf{B}$ will generate nonzero velocity components for the x and y directions. The linearized wave for the plane wave solutions $n_1 = \tilde{n}_1 \exp(-i\omega t + ikx)$ are



$$\begin{aligned} -\omega n_{e1} + kn_0 u_{ex} &= 0 \\ -i\omega m_e u_{ex} &= -eE_1 - eu_{ey}B_0 \\ -i\omega m_e u_{ey} &= eu_{ex}B_0 \\ ikE_1 &= -\frac{e}{\epsilon_0} n_{e1} \end{aligned}$$

First we use the third equation to eliminate u_{ey} from the second equation.

$$i\omega m_e \left(1 - \frac{e^2 B_0^2}{\omega^2 m_e^2}\right) u_{ex} = i\omega m_e \frac{\omega^2 - \omega_{ge}^2}{\omega^2} u_{ex} = eE_1$$

or

$$u_{ex} = -i\frac{e}{\omega m_e}\frac{\omega^2}{\omega^2 - \omega_{ge}^2} E_1$$

substitution into the continuity equation

$$n_{e1} = \frac{kn_0}{\omega} u_{ex} = -i\frac{e}{\omega m_e}\frac{\omega^2}{\omega^2 - \omega_{ge}^2}\frac{kn_0}{\omega} E_1$$

which we can now substitute into Poisson's equation:

$$\begin{aligned} ikE_1 &= -\frac{e}{\epsilon_0} n_{e1} \\ &= i\frac{n_0 e^2}{m_e \epsilon_0}\frac{1}{\omega^2 - \omega_{ge}^2} kE_1 \\ &= \frac{\omega_{pe}^2}{\omega^2 - \omega_{ge}^2} ikE_1 \end{aligned}$$

With the dispersion relation

$$\omega^2 = \omega_{ge}^2 + \omega_{pe}^2 \equiv \omega_{uh}^2$$

These waves are called upper hybrid waves and the frequency ω_{uh}^2 is the upper hybrid frequency. As in the case of Langmuir waves the dispersion relation does not depend on the wave vector k. The electrons again perform an oscillation in the magnetic field, however in this case it is modified throught the gyro motion.

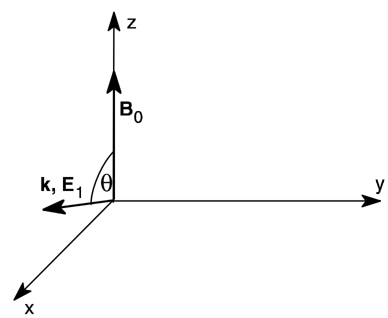
Exercise: Assume that the velocity along x is the real part of $u_{x1} = \tilde{u}_{x1} \exp(-i\omega t + ikx)$. Compute the real parts of the x and y components of the velocity, the density perturbation and the elctric field. What are the electron orbits in the wave?

Electrostatic ion waves

As in the section on unmagnetized waves we are now looking for the low frequency ion wave. In the section on unmagnetized wave we have first neglected Poisson's equation using the argument that the electrons very efficiently neutralize the ion motion which lead to the ion-acoustic wave. We will also use which simplifies the equations considerably. The linearized equations in this case are

$$\begin{aligned}
\partial_t n_e &= -n_0 \nabla \cdot \mathbf{u}_e \\
m_e n_0 \partial_t \mathbf{u}_e &= -\gamma_e k_B T_e \nabla n_e - e n_0 \mathbf{E} - e n_0 \mathbf{u}_e \times \mathbf{B}_0 \\
\partial_t n_i &= -n_0 \nabla \cdot \mathbf{u}_i \\
m_i n_0 \partial_t \mathbf{u}_i &= -\gamma_i k_B T_i \nabla n_i + e n_0 \mathbf{E} + e n_0 \mathbf{u}_i \times \mathbf{B}_0
\end{aligned}$$

Note that neutrality implies $n_e = n_i = n_1$. Now assuming the magnetic field along the z direction $\mathbf{B}_0 = B_0 \mathbf{e}_z$ and the wave vector in the x, z plane (note that $\mathbf{E}$ is along the $\mathbf{k}$ vector because we discuss electrostatic waves) we obtain



$$\begin{aligned}
\partial_t n_1 &= -n_0 \nabla \cdot \mathbf{u}_e \\
m_e \partial_t \mathbf{u}_e &= -\frac{\gamma_e k_B T_e}{n_0} \nabla n_1 - e\mathbf{E} - e\mathbf{u}_e \times \mathbf{B}_0 \\
m_i \partial_t \mathbf{u}_i &= -\frac{\gamma_i k_B T_i}{n_0} \nabla n_1 + e\mathbf{E} + e\mathbf{u}_i \times \mathbf{B}_0
\end{aligned}$$

Components

$$\begin{aligned}
-i\omega m_e \mathbf{u}_e &= -i\mathbf{k}\frac{\gamma_e k_B T_e}{n_0} n_1 - e\mathbf{E} - eB_0 \mathbf{u}_e \times \mathbf{e}_z \\
-i\omega m_i \mathbf{u}_i &= -i\mathbf{k}\frac{\gamma_i k_B T_i}{n_0} n_1 + e\mathbf{E} + eB_0 \mathbf{u}_i \times \mathbf{e}_z
\end{aligned}$$

and the continuity equation

$$\omega \frac{n_1}{n_0} = \mathbf{k} \cdot \mathbf{u}_e = \mathbf{k} \cdot \mathbf{u}_i$$

Sum of the equations

$$-i\omega \left(m_i \mathbf{u}_i + m_e \mathbf{u}_e \right) = -i\mathbf{k} \left(\gamma_e k_B T_e + \gamma_i k_B T_i \right) \frac{n_1}{n_0} + eB_0 \left(\mathbf{u}_i - \mathbf{u}_e \right) \times \mathbf{e}_z$$

and dot product with $\mathbf{k}$ (divergence)

$$
\begin{aligned}
-i\omega^2\frac{n_1}{n_0} &= -ik^2\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}\frac{n_1}{n_0} + \frac{eB_0}{m_i}\mathbf{k}\cdot((\mathbf{u}_i - \mathbf{u}_e)\times\mathbf{e}_z) \\
&= -ik^2 c_s^2\frac{n_1}{n_0} + \omega_{gi}\mathbf{e}_z\cdot(\mathbf{k}\times(\mathbf{u}_i - \mathbf{u}_e)) \\
&= -ik^2 c_s^2\frac{n_1}{n_0} + \omega_{gi}k_x(u_{iy} - u_{ey})
\end{aligned}
$$

Cross product of the momentum equation with $\mathbf{k}$

$$
\begin{aligned}
-i\omega\mathbf{k}\times\mathbf{u}_e &= -\omega_{ge}\mathbf{k}\times(\mathbf{u}_e\times\mathbf{e}_z) = -\omega_{ge}k_z\mathbf{u}_e + \omega_{ge}\mathbf{e}_z(\mathbf{k}\cdot\mathbf{u}_e) \\
-im_i\mathbf{k}\times\mathbf{u}_i &= \omega_{gi}\mathbf{k}\times(\mathbf{u}_i\times\mathbf{e}_z) = \omega_{gi}k_z\mathbf{u}_e - \omega_{gi}\mathbf{e}_z(\mathbf{k}\cdot\mathbf{u}_e)
\end{aligned}
$$

Components for the electron equation

$$
\begin{aligned}
i\omega k_z u_{ey} &= -\omega_{ge}k_z u_{ex} \\
-i\omega(k_z u_{ex} - k_x u_{ez}) &= -\omega_{ge}k_z u_{ey} \\
-i\omega k_x u_{ey} &= \omega_{ge}k_x u_{ex}
\end{aligned}
$$

With the solution

$$
\begin{aligned}
u_{ex} &= -i\frac{\omega}{\omega_{ge}}u_{ey} \\
u_{ez} &= i\frac{\omega_{ge}^2 - \omega^2}{\omega\omega_{ge}}\frac{k_z}{k_x}u_{ey}
\end{aligned}
$$

such that

$$
\begin{aligned}
\omega\frac{n_1}{n_0} &= k_x u_{ex} + k_z u_{ez} \\
&= i\left(-\frac{\omega}{\omega_{ge}}k_x + \frac{\omega_{ge}^2 - \omega^2}{\omega\omega_{ge}}\frac{k_z^2}{k_x}\right)u_{ey}
\end{aligned}
$$

or

$$
u_{ey} = i\frac{\omega}{\frac{\omega}{\omega_{ge}}k_x - \frac{\omega_{ge}^2-\omega^2}{\omega\omega_{ge}}\frac{k_z^2}{k_x}}\frac{n_1}{n_0}
$$

and for the ions similarly

$$
u_{iy} = -i\frac{\omega}{\frac{\omega}{\omega_{gi}}k_x - \frac{\omega_{gi}^2-\omega^2}{\omega\omega_{gi}}\frac{k_z^2}{k_x}}\frac{n_1}{n_0}
$$

Substitution into

$$\begin{aligned}\left(\omega^2 - k^2 c_s^2\right) &= i\omega_{gi} k_x \left(u_{iy} - u_{ey}\right)\\ &= \omega_{gi} k_x \left(\frac{\omega}{\frac{\omega}{\omega_{gi}} k_x - \frac{\omega_{gi}^2 - \omega^2}{\omega \omega_{gi}} \frac{k_z^2}{k_x}} + \frac{\omega}{\frac{\omega}{\omega_{ge}} k_x - \frac{\omega_{ge}^2 - \omega^2}{\omega \omega_{ge}} \frac{k_z^2}{k_x}} \right) \frac{n_1}{n_0}\end{aligned}$$

or for the dispersion relation

$$1 - \frac{k^2 c_s^2}{\omega^2} - \frac{\omega_{gi}}{\omega} \left(\frac{1}{\frac{\omega}{\omega_{ge}} - \frac{\omega_{ge}^2 - \omega^2}{\omega \omega_{ge}} \frac{k_z^2}{k_x^2}} + \frac{1}{\frac{\omega}{\omega_{gi}} - \frac{\omega_{gi}^2 - \omega^2}{\omega \omega_{gi}} \frac{k_z^2}{k_x^2}} \right) = 0$$

for electrostatic ion waves. We consider this relation in various limits for a better understanding.

a) Assume the wave vector along $\mathbf{B}_0$, i.e., $\mathbf{k} = k\mathbf{e}_z$ and the limit of $k_x \to 0$. for this limit the denominator of the terms in brackets assumes infinity provided that $\omega \neq \pm\omega_{gi}, \pm\omega_{ge}$. In this case the dispersion relation reduces to

$$\omega^2 = k_z^2 c_s^2$$

which is again the ion-acoustic wave now for propagation along the magnetic field in a magnetized plasma. For propagation of waves along the magnetic field one would expect to recover many of the properties of unmagnetized waves because the magnetic field does not influence the wave properties. However, one has to be careful in applying this as a rule. For instance properties of the adiabatic coeffients may depend on the motion parallel to $\mathbf{B}_0$ and be different from an unmagnetized plasma.

b) Let us now consider the case for $\omega \to \omega_{gi}$ in the limit of $k_x \to 0$. In this case the first denominator in brackets goes to infinity such that this term converges to zero. The remainder of the dispersion relation is

$$1 - \frac{k^2 c_s^2}{\omega_{gi}^2} - \frac{1}{1 - \frac{\omega_{gi}^2 - \omega^2}{\omega_{gi}^2} \frac{k_z^2}{k_x^2}} = 0$$

Here we can always find a solution for ω sufficiently close to ω_{gi} which satisfies the dispersion relation and $\omega \approx \omega_{gi}$ is a solution. These waves are the ion-cyclotron waves.

Exercise: Similarly one can show that $\omega \approx \omega_{ge}$ is a solution of the dispersion relation.

c) Let us now consider a wave vector perpendicular to the magnetic field $k_z \to 0$. In this case the dispersion relation is

$$1 - \frac{k^2 c_s^2}{\omega^2} - \frac{\omega_{ge}\omega_{gi}}{\omega^2} - \frac{\omega_{gi}^2}{\omega^2} = 0$$

Since $\omega_{gi} \ll \omega_{ge}$ one can neglect the last term with the solution

$$\omega^2 = k^2 c_s^2 + \omega_{ge}\omega_{gi}$$

The corresponding waves are the lower hybrid waves and the frequency

$$\omega = \sqrt{\omega_{ge}\omega_{gi}} \equiv \omega_{lh}$$

is the lower hybrid frequency.

The physical interpretation of the lower hybrid waves is as follows. The electric field is along $\mathbf{k}$ and the $\mathbf{k}$ vector is perpendicular to $\mathbf{B}_0$ such that it is conceivable that the heavy ions follow the electric field and the electrons perform the $\mathbf{E} \times \mathbf{B}$ and the polarization drift. The x displacement of the ions is identical to that of the electrons only if $\omega = \omega_{lh} = \sqrt{\omega_{ge}\omega_{gi}}$.

d) Let us finally consider the case that $\mathbf{k}$ is almost perpendicular such that $k_x \gg k_z$. In this case the frequency should be close to the ion cyclotron frequency $\omega \sim \omega_{gi}$ and $\omega/\omega_{ge} \ll 1$.

$$1 - \frac{k^2 c_s^2}{\omega^2} + \frac{\omega_{gi} k_x^2}{\omega_{ge} k_z^2} - \frac{1}{\frac{\omega^2}{\omega_{gi}^2} - \frac{\omega_{gi}^2 - \omega^2}{\omega_{gi}^2}\frac{k_z^2}{k_x^2}} = 0$$

We can neglect the second term in the denominator because $k_z/k_x \ll 1$ and for $k_x/k_z \ll (m_i/m_e)^{1/2}$ we can neglect the third term in the above relation such that we obtain

$$\omega^2 = k^2 c_s^2 + \omega_{gi}^2$$

This is the dispersion relation for electrostatic ion-cyclotron waves.

Summary of the results for the electrostatic ion waves:

Orientation of $\mathbf{k}$	Dispersion relation	Wave
$\theta = 0, \quad k_x = 0$	$\omega^2 = k_z^2 c_s^2$	Ion-acoustic
$\theta = 0, \quad k_x = 0$	$\omega^2 = \omega_{gi}^2$	Ion-cyclotron
$\theta = 0, \quad k_x = 0$	$\omega^2 = \omega_{ge}^2$	Electron-cyclotron
$\theta < \pi/2, \quad 1 \gg k_z/k_x \gg (m_i/m_e)^{1/2}$	$\omega^2 = k^2 c_s^2 + \omega_{gi}$	Ion-cyclotron
$\theta = \pi/2, \quad k_z = 0$	$\omega^2 = k^2 c_s^2 + \omega_{ge}\omega_{gi}$	Lower hybrid
$\theta = \pi/2, \quad k_z = 0$	$\omega = 0$	

Exercise: Show that $\omega = 0$ is a solution for the case $k_z = 0$.

4.5.2 High frequency electromagnetic magnetized waves

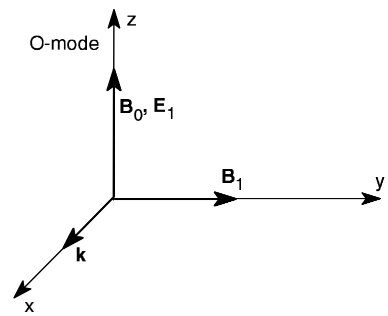
Waves perpendicular to $\mathbf{B}_0$

As before for the unmagnetized waves it is necessary to extend the discussion to electromagnetic waves. We are looking for high frequency wave such that the ion dynamics will be negected. For simplicity we will also assume a cold plasma where the electron pressure can be ignored. The linearized equations are therefore

$$
\begin{aligned}
m_e n_0 \partial_t \mathbf{u} &= -en_0\mathbf{E} - en_0\mathbf{u}\times\mathbf{B}_0 \\
\nabla\times\mathbf{E} &= -\partial_t\mathbf{B} \\
\nabla\times\mathbf{B} &= -\mu_0 e n_0\mathbf{u} + \frac{1}{c^2}\partial_t\mathbf{E}
\end{aligned}
$$

where $\mathbf{E} = \mathbf{E}_1$, $\mathbf{B} = \mathbf{B}_1$, and $\mathbf{u} = \mathbf{u}_{e1}$. Note that we don't use Poisson's equation because the above equations do not depend on n_{e1}. This would change if one uses a warm plasma. In that case the perturbed pressure is a function of n_{e1} and one needs Poisson's equation as an additional equation for n_{e1}.

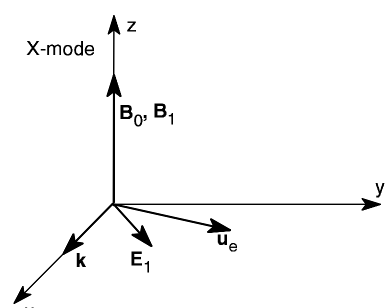
The basic coordinate system uses $\mathbf{B}_0 = B_0\mathbf{e}_z$, and we are first looking for wave with $\mathbf{k} = k\mathbf{e}_x$. There are two possibilities for the electric field: It can either be along the magnetic field $\mathbf{B}_0$ with $\mathbf{E}_1 = E_1\mathbf{e}_z$ (the ordinary or O-mode), or it can be in the x, y plane perpendicular to $\mathbf{B}_0$ (the extraordinary or X-mode).



O(ordinary)-mode: In the first case the electric field generates a velocity along the z direction and the $\mathbf{u}\times\mathbf{B}_0$ force decouples from the equations which give the dispersion relation for electromagnetic waves in an unmagnetized plasma.

$$
\omega^2 = \omega_{pe}^2 + k^2c^2
$$

X(extraordinary)-mode: If the electric field is in the the x, y plane the electric field generates a velocity $\mathbf{u}$ in the x, y plane and the $\mathbf{u}\times\mathbf{B}_0$ term generates a component perpendicular to the original velocity and perpendicular to $\mathbf{B}_0$ such that this velocity is also entirely in the x, y plane. Thus we do not need to consider the z component of the momentum equation. This illustrates that the ordinary and extraordinary modes separate. In this case with $\mathbf{E}_1 = E_x\mathbf{e}_x + E_y\mathbf{e}_y$ the components of the linear equations are



$$
\begin{aligned}
-i\omega m_e u_x &= -eE_x - eu_yB_0 \\
-i\omega m_e u_y &= -eE_y + eu_xB_0 \\
ikE_y &= i\omega B_1 \\
0 &= -\mu_0 e n_0 u_x - \frac{i\omega}{c^2}E_x \\
-ikB_1 &= -\mu_0 e n_0 u_y - \frac{i\omega}{c^2}E_y
\end{aligned}
$$

These are 5 equations for 5 unknowns. Using the last three equations one can express the velocities in terms of the electric field which can then be used in the first two equations. This yields two equations for E_x and E_y. Writing these in matrix form solutions are determined by setting the determinant of the coefficient matrix to 0.

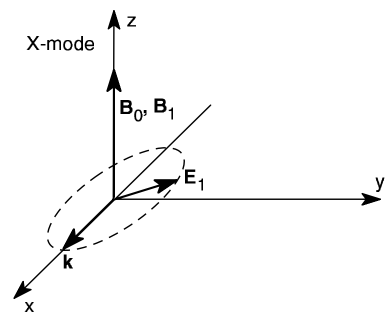
With some manipulation we can rewrite the dispersion relation as

$$n^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{uh}^2}$$

with $\omega_{uh}^2 = \omega_{pe}^2 + \omega_{ge}^2$ for the extraordinary or X-mode. Here n is the index of refraction.

Exercise: Derive the dispersion relation

As discussed at the beginning of this derivation the X-mode has a $\mathbf{k}$ vector perpendicular to the magnetic field and is partially transverse $\mathbf{k} \times \mathbf{E}_1 \neq 0$ and partially longitudinal $\mathbf{k} \cdot \mathbf{E}_1 \neq 0$. Solving the dispersion relation for ω and substitution in one of the electric field equation shows that E_x and E_y are out of phase such that the electric field vectors performs an elliptical rotation in the x, y plane.



Cutoffs and resonances: Two important properties of waves are cutoffs and resonances.

- A cutoff is any frequency where $k \to 0$.
- A resonance is any frequency where $k \to \pm\infty$.

For the dispersion relation in the form above the resonances are easy to determine in that they are the frequencies where the terms in the denominator become 0.

Therefore the resonances are at $\omega = 0$ and $\omega = \pm\omega_{uh}$.

The cutoffs are determined by setting the rhs to zero such that we need to solve

$$\omega^2\left(\omega^2 - \omega_{uh}^2\right) - \omega_{pe}^2\left(\omega^2 - \omega_{pe}^2\right) = 0$$

Since this is a quadratic equation in ω^2 the solutions are given by

$$\omega = \left(\omega_{pe}^2 + \frac{\omega_{ge}^2}{2} \pm \omega_{ge}\sqrt{\omega_{pe}^2 + \omega_{ge}^2/4}\right)^{1/2}$$

which can also be expressed as

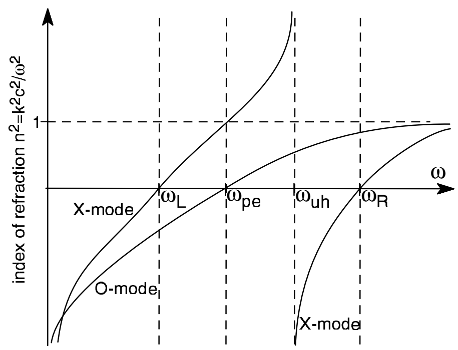
$$\omega_{\binom{R}{L}} = \pm\frac{\omega_{ge}}{2} + \sqrt{\omega_{pe}^2 + \omega_{ge}^2/4}$$

Where L and R refer to left and right which will become clear in the next section.

Exercise: Derive the resonance frequencies from the dispersion relation and show that the two equations for the solutions to the resonances are equivalent.

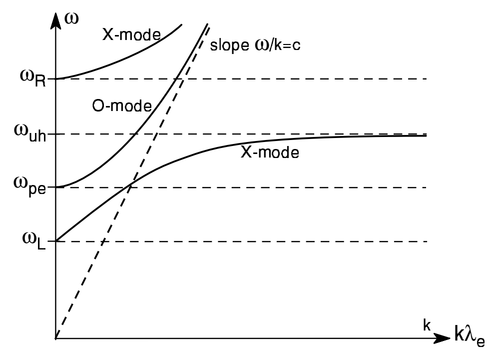
With the knowledge of the cutoffs and resonances we can draw the dispersion diagram for the index of refraction as a function of frequency. The ordinary mode can be included in this diagram by noting that the refractive index is

$$n^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}$$



The understanding of these modes is complemented by drawing also the usual dispersion daigram $\omega(k)$.

Exercise: Solve the usual dispersion relation for the X-mode, i.e., $\omega(k)$.



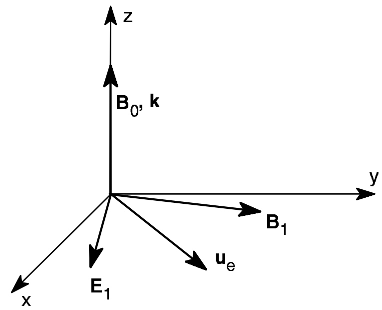
This diagram shows that there are frequency ranges (bands) in which the dispersion relation has no real solution for ω. These ranges are called stop bands (originating from radio engineering). The stop bands are the ranges $[0, \omega_{pe}]$ for the O-mode, and $[0, \omega_L]$, $[\omega_{uh}, \omega_R]$ for the X-mode.

The dispersion diagramm for n^2 shows that in these ranges the refractive index and thus k are negative and therefor we conclude that the modes are evanescent in these frequency ranges and are reflected when they encounter a plasma region in which the condition is satisfied. The other ranges are called pass bands because waves can propagate.

Electromagnetic waves along $\mathbf{B}_0$

To discuss elctromagnetic waves along $\mathbf{B}_0$ we use the same set of linear equations as in the case of waves perpendicular to $\mathbf{B}_0$ which for the plane wave solutions assume the form

$$
\begin{aligned}
i\omega m_e n_0 \mathbf{u} &= -en_0\mathbf{E} - en_0\mathbf{u}\times\mathbf{B}_0 \\
i\mathbf{k}\times\mathbf{E} &= i\omega\mathbf{B} \\
i\mathbf{k}\times\mathbf{B} &= -\mu_0 e n_0\mathbf{u} - \frac{i\omega}{c^2}\mathbf{E}
\end{aligned}
$$



and $\mathbf{E} = \mathbf{E}_1$, $\mathbf{B} = \mathbf{B}_1$, $\mathbf{u} = \mathbf{u}_{e1}$, and $\mathbf{B}_0 = B\,\mathbf{e}_z$. For $\mathbf{k} = k\,\mathbf{e}_z$ a consistent solution can be found by assuming that all perturbations $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{u}$ are in the $x\ y$ plane. Writing out the components of the linear equations

$$
\begin{aligned}
-i\omega m_e u_x &= -eE_x - eu_y B_0 \\
-i\omega m_e u_y &= -eE_y + eu_x B_0 \\
-ikE_y &= i\omega B_x \\
ikE_x &= i\omega B_y \\
-ikB_y &= -\mu_0 e n_0 u_x - \frac{i\omega}{c^2}E_x \\
ikB_x &= -\mu_0 e n_0 u_y - \frac{i\omega}{c^2}E_y
\end{aligned}
$$

Using the induction equation we can substitute B_x and B_y in the last two equations to obtain the velocities in terms of the electric field. These can be used to eliminate the velicities in the first two equations which yields two equations for E_x and E_y. Writing these in matrix form solutions are determined by setting the determinant of the coefficient matrix to 0.

With some manipulation we can rewrite the dispersion relation as

$$
1 - \frac{\omega}{\omega_{pe}^2}\left(\omega - \frac{c^2k^2}{\omega}\right) = \pm\frac{\omega_{ge}}{\omega_{pe}^2}\left(\omega - \frac{c^2k^2}{\omega}\right)
$$

or

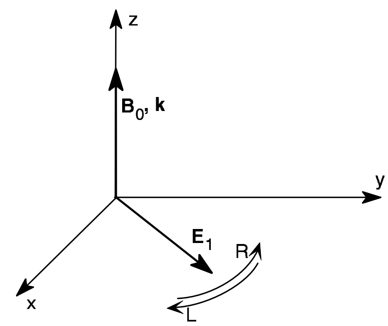
$$
1 = \frac{1}{\omega_{pe}^2}\left(\omega \pm \omega_{ge}\right)\left(\omega - \frac{c^2k^2}{\omega}\right)
$$

or for the refractive index

$$
n^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2/\omega^2}{1 \pm \omega_{ge}/\omega}
$$

Exercise: Derive the dispersion relation

Here the wave corresponding to the "+" sign is called the L-wave meaning left circularly polarized and the wave corresponding to the "$-$" sign is called the R-wave implying right circular polarization. These term originate from the rotation of the electric field vector (R corresponds to the right hand rule meaning the thumb of the right hand points along the k vector and the fingers point along the direction of rotation of the electric field). The situation is cylindrically symmetric which implies the circular polarization rather then an elliptic polarization for the X-mode.

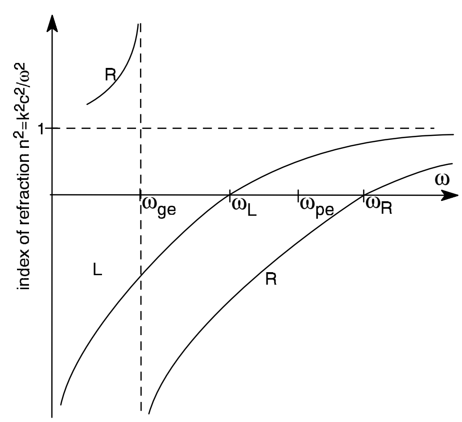


The rotation of the electric field for the R-mode corresponds to the rotation of the gyro motion for electrons. For $\omega = \omega_{ge}$ the electrons are in pahse with the wave and are continuously accelerated which is the reason for the resonance $k \to \infty$ at this frequency. The L-mode has no resonances because it rotates in the direction opposite to the electron gyration.

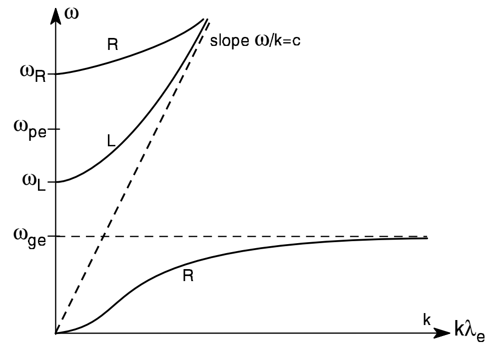
The cutoffs of these waves are determined by $k = 0$ which yields

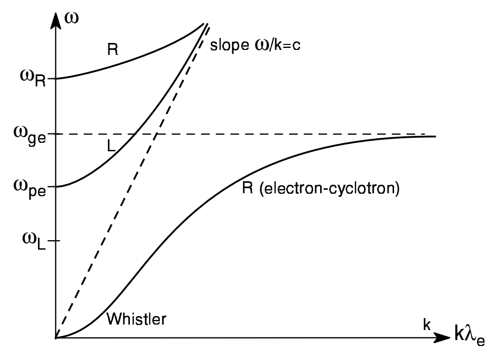
$$\omega_{\left(\begin{array}{c} R \\ L \end{array}\right)} = \pm\frac{\omega_{ge}}{2} + \sqrt{\omega_{pe}^2 + \omega_{ge}^2/4}$$

which are the same as for the case of the X-mode. Note that $\omega_R > \omega_{ge}$ always but depending on the value of ω_{pe} we can have $\omega_L > \omega_{ge}$ or $\omega_L < \omega_{ge}$. The dispersion properties are slightly different for the two cases. For the case $\omega_L > \omega_{ge}$ the dispersion relation $\omega(k)$ and the index of refraction is shown in the next two figures.



There are two pass bands for the R mode $[0, \omega_{ge}]$ and $[\omega_R, \infty]$ with the cutoff in between. The low frequency part of the R mode is often called electron cyclotron wave. Note that the properties of this wave are quite different then for the electrostatic electron-cyclotron wave. For the lowest frequencies of the R-mode $\omega(k)$ has a positive second derivative. Thus the phase velocity and the group velocity increase with increasing frequency. These waves are called whistler waves because higher freuencies travel faster then the lower frequncies and thus the corresponding radiowave generates a whistle starting at high frequencies and descending to lower frequencies.





The L mode has its pass band $[\omega_L, \infty]$ with the stop band in the range $[0, \omega_L]$. In the case with $\omega_L < \omega_{ge}$ the L-mode and the low frequncy R-mode have a frequency range in which both overlap which is not present for $\omega_L > \omega_{ge}$.

The high frequency R-wave has always a higher phase speed than the L-wave. Thus the polarization of the R and the L components rotate along the path of a plane wave which is known as Faraday rotation and used to determine the plasma densities of laboratory and space plasmas.

This concludes our discussion of the high frequency electromagnetic waves. Just as a reminder the spectrum of low frequency electromagnetic waves is dominated by the MHD waves. A complete discussion of electromagnetic waves including electron and ion dynamics shows that the MHD waves are the low frequncy branches of the electromagnetic waves.

4.6 Two-stream instability

Before we conclude this chapter on two-fluid properties and waves we want to address two further topics. The first of these are instabilities driven by cold plasma beams. The second topic (in the next section are waves caused by plasma drifts. To consider the case of a beam of electrons consider the following situation. We have the ion population at rest and the electrons moving at a velocity of V_0 along the x direction through the ions. In this case it is sufficient to consider only the x derivative with the definitions $\mathbf{u}_e = u_{e1}\mathbf{e}_x$ $\mathbf{E}_1 = E\mathbf{e}_x$. The linearized equations for this case are

$$
\begin{aligned}
\partial_t n_{e1} + V_0 \partial_x n_{e1} &= -n_0 \partial_x u_e \\
m_e n_0 \partial_t u_e + m_e n_0 V_0 \partial_x u_e &= -\gamma_e k_B T_e \partial_x n_{e1} - e n_0 E
\end{aligned}
$$

$$\begin{aligned}\partial_t n_{i1} &= -n_0 \partial_x u_i \\ m_i n_0 \partial_t u_i &= -\gamma_i k_B T_i \partial_x n_{i1} + e n_0 E \\ \partial_x E &= \frac{e}{\epsilon_0} \left(n_{i1} - n_{e1}\right)\end{aligned}$$

Using again a plane wave approach one obtains 5 equations for n_{e1}, n_{i1}, u_e, u_i, and E: Using the continuity equations on can express the velocities in terms of n_{e1} and n_{i1}. These can be used in the momentum equations to obtain relations between n_{e1} and E, and n_{i1} and E which substituted in Poisson's equation for the densities yield the dispersion relation.

$$ikE = ikE \left[\frac{e^2 n_0}{\epsilon_0 m_i} \left(\omega^2 - k^2 c_{si}^2\right)^{-1} + \frac{e^2 n_0}{\epsilon_0 m_e} \left(\left(\omega - kV_0\right)^2 - k^2 c_{se}^2\right)^{-1}\right]$$

or

$$1 = \omega_{pi}^2 \left(\omega^2 - k^2 c_{si}^2\right)^{-1} + \omega_{pe}^2 \left(\left(\omega - kV_0\right)^2 - k^2 c_{se}^2\right)^{-1}$$

Cold plasma approximation dispersion relation

$$\epsilon\left(k, \omega\right) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\left(\omega - kV_0\right)^2} = 0$$

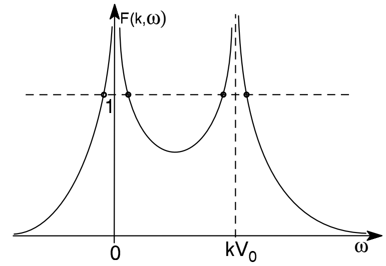
This is a real equation of fourth order in ω. This implies that the complex conjugate is also a solution (in case the equation has any complex roots). Therefore there is always an instability if the equation has any complex roots. It is illustrative to consider the case of infinitely massive ions. In this case $\omega_{pi} \approx 0$ such that the dispersion relation has the roots

$$\omega = kV_0 \pm \omega_{pe}$$

Thus ions should be important for an instability. Choosing the "$-$" sign and assuming that we need a low frequncy (for the ions) in the laboratory frame we expect $kV_0 \approx \omega_{pe}$. To address the problem of complex roots and instability consider the function

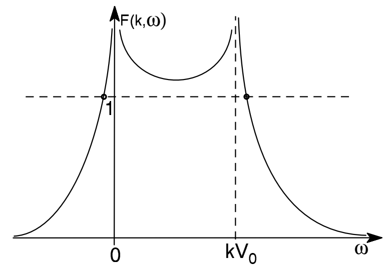
$$f\left(k, \omega\right) = \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\left(\omega - kV_0\right)^2}$$

This allows a graphical way of finding the 4 solutions if $f(k,\omega) = 1$. However as indicated there may be only two solutions if $f_{\min} > 1$. This minimum is determined by



$$
\begin{aligned}
\frac{\partial f}{\partial \omega} &= 0 \\
&= -2\frac{\omega_{pi}^2}{\omega^3} - 2\frac{\omega_{pe}^2}{(\omega - kV_0)^3}
\end{aligned}
$$

or



$$
\begin{aligned}
\omega_{pi}^2 (\omega - kV_0)^3 + \omega_{pe}^2 \omega^3 &= \\
\omega^3 \left(\omega_{pi}^2 + \omega_{pe}^2\right) - 3\omega_{pi}^2 kV_0 \omega^2 + 3\omega_{pi}^2 k^2 V_0^2 \omega - \omega_{pi}^2 k^3 V_0^3 &= 0
\end{aligned}
$$

Since $kV_0 \approx \omega_{pe}$ and $\omega_{pe}^2 \gg \omega_{pi}^2$ the first and the last term in this equation should dominate with the solution of

$$
\omega \approx \left(\frac{m_e}{m_i}\right)^{1/3} kV_0
$$

and

$$
f_{\min}(k,\omega) \cong \frac{\omega_{pi}^2}{(m_e/m_i)^{2/3} k^2 V_0^2} + \frac{\omega_{pe}^2}{k^2 V_0^2} = \left(\left(\frac{m_e}{m_i}\right)^{1/3} + 1\right) \frac{\omega_{pe}^2}{k^2 V_0^2}
$$

Thus we expect instability in the range of

$$
|kV_0| \leq \omega_{pe}
$$

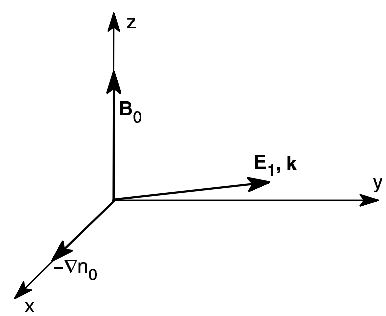
These types of instabilities are very common. The free energy for these is the relative streaming of two fairly cold plasma beams relative to each other. Note that the dispersion relation including the pressure terms is more stable and the instability depends on the ratio of thermal to drift velocity. In terms of the energy of the two beams an equilibrium would consit of a single Maxwellian of equal temperature. Thus the instability is just the mechanism by which a plasma configuration which is far from local thermal equilibrium relaxes fast into this equilibrium. Here it is remarkable that this relaxation occurs even without any collisions. Note that the equations we have used do not consider any dissipation (resistivity, viscosity etc.).

Note that the analysis of streaming instabilities is usually done in a local approximation. This means that one assumes that the streaming particle population has no spatial structure in this approximation. Clearly a stream of electrons through ions would create a magnetic field with a gradient. However, in this approximation we do not consider the magnetic field is negligible and assume that the gradient has a length scale much larger than any wave or instability which is considered here such that the magnetic gradient can be neglected. If one would actually would consider the spatial inhomogeneity the analysis would be considerably more complex. In particular a solution in terms of fourier modes is possible only for the directions with ignorable coordinates. For instance, if an equilibrium is x dependent one can use the fourier of wave aproach only for the y and z dependence of the perturbations while the x dependence requires the actual solution of an Eigenvalue problem.

4.7 Drift waves

Drift waves exist because of spatial inhomogeneities. Here it is important to understand that similar to to the local approximation in the case of the two-stream instability we consider only the waves driven by the local gradient such that the wave lengths of the solutions have to be much smaller than the typical gradient length scale.

In the following example we consider a constant density gradient. The typical frequencies are assumed small enough to have the electrons carry out an $\mathbf{E} \times \mathbf{B}$ drift but large enough to neglect the ion dynamics. The gradient is assumed in the x direction, the magnetic field is along the z direction, and the wave vector is mostly along y with a small component in the z direction which allows the electrons to move along the magnetic field. We will also use the elctrostatic approximation, i.e., $\nabla \times \mathbf{E} = 0$. The basic equations are



$$\begin{aligned} \partial_t n_e &= -\nabla \cdot n_0 \mathbf{u}_e \\ m_e n_0 \partial_t \mathbf{u}_e &= -\gamma_e k_B T_e \nabla n_e - e n_0 \mathbf{E} - e n_0 \mathbf{u}_e \times \mathbf{B}_0 \\ \nabla \times \mathbf{E} &= 0 \end{aligned}$$

or in linearized form

$$\begin{aligned} \partial_t n_{e1} &= -u_{ex} \partial_x n_0 - n_0 \partial_y u_{ey} - n_0 \partial_z u_{ez} \\ m_e n_0 \partial_t u_{xe} &= -e n_0 u_{ey} B_0 \\ m_e n_0 \partial_t u_{ye} &= -\gamma_e k_B T_e \partial_y n_{e1} - e n_0 E_y + e n_0 u_{ex} B_0 \\ m_e n_0 \partial_t u_{ze} &= -\gamma_e k_B T_e \partial_z n_{e1} - e n_0 E_z \\ k_y E_z - k_z E_y &= 0 \end{aligned}$$

Now we have to be careful regarding the approximation. Assuming that the frequncy is sufficiently small to neglect the electron inertia the equations are

$$\partial_t n_{e1} = -u_{ex} \partial_x n_0 - n_0 \partial_y u_{ey} - n_0 \partial_z u_{ez}$$

$$
\begin{aligned}
0 &= -en_0 u_{ey} B_0 \\
0 &= -\gamma_e k_B T_e \partial_y n_{e1} - en_0 E_y + en_0 u_{ex} B_0 \\
0 &= -\gamma_e k_B T_e \partial_z n_{e1} - en_0 E_z \\
k_y E_z - k_z E_y &= 0
\end{aligned}
$$

This demonstrates $u_{ey} = 0$. Assuming $k_z \ll \frac{1}{n_0}\partial_x n_0$, the pressure gradient force being much smaller then the $\mathbf{E} \times \mathbf{B}$ drift and substituting $E_z = \frac{k_z}{k_y} E_y$ yields

$$
\begin{aligned}
-i\omega n_{e1} &= -u_{ex} \partial_x n_0 \\
0 &= -en_0 E_y + en_0 u_{ex} B_0 \\
0 &= -ik_z \gamma_e k_B T_e n_{e1} - en_0 \frac{k_z}{k_y} E_y
\end{aligned}
$$

such that

$$u_{ex} = \frac{1}{B_0} E_y$$

and substituted into the continuity equation

$$\frac{n_{e1}}{n_0} = -i \frac{\partial_x n_0}{\omega n_0} u_{ex} = -i \frac{\partial_x n_0}{\omega n_0} \frac{1}{B_0} E_y$$

The z component of the momentum equation yields

$$\frac{n_{e1}}{n_0} = i \frac{e}{k_y \gamma_e k_B T_e} E_y$$

Equating the two equations

$$\omega = -k_y \frac{\partial_x n_0}{n_0} \frac{c_{se}^2}{\omega_{ge}}$$

Defining the gradient length scale as

$$L_\nabla = \left(\frac{\partial_x n_0}{n_0} \right)^{-1}$$

and defining the diamagnetic drift speed as

$$v_{de} = \frac{1}{L_\nabla} \frac{c_{se}^2}{\omega_{ge}}$$

the dispersion relation for electrostatic drift wave is

$$\omega = -k_y v_{de}$$

There is a large variiety of drift waves corresponding to various plasma regimes and approximations similar to the ordinary plasma waves. Drift waves are important in space and laboratory plasmas.

An important property of drift waves are instabilities at very strong gradients. Such gradients exist at very thin plasma boundaries which implies high drift speeds. In those cases the waves can become unstable. The instability has two possible effects. In both cases the instability tends to lower the drift speed and to thermalize the corresponding particle motion. In the presence of a strong density gradient this will cause diffusion such that the instability tends to lower the gradient. In the case of a magnetic field gradient the drift instabilty is driven by the strang current density. The ffect of the instability is to lower the current density. Thus the macroscopic effect is that of a resisitvity evne though the basic plasma is collisionless. The resistivity resulting from the wave turbulence is therefore called an anomalous resistivity.