# Chapter 5

# Plasma Kinetic Theory

## 5.1 Klimontovich Equation

### 5.1.1 Introduction

Start from first principles => exact plasma description

**Single particle:**

- location $\mathbf{X}_1(t)$, velocity $\mathbf{V}_1(t)$
  => 6 degree of freedom
  => six-dimensional space
- Density of the particle in this space: $N(\mathbf{x}, \mathbf{v}, t) = \delta(\mathbf{x} - \mathbf{X}_1(t))\, \delta(\mathbf{v} - \mathbf{V}_1(t))$

with: $\delta(\mathbf{x} - \mathbf{X}_1(t)) = \delta(x - X_1(t))\, \delta(y - Y_1(t))\, \delta(z - Z_1(t))$

$\delta -$ Dirac delta function

Consider $N_{0s}$ **particles** of species $s$:

Density of this distribution in phase space:

$$N_s(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N_{0s}} \delta(\mathbf{x} - \mathbf{X}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t))$$

and for all species:

$$N = \sum_s N_s(\mathbf{x}, \mathbf{v}, t)$$

**Particle motion:**

$$\begin{aligned} \dot{\mathbf{X}}_i &= \mathbf{V}_i(t) \\ \dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\mathbf{E}^m(\mathbf{X}_i(t), t) + \frac{q_s}{m_s}\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{X}_i(t), t) \end{aligned}$$

To solve the equations of motion we need Maxwell's equations

$$\begin{aligned}
\nabla\cdot\mathbf{E}^m(\mathbf{x},t) &= \frac{1}{\epsilon_0}\rho_c(\mathbf{x},t)\\
\nabla\cdot\mathbf{B}^m(\mathbf{x},t) &= 0\\
\nabla\times\mathbf{E}^m(\mathbf{x},t)+\frac{\partial\mathbf{B}^m(\mathbf{x},t)}{\partial t} &= 0\\
\nabla\times\mathbf{B}^m(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial\mathbf{E}^m(\mathbf{x},t)}{\partial t} &= \mu_0\mathbf{j}^m(\mathbf{x},t)
\end{aligned}$$

($m$ stands for microscopic fields) with the charge and current densities (sources)

$$\begin{aligned}
\rho_c^m(\mathbf{x},t) &= \sum_s q_s\int_{-\infty}^{\infty}d^3vN_s(\mathbf{x},\mathbf{v},t)\\
\mathbf{j}^m(\mathbf{x},t) &= \sum q_s\int_{-\infty}^{\infty}d^3v\mathbf{v}N_s(\mathbf{x},\mathbf{v},t)
\end{aligned}$$

The above equations fully determine the entire sytem of particles.

Initial value problem:

$$N_s(\mathbf{x},\mathbf{v},t=0)\quad => \quad \mathbf{E}^m(\mathbf{x},t=0)\,,\ \mathbf{B}^m(\mathbf{x},t=0)$$

=> Integrate equations in time.

### 5.1.2 Klimontovich Equation

Time evolution of the distribution function $N_s(\mathbf{x},\mathbf{v},t)$:

$$\begin{aligned}
\frac{\partial N_s(\mathbf{x},\mathbf{v},t)}{\partial t} = &-\sum_{i=1}^{N_{0s}}\dot{\mathbf{X}}_i\cdot\nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))\\
&-\sum_{i=1}^{N_{0s}}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}}\delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))
\end{aligned}$$

Note:

$$\begin{aligned}
\frac{\partial f(a-b)}{\partial a} &= -\frac{\partial f(a-b)}{\partial b}\\
\frac{df(g(t))}{dt} &= \frac{df}{dg}\frac{dg}{dt}
\end{aligned}$$

Substitute: $\dot{\mathbf{X}}_i$ and $\dot{\mathbf{V}}_i$

$$\frac{\partial N_s(\mathbf{x},\mathbf{v},t)}{\partial t} = -\sum_{i=1}^{N_{0s}} \mathbf{v}\cdot\nabla_{\mathbf{x}}\delta\left(\mathbf{x}-\mathbf{X}_i(t)\right)\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right) -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{x},t)+\frac{q_s}{m_s}\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)\right\}\cdot\nabla_{\mathbf{v}}\delta\left(\mathbf{x}-\mathbf{X}_i(t)\right)\delta\left(\mathbf{v}-\mathbf{V}_i(t)\right)$$

where we used $f(a)\delta(a-b)=f(b)\delta(a-b)$.

**Exercise:** Prove that the last equation for $N_s(\mathbf{x},\mathbf{v},t)$ is correct and in particular that one can replace $\mathbf{V}_i(t)\times\mathbf{B}^m(\mathbf{X}_i(t),t)$ with $\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)$ in this equation.

As the final step we can now take the $\mathbf{v}\cdot\nabla_{\mathbf{x}}$ and $\left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{x},t)+\frac{q_s}{m_s}\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)\right\}$ in front of the summation which yields the **Klimontovich equation**

$$\frac{\partial N_s}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{x}}N_s+\frac{q_s}{m_s}\left(\mathbf{E}^m+\mathbf{v}\times\mathbf{B}^m\right)\cdot\nabla_{\mathbf{v}}N_s=0$$

Together with the Maxwell's equation and the definitions for charge and current densities this provides a full description of the plasma dynamics!

However, since the distribution is a distribution of delta functions it still requires basically to follow all individual particles which in typical application is not feasible even on modern supercomputers.

**Properties of the Klimontovich equation**

- Incompressibility in phase space: Hypothetical point particle at $\mathbf{x}$, $\mathbf{v}$
  total time derivative

  $$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{d\mathbf{x}}{dt}\cdot\nabla_{\mathbf{x}}+\frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}}$$

  => Klimontovich equation

  $$\frac{DN_s(\mathbf{x},\mathbf{v},t)}{Dt}=0$$

  => along each (hypothetical) path $N_s is$ constant!

- Conservation of particles (continuity): $\partial f/\partial t+\nabla_{\mathbf{r}}\cdot(\mathbf{v}f)=0$
  In 6-dimensional phase space we can define $\nabla_{\mathbf{R}}=(\nabla_{\mathbf{x}},\nabla_{\mathbf{v}})$ and $\mathbf{V}=(d\mathbf{x}/dt,\, d\mathbf{v}/dt)$ =>

  $$\frac{\partial N_s}{\partial t}+\nabla_{\mathbf{R}}\cdot(\mathbf{V}N_s)=0$$

  Klimontovich eq. must satisfy continuity!

### 5.1.3 Plasma Kinetic Equation

The Klimontovich distribution is a distribution of $\delta$ functions => need to reduce amount of information (we know that a plasma behave collectively so it is not necessary to follow each individual particle.).

=> generate smooth distribution using an appropriate average

Rigorous way:

- Ensemble average over infinite number of realizations, e.g., with a temperature contact => statistical mechanics

Alternatively:

- Define boxes size $\Delta x$, $\Delta v$ with $\Delta x \ll \lambda_{de}$ and count particles in range $[\mathbf{x}, \mathbf{v}]$ to $[\mathbf{x}+\Delta\mathbf{x}, \mathbf{v}+\Delta\mathbf{v}]$ => $f_s = \frac{n_s}{\Delta x^3 \Delta v^3}$

Define fluctuations

$$\begin{aligned} N_s(\mathbf{x},\mathbf{v},t) &= f_s(\mathbf{x},\mathbf{v},t) + \delta N_s(\mathbf{x},\mathbf{v},t) \\ \mathbf{E}^m(\mathbf{x},t) &= \mathbf{E}(\mathbf{x},t) + \delta\mathbf{E}(\mathbf{x},t) \\ \mathbf{B}^m(\mathbf{x},t) &= \mathbf{B}(\mathbf{x},t) + \delta\mathbf{B}(\mathbf{x},t) \end{aligned}$$

such that: $\langle\delta N_s\rangle$, $\langle\delta\mathbf{E}\rangle$, $\langle\delta\mathbf{B}\rangle = 0$

=>

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = \frac{q_s}{m_s}\langle(\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B})\cdot\nabla_{\mathbf{v}}\delta N_s\rangle$$

- Left side - collective effects
- right side -collisional effects

Continuum limit: $N_0 \to \infty$

- right side: fluctuations $\delta N_s \sim N_0^{1/2}$ (statistical mechanics)
  $\delta\mathbf{E} \sim e\delta N_s \sim \frac{1}{N_0} N_0^{1/2} \sim N_0^{-1/2}$

=> right side $\to$ const

=> left side $\sim N_0 \to \infty$

Which yields the collisionless **Boltzmann** equations:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = 0$$

Complemented with Maxwells equations and with the definitions for charge and current density

$$
\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{x},t) &= \frac{1}{\epsilon_0}\rho_c(\mathbf{x},t) \\
\nabla \cdot \mathbf{B}(\mathbf{x},t) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{x},t) &= -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} \\
\nabla \times \mathbf{B}(\mathbf{x},t) &= \mu_0 \mathbf{j}(\mathbf{x},t) + \frac{1}{c^2}\frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t} \\
\rho_c(\mathbf{x},t) &= \sum_s q_s \int_{-\infty}^{\infty} d^3v f_s(\mathbf{x},\mathbf{v},t) \\
\mathbf{j}(\mathbf{x},t) &= \sum q_s \int_{-\infty}^{\infty} d^3v \mathbf{v} f_s(\mathbf{x},\mathbf{v},t)
\end{aligned}
$$

yield the **Vlasov** equations.

# 5.2 Liouville Equation

## 5.2.1 Concept of a system

**Motivation:**

Use Liouville equation => derivation of a kinetic equation (right hande side of the Boltzmann equation)

Note: Klimontovich equation - Behaviour of individual particles

***One particle:***

- spatial coordinate of the system $\mathbf{x}_1 = (x_1, y_1, z_1)$
- velocity coordinate of the system $\mathbf{v}_1 = (v_{x1}, v_{y1}, v_{z1})$
- Particle orbit (as before) by $\mathbf{X}_1(t)$ and $\mathbf{V}_1(t)$
- System coordinates: $(\mathbf{x}_1, \mathbf{v}_1) = (x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1})$ (6 coord)
- Density of systems:
  $N(\mathbf{x},\mathbf{v},t) = \delta(\mathbf{x}_1 - \mathbf{X}_1(t))\,\delta(\mathbf{v}_1 - \mathbf{V}_1(t))$
  One system consisting of one particle

***2 particles:***

- 12 coordinates for our system
- Phase space: $(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2) = (x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, x_2, y_2, z_2, v_{x2}, v_{y2}, v_{z2})$

- Density:
  $N(\mathbf{x},\mathbf{v},t)=\delta(\mathbf{x}_1-\mathbf{X}_1(t))\,\delta(\mathbf{v}_1-\mathbf{V}_1(t))\,\delta(\mathbf{x}_2-\mathbf{X}_2(t))\,\delta(\mathbf{v}_2-\mathbf{V}_2(t))$
  1 system consiting of 2 particles

***Generalisation to $N_0$ particles*** => Phase space has 6N coordinates

Density:

$$N(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2,...\mathbf{x}_{N_0},\mathbf{v}_{N_0},t)=\prod_{i=0}^{N_0}\delta(\mathbf{x}_i-\mathbf{X}_i(t))\,\delta(\mathbf{v}_i-\mathbf{V}_i(t))$$

### 5.2.2 Liouville equation

Interested in the time evolution of $N$ and with

$$\frac{\partial\delta(\mathbf{x}_i-\mathbf{X}_i(t))}{\partial t}=-\frac{d\mathbf{X}_i(t)}{dt}\cdot\nabla_{\mathbf{x}_i}\delta(\mathbf{x}_i-\mathbf{X}_i(t))$$

=>

$$\begin{aligned}\frac{\partial N(\mathbf{x},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_0}\dot{\mathbf{X}}_i\cdot\nabla_{\mathbf{x}_i}\prod_{i=0}^{N_0}\delta(\mathbf{x}_i-\mathbf{X}_i(t))\,\delta(\mathbf{v}_i-\mathbf{V}_i(t))\\ &\quad -\sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}\prod_{i=0}^{N_0}\delta(\mathbf{x}_i-\mathbf{X}_i(t))\,\delta(\mathbf{v}_i-\mathbf{V}_i(t))\end{aligned}$$

As before we can substitute (note $\mathbf{x}_i$ instead of $\mathbf{X}_i(t)$ and $\mathbf{v}_i$ instead of $\mathbf{V}_i(t)$ in the Lorentz force!):

$$\begin{aligned}\dot{\mathbf{X}}_i &= \mathbf{V}_i(t)\\ \dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\left[\mathbf{E}^m(\mathbf{x}_i,t)+\mathbf{v}_i\times\mathbf{E}^m(\mathbf{x}_i,t)\right]\end{aligned}$$

Such that

$$\frac{\partial N}{\partial t}+\sum_{i=1}^{N_0}\mathbf{v}_i\cdot\nabla_{\mathbf{x}_i}N+\sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}N=0$$

which is **Liouville's equation.**

Properties:

(a)

$$\frac{D}{Dt}N(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2,...\mathbf{x}_{N_0},\mathbf{v}_{N_0},t)=0$$

with

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\sum_{i=1}^{N_0}\mathbf{v}_i\cdot\nabla_{\mathbf{x}_i}+\sum_{i=1}^{N_0}\dot{\mathbf{V}}_i\cdot\nabla_{\mathbf{v}_i}$$

=> incompressibility

(b) Continuity

$$\begin{aligned} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} N &= \nabla_{\mathbf{x}_i} \cdot (\mathbf{v}_i N) \\ \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i} &= \nabla_{\mathbf{v}_i} \cdot \left(\dot{\mathbf{V}}_i N\right) \end{aligned}$$

because

$$\nabla_{\mathbf{v}_i} \cdot \dot{\mathbf{V}}_i = \nabla \cdot \left\{ \frac{q_s}{m_s} \left[\mathbf{E}^m (\mathbf{x}_i, t) + \mathbf{v}_i \times \mathbf{E}^m (\mathbf{x}_i, t)\right] \right\} = 0$$

=>

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \nabla_{\mathbf{x}_i} \cdot (\mathbf{v}_i N) + \sum_{i=1}^{N_0} \nabla_{\mathbf{v}_i} \cdot \left(\dot{\mathbf{V}}_i N\right) = 0$$

**Probability density:**

Ensemble of systems $N$:

Def.:

$$f_{N_0} (\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, ...\mathbf{x}_{N_0}, \mathbf{v}_{N_0}, t)\, d\mathbf{x}_1 d\mathbf{v}_1 d\mathbf{x}_2 d\mathbf{v}_2 ... d\mathbf{x}_{N_0} d\mathbf{v}_{N_0}$$

is the proability that

$\mathbf{X}_1 (t)$ is in the interval $[\mathbf{x}_1, \mathbf{x}_1 + d\mathbf{x}_1]$, $\mathbf{X}_2 (t)$ is in the interval $[\mathbf{x}_2, \mathbf{x}_2 + d\mathbf{x}_2]$, ..

$\mathbf{V}_1 (t)$ is in the interval $[\mathbf{v}_1, \mathbf{v}_1 + d\mathbf{v}_1]$, $\mathbf{V}_2 (t)$ is in the interval $[\mathbf{v}_2, \mathbf{v}_2 + d\mathbf{v}_2]$, ..

Probability is conserved along trajectory:

each fluid element moves along the trajectory as a probability

With $\nabla_{\mathbf{x}_i} \cdot \mathbf{v}_i = 0$ and $\nabla_{\mathbf{v}_i} \cdot \dot{\mathbf{V}}_i = 0$ =>

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0} + \sum_{i=1}^{N_0} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i} f_{N_0} = 0$$

or

$$\frac{D}{Dt} f_{N_0} (\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, ...\mathbf{x}_{N_0}, \mathbf{v}_{N_0}, t) = 0$$

We have now a probability distribution function which determines the kinetic evolution exactly, however, this distribution is in a $6N_0$ dimensional space. Thus there is now reduction in complexity compared to Klimotovich equation!

### 5.2.3 BBGKY Hierarchy

BBGKY -Bogoliubov, Born and Green, Kirkwood, Yvon

*Motivation:* Reduction of complexity

Probability density:

$$f_{N_0}\left(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2,...\mathbf{x}_{N_0},\mathbf{v}_{N_0},t\right)d\mathbf{x}_1 d\mathbf{v}_1 d\mathbf{x}_2 d\mathbf{v}_2 ... d\mathbf{x}_{N_0} d\mathbf{v}_{N_0}$$

is the proability to find

particle 1 in $I_1 = [(\mathbf{x}_1,\mathbf{v}_1)\,,\,(\mathbf{x}_1 + d\mathbf{x}_1\mathbf{v}_1 + d\mathbf{v}_1)]$,

particle 2 in $I_2 = [(\mathbf{x}_2,\mathbf{v}_2)\,,\,(\mathbf{x}_2 + d\mathbf{x}_2\mathbf{v}_2 + d\mathbf{v}_2)]$,

etc.

**Reduced probability distributions:**

$$f_k\left(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2,...\mathbf{x}_k,\mathbf{v}_k,t\right) \equiv V^k \int_{-\infty}^{\infty} d\mathbf{x}_{k+1} d\mathbf{v}_{k+1} d\mathbf{x}_{k+2} d\mathbf{v}_{k+2} ... d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} f_{N_0}$$

= prob. to find particle 1 in $I_1$, ...particle k in $I_k$ irrespective of the locatins for p$_{k+1}$ to p$_{N_0}$.

- $V$ spatial volume occupied by particles
- $V^k$ needed for normalization -> later
- $f_{N_0}$ is a probability => $\int_{-\infty}^{\infty} d\mathbf{x}_1 d\mathbf{v}_1 ... d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} f_{N_0} = 1$
- boundary conditions: $f_{N_0} \to 0$ for $|\mathbf{x}_i| \to \pm\infty$
  $f_{N_0} \to 0$ for $|\mathbf{v}_i| \to \pm\infty$
- Symmetry regarding particle labels
  $f_{N_0}\left(\mathbf{x}_1,\mathbf{v}_1,...\mathbf{x}_k,\mathbf{v}_k,...\mathbf{x}_{N_0},\mathbf{v}_{N_0},t\right) = f_{N_0}\left(\mathbf{x}_k,\mathbf{v}_k,...\mathbf{x}_1,\mathbf{v}_1,...\mathbf{x}_{N_0},\mathbf{v}_{N_0},t\right)$
- Assume Coulomb model:
  $\mathbf{V}_i(t) = \sum_{j=1}^{N_0} \mathbf{a}_{ij}$ with $\mathbf{a}_{ij} = \frac{q^2}{4\pi\epsilon_0 m |\mathbf{x}_i - \mathbf{x}_j|^3}\left(\mathbf{x}_i - \mathbf{x}_j\right)$ and $\mathbf{a}_{ii} = 0$

***Lioville equation***

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0} + \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} = 0$$

**(a) Reduced distribution function $f_{N_0-1}$:**

Integrate over $d\mathbf{x}_{N_0} d\mathbf{v}_{N_0}$

$$\begin{aligned}\int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \frac{\partial f_{N_0}}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0} \\ + \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \quad &= \quad 0\end{aligned}$$

First term

$$T_1 = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} f_{N_0} = V^{1-N_0}\frac{\partial f_{N_0-1}}{\partial t}$$

Second term

$$\begin{aligned} T_2 &= \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} f_{N_0} + \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \mathbf{v}_{N_0} \cdot \nabla_{\mathbf{x}_{N_0}} f_{N_0} \\ &= V^{1-N_0} \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0-1} + \int_{-\infty}^{\infty} dy_{N_0} dz_{N_0} d\mathbf{v}_{N_0} v_{xN_0} \cdot f_{N_0}\big|_{x_{N_0}=-\infty}^{x_{N_0}=\infty} + \ldots \\ &= V^{1-N_0} \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0-1} \end{aligned}$$

Third term: Split

$$\sum_{i=1}^{N_0}\sum_{j=1}^{N_0} \mathbf{a}_{ij} = \sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \mathbf{a}_{ij} + \sum_{j=1}^{N_0-1} \mathbf{a}_{N_0 j} + \sum_{i=1}^{N_0-1} \mathbf{a}_{iN_0} + \mathbf{a}_{N_0 N_0}$$

such that

$$\begin{aligned} T_3 &= \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \\ &= \sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \\ &\quad + \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \sum_{j=1}^{N_0-1} \mathbf{a}_{N_0 j} \cdot \nabla_{\mathbf{v}_{N_0}} f_{N_0} + \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \sum_{i=1}^{N_0-1} \mathbf{a}_{iN_0} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \\ &= V^{1-N_0} \sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} + \sum_{i=1}^{N_0-1} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{iN_0} \cdot \nabla_{\mathbf{v}_i} f_{N_0} \end{aligned}$$

Collect terms and multiply with $V^{N_0-1}$ to obtain the equation for the reduced distributions function $f_{N_0-1}$

$$\begin{aligned} \frac{\partial f_{N_0-1}}{\partial t} + \sum_{i=1}^{N_0-1} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0-1} + \sum_{i=1}^{N_0-1}\sum_{j=1}^{N_0-1} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} & \\ + V^{N_0-1} \sum_{i=1}^{N_0-1} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0} d\mathbf{v}_{N_0} \mathbf{a}_{iN_0} \cdot \nabla_{\mathbf{v}_i} f_{N_0} &= 0 \end{aligned}$$

This is the equation for the reduced distribution function $f_{N_0-1}$.

**(b) Reduced distribution function $f_{N_0-2}$:**

Note recurrence relation

$$f_{k-1} = V^{-1}\int_{-\infty}^{\infty} d\mathbf{x}_k d\mathbf{v}_k f_k$$

Integrate the equation for $f_{N_0-1}$ over $d\mathbf{x}_{N_0-1} d\mathbf{v}_{N_0-1}$

Term 1:

$$T_1 = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0-1} d\mathbf{v}_{N_0-1} f_{N_0-1} = V \frac{\partial}{\partial t} f_{N_0-2}$$

Term 2:

$$T_2 = V \sum_{i=1}^{N_0-2} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0-2}$$

Term 3:

$$T_3 = V \sum_{i=1}^{N_0-2} \sum_{j=1}^{N_0-2} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-2} + \int_{-\infty}^{\infty} d\mathbf{x}_{N_0-1} d\mathbf{v}_{N_0-1} \sum_{i=1}^{N_0-2} \mathbf{a}_{i,N_0-1} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1}$$

Term 4:

$$T_4 = \sum_{i=1}^{N_0-2} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0-1} d\mathbf{v}_{N_0-1} \mathbf{a}_{iN_0-1} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1}$$

Collect terms and divide by $V$ to obtain the equation

$$\begin{aligned} \frac{\partial f_{N_0-2}}{\partial t} + \sum_{i=1}^{N_0-2} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0-2} + \sum_{i=1}^{N_0-2} \sum_{j=1}^{N_0-2} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_{N_0-2} & \\ + \frac{2}{V} \sum_{i=1}^{N_0-2} \int_{-\infty}^{\infty} d\mathbf{x}_{N_0-1} d\mathbf{v}_{N_0-1} \mathbf{a}_{iN_0-1} \cdot \nabla_{\mathbf{v}_i} f_{N_0-1} &= 0 \end{aligned}$$

for the reduced distributions function $f_{N_0-2}$.

=> **Recurrence relation for** $f_k$

$$\begin{aligned} \frac{\partial f_k}{\partial t} + \sum_{i=1}^{k} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_k + \sum_{i=1}^{k} \sum_{j=1}^{k} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_k & \\ + \frac{N_0 - k}{V} \sum_{i=1}^{k} \int_{-\infty}^{\infty} d\mathbf{x}_{k+1} d\mathbf{v}_{k+1} \mathbf{a}_{i,k+1} \cdot \nabla_{\mathbf{v}_i} f_{k+1} &= 0 \end{aligned}$$

This is the **BBGKY Hierarchy** of kinetic equations.

This is a complete description without any reduction in the physics because each reduced distribution function couples to the next level.

**Example** $k = 1$**:**

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1 + \frac{N_0 - 1}{V} \int_{-\infty}^{\infty} d\mathbf{x}_2 d\mathbf{v}_2 \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} f_2 \left(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t\right) = 0$$

where

- $f_1(\mathbf{x}_1, \mathbf{v}_1, t)\, d\mathbf{x}_1 d\mathbf{v}_1$ is the proability to find particle 1 in $I_1 = [(\mathbf{x}_1, \mathbf{v}_1),\ (\mathbf{x}_1 + d\mathbf{x}_1, \mathbf{v}_1 + d\mathbf{v}_1)]$,
- $f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t)\, d\mathbf{x}_1 d\mathbf{v}_1$ is the proability to find particles 1 and 2 in $I_2 = [(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2),\ (\mathbf{x}_1 + d\mathbf{x}_1, \mathbf{x}_2 + d\mathbf{x}_2, \mathbf{v}_1 + d\mathbf{v}_1, \mathbf{v}_2 + d\mathbf{v}_2)]$,

**Example:**

Single dice: $P_1(x) = \delta(x-5)$ is the probability to throw a 5.

Two dice: $P_2(x, y) = \delta(x-5)\,\delta(y-5)$ is the joint probability to throw two 5's. No correlation => $P_2(x, y) = P_1(x)\, P_2(y)$

However, if the first throw sets a constraint for the second then

$$P_2(x, y) = P_1(x)\, P_2(y) + \delta P(x, y)$$

=> introduce **correlation function** $g(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t)$ with

$$f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = f_1(\mathbf{x}_1, \mathbf{v}_1, t)\, f_1(\mathbf{x}_2, \mathbf{v}_2, t) + g(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t)$$

or symbolic:

$$f_2(1, 2, t) = f_1(1, t)\, f_1(2, t) + g(1, 2, t)$$

Substitute $f_2$ in the first equation of the hierarchie:

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1 + n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} [f_1(1, t)\, f_1(2, t) + g(1, 2, t)] = 0$$

where $1 \equiv (\mathbf{x}_1, \mathbf{v}_1)$. With

$$\left\{ n_0 \int_{-\infty}^{\infty} d2\, f_1(2, t)\, \mathbf{a}_{1,2} \right\} \cdot \nabla_{\mathbf{v}_1} f_1(1, t) = \langle \mathbf{a}_{1,2} \rangle_{f_1} \cdot \nabla_{\mathbf{v}_1} f_1(1, t)$$

we obtain

$$\frac{\partial f_1(1, t)}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1(1, t) + \mathbf{a} \cdot \nabla_{\mathbf{v}_1} f_1(1, t) = -n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g(1, 2, t)$$

with $\mathbf{a} = \langle \mathbf{a}_{1,2} \rangle_{f_1}$.

Collisionless Boltzmann equation for $g = 0$ => single particle distribution function.

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1 + \mathbf{a} \cdot \nabla_{\mathbf{v}_1} f_1 = 0$$

+ Maxwell's equations => Vlasov equations!

Collisions are determined only by the term $-n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g\left(1,2,t\right)$ that is by the correlation $g\left(1,2,t\right)$. Thus there is need to determine $g = f_2 - f_1 f_1$.

=> $k = 2$ in BBGKY hierarchy:

$$\frac{\partial f_2}{\partial t} + \left(\mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} + \mathbf{v}_2 \cdot \nabla_{\mathbf{x}_2}\right) f_2 + \left(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2}\right) f_2 \\ +n_0 \int_{-\infty}^{\infty} d3 \,\left(\mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2}\right) f_3\left(1,2,3,t\right) = 0$$

Now one can factor $f_3$ similar to $f_2(1,2,t) = f_1\left(1\right) f_1\left(2\right) + g\left(1,2\right)$ (Mayer cluster expansion):

$$\begin{aligned} f_3\left(1,2,3\right) &= f_1\left(1\right) f_1\left(2\right) f_1\left(3\right) + f_1\left(1\right) g\left(2,3\right) \\ &\quad + f_1\left(2\right) g\left(1,3\right) + f_1\left(3\right) g\left(1,2\right) + h\left(1,2,3\right) \end{aligned}$$

where $h$ represents the so-called three-particle correlations. Note that although not explicitly stated all of these are also time dependent.

Assume that three-particle correlations are negligible, i.e., $h = 0$ and replace $f_2$ and $f_3$ in the equation for $k = 2$:

Term 1:

$$\begin{aligned} T_1 &= \frac{\partial}{\partial t}\left(f_1\left(1\right) f_1\left(2\right) + g\left(1,2\right)\right) \\ &= f_1\left(1\right) \partial_t f_1\left(2\right) + f_1\left(2\right) \partial_t f_1\left(1\right) + \partial_t g\left(1,2\right) \end{aligned}$$

Term 2:

$$T_2 = f_1\left(2\right) \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1\left(1\right) + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} g\left(1,2\right) + \left\{1 \longleftrightarrow 2\right\}$$

Term 3:

$$T_3 = f_1\left(2\right) \mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} f_1\left(1\right) + \mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} g\left(1,2\right) + \left\{1 \longleftrightarrow 2\right\}$$

Term 4:

$$\begin{aligned} T_4 &= n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} \left\{f_1\left(1\right) f_1\left(2\right) f_1\left(3\right) + f_1\left(1\right) g\left(2,3\right) \right. \\ &\quad \left. + f_1\left(2\right) g\left(1,3\right) + f_1\left(3\right) g\left(1,2\right)\right\} + \left\{1 \longleftrightarrow 2\right\} \end{aligned}$$

where $\left\{1 \longleftrightarrow 2\right\}$ indicates that the same terms as before sould be added with the index 1 and 2 interchanged.

Most terms in the equation for $k = 2$ cancel each other. For instance

$$f_1\left(2\right)\left[\partial_t f_1\left(1\right) + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1\left(1\right) + n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1}\left(f_1\left(1\right) f_1\left(3\right) + g\left(1,3\right)\right)\right] = 0$$

The remaining terms yield:

$$
\begin{aligned}
\frac{\partial g(1,2,t)}{\partial t} &+ (\mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} + \mathbf{v}_2 \cdot \nabla_{\mathbf{x}_2})\, g(1,2,t) \\
&= -(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2})\, [f_1(1,t)\, f_1(2,t) + g(1,2,t)] \\
&\quad -n_0 \int_{-\infty}^{\infty} d3\; (\mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1})\, [f_1(1,t)\, g(2,3,t) + f_1(3,t)\, g(1,2,t)] \\
&\quad + \{1 \longleftrightarrow 2\}
\end{aligned}
$$

This equation and the equation for $f_1$

$$
\frac{\partial f_1(1,t)}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f_1(1,t) + \mathbf{a} \cdot \nabla_{\mathbf{v}_1} f_1(1,t) = -n_0 \int_{-\infty}^{\infty} d2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g(1,2,t)
$$

- => 2 equations for $f_1$ and $g$
- Truncation ignores three-particle correlations (3 body collsions)
- Derivation requires $N_0 \gg 1$
- Equations are almost exact (approximations are usually very well satisfied.

### 5.2.4 Scaling of the BBGKY hierarchy

Recall

$$
\begin{aligned}
\frac{\partial f_k}{\partial t} + \sum_{i=1}^{k} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_k + \sum_{i=1}^{k}\sum_{j=1}^{k} \mathbf{a}_{ij} \cdot \nabla_{\mathbf{v}_i} f_k & \\
+ \frac{N_0 - k}{V} \sum_{i=1}^{k} \int_{-\infty}^{\infty} d\mathbf{x}_{k+1} d\mathbf{v}_{k+1} \mathbf{a}_{i,k+1} \cdot \nabla_{\mathbf{v}_i} f_{k+1} &= 0
\end{aligned}
$$

Assume Yukawa type potential for individual charges with minimum at $\phi_0$ at distance $R$

Normalization

$$
\begin{aligned}
\phi(r) = \phi_0 \varphi(\xi) &\qquad r = R\xi \\
v = v_{th} u &\qquad t = \tau_0 \tau \\
\tau_0 = \frac{R}{v_{th}} &\qquad a_{ij} = -\frac{1}{m} \nabla_{\mathbf{x}} \phi_{ij} = -\frac{\phi_0}{mR} \nabla_{\xi} \varphi = \frac{\phi_0}{mR} \tilde{a}_{ij}
\end{aligned}
$$

Multiply equation with $\tau_0$ (and note $f_{k+1} \sim f_k / v_{th}^3$

$$
\begin{aligned}
\frac{\partial f_k}{\partial t} + \sum_i \mathbf{u}_i \cdot \nabla_{\xi_i} f_k + \frac{R}{v_{th}^2} \frac{\phi_0}{mR} \sum_{i,j} \tilde{\mathbf{a}}_{ij} \cdot \nabla_{\mathbf{v}_i} f_k & \\
+ n_0 \frac{R}{v_{th}} \frac{\phi_0}{mR} R^3 \sum_i \int_{-\infty}^{\infty} d\xi_{k+1} d\mathbf{u}_{k+1} \tilde{\mathbf{a}}_{i,k+1} \cdot \nabla_{\mathbf{u}_i} \tilde{f}_{k+1} &= 0
\end{aligned}
$$

- where $k \ll N_0$ is assumed.
- Coefficient for the third term: $c_3 = \phi_0 / \left(m v_{th}^2\right) = \alpha$
- Coefficient for the fourth term: $c_4 = n_0 R^3 \phi_0 / \left(m v_{th}^2\right) = \alpha\beta$

**Special cases:**

a) Knudsen gas (rarified gas)

$\alpha = \phi_0 / \left(k_B T\right) = O(1)$

$\beta = n_0 R^3 \ll 1$ => Boltzmann equation

b) Weak interaction

$\alpha \ll 1$ , $\beta \leq O(1)$ => Landau equation

c) Plasma case

$\alpha = \phi_0 / \left(k_B T\right) \ll 1, \beta \sim 1/\alpha$ => Lenard-Balescu equation.

## 5.3 Lenard Balescu Equation

### 5.3.1 Bogoliubov's Hypothesis

**Motivation:** Simplify equation for $f = f_1$ and two particle correlations $g$.

Consider the basic case of a homogeneous plasma:

$$\begin{aligned} f\left(\mathbf{x}, \mathbf{v}, t\right) &= f\left(\mathbf{v}, t\right) \\ g\left(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t\right) &= g\left(\left|\mathbf{x}_1 - \mathbf{x}_2\right|, \mathbf{v}_1, \mathbf{v}_2, t\right) \\ \text{and} \quad \left\langle \mathbf{a}_{1,2} \right\rangle_f &= n_0 \int_{-\infty}^{\infty} d\mathbf{x}_2 d\mathbf{v}_2\, f\left(2, t\right) \mathbf{a}_{1,2} = 0 \end{aligned}$$

which yields

$$\frac{\partial f_1\left(\mathbf{v}_1, t\right)}{\partial t} = -n_0 \int_{-\infty}^{\infty} d\mathbf{x}_2 d\mathbf{v}_2\, \mathbf{a}_{1,2} \cdot \nabla_{\mathbf{v}_1} g\left(\left|\mathbf{x}_1 - \mathbf{x}_2\right|, \mathbf{v}_1, \mathbf{v}_2, t\right)$$

$$\begin{aligned} \frac{\partial g\left(\Delta\mathbf{x}_{12}, \mathbf{v}_1, \mathbf{v}_2, t\right)}{\partial t} &+ \left(\mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} + \mathbf{v}_2 \cdot \nabla_{\mathbf{x}_2}\right) g\left(1,2\right) \\ &+ \left(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2}\right) g\left(1,2\right) \\ &+ n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} f\left(1\right) g\left(2,3\right) + n_0 \int_{-\infty}^{\infty} d3\, \mathbf{a}_{2,3} \cdot \nabla_{\mathbf{v}_2} f\left(2\right) g\left(1,3\right) \\ &= -\left(\mathbf{a}_{12} \cdot \nabla_{\mathbf{v}_1} + \mathbf{a}_{21} \cdot \nabla_{\mathbf{v}_2}\right) f\left(1\right) f\left(2\right) \end{aligned}$$

Note terms: $\int_{-\infty}^{\infty} d3\, \mathbf{a}_{1,3} \cdot \nabla_{\mathbf{v}_1} f_1\left(3\right) g\left(1,2\right) = 0$ because of $\left\langle \mathbf{a}_{1,2} \right\rangle_f = 0$

Pulverasition $g/f_1 \sim \Lambda$, $a_{12} \sim e^2/m \sim \Lambda^{-1}$ => term 3 $= O\left(\Lambda^{-2}\right)$; other terms $= O\left(\Lambda^{-1}\right)$

Or normalization: Term 3 $= O\left(\Lambda^{-1}\right)$

=>

$$\begin{aligned}
\partial_t f_1\left(\mathbf{v}_1,t\right) &= -n_0\int_{-\infty}^{\infty} d\mathbf{x}_2 d\mathbf{v}_2\,\mathbf{a}_{1,2}\cdot\nabla_{\mathbf{v}_1} g\left(\left|\mathbf{x}_1-\mathbf{x}_2\right|,\mathbf{v}_1,\mathbf{v}_2,t\right)\\
\partial_t g\left(1,2\right)+V_1 g+V_2 g &= S
\end{aligned}$$

with

$$\begin{aligned}
V_1 g &= \mathbf{v}_1\cdot\nabla_{\mathbf{x}_1} g\left(1,2\right)+\left(n_0\int_{-\infty}^{\infty} d3\, g\left(2,3\right)\mathbf{a}_{1,3}\right)\cdot\nabla_{\mathbf{v}_1} f\left(1\right)\\
V_2 g &= \mathbf{v}_2\cdot\nabla_{\mathbf{x}_2} g\left(1,2\right)+\left(n_0\int_{-\infty}^{\infty} d3\, g\left(1,3\right)\mathbf{a}_{2,3}\right)\cdot\nabla_{\mathbf{v}_2} f\left(2\right)\\
S\left(\Delta\mathbf{x}_{12},\mathbf{v}_1,\mathbf{v}_2,t\right) &= \left(\mathbf{a}_{12}\cdot\nabla_{\mathbf{v}_1}+\mathbf{a}_{21}\cdot\nabla_{\mathbf{v}_2}\right) f\left(1\right) f\left(2\right)
\end{aligned}$$

**Bogoliubov's hierarchy of time scales:**

$\tau_{diff}$ - Diffusion time (macroscopic),e.g., heat conduction or magnetic diffusion

$\ll \tau_{hydr}$ - Macroscopic dynamica time $\quad \tau_{hydr} = L/c_s$

$\ll \tau_{kin}$ Relaxation time of the one-particle distribution $\tau_{kin} \approx L_c/v_{the} = 1/\nu_{coll}$

$\ll \tau_{corr}$ Relaxation time of the correlation function, $\tau \approx \lambda_D/v_{th} \approx 1/\omega_{pe}$

**Hypothesis:** => $f_1(1,t)$ varies slow compared to $g$

($g$ relaxes fast)

=> Source term $S$ can be treated time independent

**Procedure to derive the Lenard-Balescu equation:**

Equation for $g\left(1,2,t\right)$ is a linear equation

=> Fourier transformation to solve for $g\left(1,2,t\right)$.

Need $g\left(t\to\infty\right)$ for the equation for $f$ because $g$ relaxes fast

=> Laplace transformation and substitution in equation

Fourier transf.:

$$\begin{aligned}
\tilde{f}\left(\mathbf{k}\right) &= \frac{1}{\left(2\pi\right)^3}\int d^3x\,\exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) f\left(\mathbf{x}\right)\\
f\left(\mathbf{x}\right) &= \int d^3k\,\exp\left(i\mathbf{k}\cdot\mathbf{x}\right)\tilde{f}\left(\mathbf{k}\right)
\end{aligned}$$

Laplace transf.:

$$\begin{aligned}
\overline{f}\left(\omega\right) &= \int_0^{\infty} dt\,\exp\left(-i\omega t\right) f\left(t\right)\\
f\left(t\right) &= \int_L \frac{d\omega}{2\pi}\exp\left(-i\omega t\right)\overline{f}\left(\omega\right)
\end{aligned}$$

with $\mathbf{x}$, $\mathbf{k}$, and $t$ along the real axis and $\omega$ along a suitable contour $L$.

Example: Acceleration

$$\mathbf{a}_{12}(\mathbf{x}) = \frac{e^2}{4\pi\epsilon_0 m_e |\mathbf{x}|^3}\mathbf{x}$$

has the Fourier transf.:

$$\begin{aligned} \tilde{\mathbf{a}}_{12}(\mathbf{k}) &= -\frac{i\mathbf{k}}{m_e}\varphi(\mathbf{k}) \\ \varphi(\mathbf{k}) &= \frac{e^2}{8\pi^3\epsilon_0\mathbf{k}^2} \end{aligned}$$

Solution to the kinetic equation: **Lenard-Balescu equation**

$$\begin{aligned} \frac{\partial f(\mathbf{v},t)}{\partial t} = & -\frac{8\pi^4 n_0}{m_e^2}\nabla_{\mathbf{v}}\cdot\int d^3k d^3v'\mathbf{k}\mathbf{k}\cdot\frac{\varphi^2(\mathbf{k})}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} \\ & \delta(\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}'))\left[(\nabla_{\mathbf{v}'}-\nabla_{\mathbf{v}})f(\mathbf{v})f(\mathbf{v}')\right] \end{aligned}$$

with the dielectric function

$$\epsilon(\mathbf{k},\omega) = 1 + \frac{\omega_e^2}{\mathbf{k}^2}\int d^3v\frac{\mathbf{k}\cdot\nabla_{\mathbf{v}}f(\mathbf{v})}{\omega-\mathbf{k}\cdot\mathbf{v}}$$

Assumptions:

- homogenous plasma
- 3 particle correlations negligible
- 2 particle correlation relaxes much faster than $f$
- “stable plasma”, dynamic processes with frequency $\approx \omega_{pe}$ excluded.

### 5.3.2 Discussion of the Lenard-Balescu equation

**(a) Problem with the equation**

Large $k$ (LB) $\sim \int \frac{dk}{k} \sim \ln k$ => (LB) diverges for $k \to \infty$

=> $k < 2\pi/\lambda_L$ $\lambda_L$- Landau length $\frac{e^2}{\lambda_L} = k_B T$

or $k < 2\pi/\lambda_{dB}$ $\lambda_{dB}$– deBroglie length $\lambda_{dB} = \frac{h}{mv_{th}}$

Why does (LB) diverge?

Assumption $|g| \ll |f_1 f_1|$ => simplification of (B2)

But: if two electron are very close => $a_{12}$ very large

=> Incorrect to assume $|g| \ll |f_1 f_1|$ for small $|\mathbf{x}_1 - \mathbf{x}_2|$

**(b) Properties**

- $f \geq 0$ at $t = 0$ => $f \geq 0$ at all times
- Particle are conserved:

$$\frac{d}{dt}\int d^3 v f\left(\mathbf{v}, t\right) = 0$$

- Momentum is conserved:

$$\frac{d}{dt}\int d^3 v \mathbf{v} f\left(\mathbf{v}, t\right) = 0$$

- Kinetic energy is conserved:

$$\frac{d}{dt}\int d^3 v \mathbf{v}^2 f\left(\mathbf{v}, t\right) = 0$$

- Any Maxwellian is a time independent solution
- As $t \to \infty$ any $f$ satisfying (a) approaches a Maxwellian.

**(c) Further simplification of the LB equation**

Re-write LB:

$$\begin{aligned} \frac{\partial f\left(\mathbf{v}, t\right)}{\partial t} &= -\nabla_{\mathbf{v}} \cdot \int d^3 v' \underline{\underline{\mathbf{Q}}}\left(\mathbf{v}, \mathbf{v}'\right) \cdot \left[\left(\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}'}\right)\right] f\left(\mathbf{v}\right) f\left(\mathbf{v}'\right) \\ &\text{with} \\ \underline{\underline{\mathbf{Q}}}\left(\mathbf{v}, \mathbf{v}'\right) &= -\frac{8\pi^4 n_0}{m_e^2}\int d^3 k \frac{\mathbf{k}\mathbf{k}\,\varphi^2\left(\mathbf{k}\right)}{\left|\epsilon\left(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}\right)\right|^2}\delta\left(\mathbf{k}\cdot\left(\mathbf{v} - \mathbf{v}'\right)\right) \end{aligned}$$

with

$$\varphi\left(\mathbf{k}\right) = \frac{e^2}{8\pi^3 \mathbf{k}^2}$$

and the Dielectric function

$$\epsilon\left(\mathbf{k}, \omega\right) = 1 + \frac{\omega_e^2}{k^2}\int d^3\tilde{v}\frac{\mathbf{k}\cdot\nabla_{\tilde{\mathbf{v}}} f\left(\tilde{\mathbf{v}}\right)}{\omega - \mathbf{k}\cdot\tilde{\mathbf{v}}}$$

or

$$\begin{aligned} \epsilon\left(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}\right) &= 1 + \frac{\psi}{k^2\lambda_{de}^2} \\ &\text{with} \\ \psi\left(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}\right) &= v_{the}^2\int d^3\tilde{v}\frac{\mathbf{k}\cdot\nabla_{\tilde{\mathbf{v}}} f\left(\tilde{\mathbf{v}}\right)}{\mathbf{k}\cdot\left(\mathbf{v} - \tilde{\mathbf{v}}\right)} \end{aligned}$$

such that

$$\underline{\underline{\mathbf{Q}}}\left(\mathbf{v}, \mathbf{v}'\right) = -\frac{n_0 e^4}{8\pi^2\epsilon_0^2 m_e^2}\int d^3 k \frac{\mathbf{k}\mathbf{k}}{k^4}\frac{\delta\left(\mathbf{k}\cdot\left(\mathbf{v} - \mathbf{v}'\right)\right)}{\left|1 + \frac{\psi}{k^2\lambda_{de}^2}\right|^2}$$

Notes:

- $\psi$ depends only on the direction not on the magnitude of $k$
- Max value of $|k|$given by impact parameter $\frac{1}{b} = \frac{4\pi\epsilon_0 mv^2}{e^2}$

**Evaluation of $\underline{\underline{\mathbf{Q}}}\left(\mathbf{v},\mathbf{v}'\right)$:**

Notes on the $\delta$ function

$$\begin{aligned}
\delta\left(f\left(x\right)\right) &= \sum_i \left[\frac{df}{dx}\left(x_i\right)\right]^{-1}\delta\left(x-x_i\right)\\
x_i &= \text{zeros of } f\left(x\right)\\
\int f\left(x\right)\frac{d\delta\left(x-a\right)}{dx}dx &= -\frac{df}{da}\left(a\right)
\end{aligned}$$

Assume: $k_1 \| \mathbf{v}-\mathbf{v}'$ =>

$$Q_{ij} = -\frac{2n_0e^4}{m_e^2}\int dk_1dk_2dk_3\frac{k_ik_j}{k^4}\frac{1}{\left|\mathbf{v}-\mathbf{v}'\right|}\frac{\delta\left(k_1\right)}{\left|1+\frac{\psi}{k^2\lambda_{de}^2}\right|^2}$$

If $i=1$ or $j=1$ => $Q_{ij}=0$

Since $k_1$ integral is trivial assume: $k_2 = k\cos\vartheta$ and $k_3 = k\sin\vartheta$

E.g.:

$$\begin{aligned}
Q_{33} &= -\frac{n_0e^4}{8\pi^2\epsilon_0^2m_e^2\left|\mathbf{v}-\mathbf{v}'\right|}\int_0^{2\pi}d\vartheta\sin^2\vartheta\int_0^{k_0}\frac{dk}{k}\frac{1}{\left|1+\psi/\left(k^2\lambda_{de}^2\right)\right|^2}\\
&= -\frac{n_0e^4}{16\pi^2\epsilon_0^2m_e^2\left|\mathbf{v}-\mathbf{v}'\right|}\int_0^{2\pi}d\vartheta\sin^2\vartheta\frac{\mathrm{Im}\left[\psi\ln\left(1+k_0\lambda_{de}^2/\psi\right)\right]}{\mathrm{Im}\,\psi}
\end{aligned}$$

$\psi$ - order unity, ignore Im => next chapter

$$1+k_0\lambda_{de}^2/\psi = 1+\lambda_{de}^2/\left(b^2\psi\right) = 1+\Lambda^2/\psi \approx \Lambda^2$$

=>

$$Q_{33}\left(\mathbf{v},\mathbf{v}'\right) = Q_{22}\left(\mathbf{v},\mathbf{v}'\right) = -\frac{n_0e^4}{8\pi\epsilon_0^2m_e^2\left|\mathbf{v}-\mathbf{v}'\right|}\ln\Lambda$$

Define $\mathbf{g}=\mathbf{v}-\mathbf{v}'$, $g=\left|\mathbf{v}-\mathbf{v}'\right|$

with $\mathbf{g}\|\mathbf{k}_1$ direction:

$$\underline{\underline{\mathbf{Q}}}\left(\mathbf{v},\mathbf{v}'\right) = -\frac{n_0e^4}{8\pi\epsilon_0^2m_e^2}\ln\Lambda\frac{g^2\underline{\underline{1}}-\mathbf{gg}}{g^3}$$

=> Landau form of $\underline{\underline{\mathbf{Q}}}$

**Derivation of a Fokker-Planck equation**

General Fokker-Planck equation -> Nicholson

Note: $\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}g = \frac{g^2\underline{\underline{1}}-\mathbf{gg}}{g^3}$ and $\nabla_{\mathbf{v}}g = -\nabla_{\mathbf{v}'}g$ and $\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{v}}g = 2/g$ =>

$$\begin{aligned}
\frac{\partial f(\mathbf{v},t)}{\partial t} &= \frac{n_0 e^4}{8\pi\epsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\cdot\int d^3v'\frac{g^2\underline{\underline{1}}-\mathbf{gg}}{g^3}\cdot\left(\nabla_{\mathbf{v}}-\nabla_{\mathbf{v}'}\right)f(\mathbf{v})f(\mathbf{v}') \\
&= \frac{n_0 e^4\ln\Lambda}{8\pi\epsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\cdot\int d^3v'\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}g\right)\cdot\left(\nabla_{\mathbf{v}}-\nabla_{\mathbf{v}'}\right)f(\mathbf{v})f(\mathbf{v}') \\
&= \frac{n_0 e^4\ln\Lambda}{8\pi\epsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\cdot\left\{\left(\nabla_{\mathbf{v}}f(\mathbf{v})\right)\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' g f(\mathbf{v}') - f(\mathbf{v})\int d^3v'\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}g\right)\cdot\nabla_{\mathbf{v}'}f(\mathbf{v}')\right\} \\
&= \frac{n_0 e^4\ln\Lambda}{8\pi\epsilon_0^2 m_e^2}\left\{\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}f(\mathbf{v})\cdot\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' g f(\mathbf{v}') - 2\left(\nabla_{\mathbf{v}}f(\mathbf{v})\right)\cdot\int d^3v'\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\cdot\nabla_{\mathbf{v}}g\right)f(\mathbf{v}')\right\} \\
&= \frac{n_0 e^4\ln\Lambda}{8\pi\epsilon_0^2 m_e^2}\left\{-4\nabla_{\mathbf{v}}\cdot f(\mathbf{v})\nabla_{\mathbf{v}}\int d^3v'\frac{f(\mathbf{v}')}{g} + \nabla_{\mathbf{v}}\nabla_{\mathbf{v}}f(\mathbf{v})\cdot\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v' g f(\mathbf{v}')\right\}
\end{aligned}$$

which yields

$$\frac{\partial f(\mathbf{v},t)}{\partial t} = -\nabla_{\mathbf{v}}\cdot\left[\mathbf{A}f(\mathbf{v})\right] + \frac{1}{2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\cdot\cdot\left[\underline{\underline{\mathbf{B}}}f(\mathbf{v})\right]$$

with

$$\begin{aligned}
\mathbf{A}(\mathbf{v},t) &= \frac{n_0 e^4\ln\Lambda}{2\pi\epsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\int d^3v'\frac{f(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} \\
\underline{\underline{\mathbf{B}}}(\mathbf{v},t) &= \frac{n_0 e^4\ln\Lambda}{4\pi\epsilon_0^2 m_e^2}\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\int d^3v'\,|\mathbf{v}-\mathbf{v}'|\,f(\mathbf{v}')
\end{aligned}$$

this is the Landau form of the Fokker-Planck equation.

Term $\mathbf{A} \hookrightarrow$ slowing of particles by many small angle collisions

Term $\underline{\underline{\mathbf{B}}} \hookrightarrow$ increase of the perpendicular velocity from small angle collisions

..Figure..

Approximation to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \nu\nabla_{\mathbf{v}}\left[\left(\mathbf{v}-\mathbf{v}_0\right)f + v_{the}^2\nabla_{\mathbf{v}}f\right]$$

where $\nu$ is a collision frequency and $v_0$ is a constant velocity. This is usually rather crude and only provides a rough idea of collisional effects.

Even more basic: Krook model

$$\frac{\partial f}{\partial t} = \nu\left(f - f_0\right)$$

which only describes the relaxation of any distribution function to $f_0$.