Chapter 11

Grid Generation

11.1 Nonuniform Grid - Motivation

It as shown that the state one-dimensional steady transport equation

$$u\frac{\partial f}{\partial x} - \alpha\frac{\partial^2 f}{\partial x^2} = 0$$

had solution of the form

$$f(x) = \frac{\exp(ux/\alpha) - 1}{\exp(u/\alpha) - 1}$$

which illustrate the presence of a thin boundary layer of width $L_\alpha = \alpha/u$. Simlarly other fluid or plasma structures lead to the formation of thin boundary layers. Frequently much of the dynamical evolution outside of these boundary layers does not require the same high resolution. However, a uniform grid would place the same high resolution everywhere independent of whether this resolution is needed everywhere or not. Therefore it is desirable to employ grids that use high resolution only inregions where this is needed and use a coarser grid inregions outside of such boundary layers.

11.1.1 Basic properties of nonuniform grids

Example: First derivative - general method

Consider a nonuniform grid with

$$\Delta x_j = x_j - x_{j-1} \qquad \Delta x_{j+1} = x_{j+1} - x_j$$
$$and \qquad r_j = \frac{\Delta x_{j+1}}{\Delta x_j}$$

Note that a unifrom grid implies $r_j = 1$. Consider a 3 point representation of

$$\frac{df}{dx} = af_{j-1} + bf_j + cf_{j+1}$$

with $f_{j-1} = f\left(x_j - \Delta x_j\right)$ and $f_{j+1} = f\left(x_j + \Delta x_{j+1}\right)$.
Taylor expansion

$$\begin{aligned}
\frac{df}{dx} &= (a+b+c)\, f|_j + \left(-a\Delta x_j + c\Delta x_{j+1}\right) \left.\frac{df}{dx}\right|_j \\
&+ \frac{1}{2}\left(a\Delta x_j^2 + c\Delta x_{j+1}^2\right) \left.\frac{d^2 f}{dx^2}\right|_j \\
&+ \frac{1}{6}\left(-a\Delta x_j^3 + c\Delta x_{j+1}^3\right) \left.\frac{d^3 f}{dx^3}\right|_j + ..
\end{aligned}$$

Evalution of the coefficients

$$\begin{aligned}
a+b+c &= 0 \\
-a+rc &= 1/\Delta x_j \\
a+r^2c &= 0
\end{aligned}$$

With the solution

$$\begin{aligned}
c &= \frac{1}{r(1+r)\Delta x_j} = \frac{1}{r}\frac{1}{\Delta x_j + \Delta x_{j+1}} \\
a &= -\frac{r}{(1+r)\Delta x_j} = -r\frac{1}{\Delta x_j + \Delta x_{j+1}} \\
b &= \left(r - \frac{1}{r}\right)\frac{1}{\Delta x_j + \Delta x_{j+1}}
\end{aligned}$$

with the error

$$\begin{aligned}
\frac{1}{6}\left(-a\Delta x_j^3 + c\Delta x_{j+1}^3\right)\left.\frac{d^3 f}{dx^3}\right|_j &= \frac{1}{6}\frac{1}{\Delta x_j + \Delta x_{j+1}}\left(r\Delta x_j^3 + \frac{1}{r}\Delta x_{j+1}^3\right)\left.\frac{d^3 f}{dx^3}\right|_j \\
&= \frac{1}{6}\frac{\Delta x_j \Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}}\left(\Delta x_j + \Delta x_{j+1}\right)\left.\frac{d^3 f}{dx^3}\right|_j \\
&= \frac{1}{6}\Delta x_j \Delta x_{j+1}\left.\frac{d^3 f}{dx^3}\right|_j
\end{aligned}$$

Thus the three point approximation of the first derivative is

$$\frac{1}{\Delta x_j + \Delta x_{j+1}} \left[\frac{1}{r} \left(f_{j+1} - f_j \right) + r \left(f_j - f_{j-1} \right) \right] = \left. \frac{df}{dx} \right|_j - \frac{1}{6} \Delta x_j \Delta x_{j+1} \left. \frac{d^3 f}{dx^3} \right|_j + ..$$

Similarly we can compute the coefficients for the 2nd derivative as

$$\begin{aligned} a &= \frac{2}{\Delta x_j \left(\Delta x_j + \Delta x_{j+1} \right)} \\ b &= \frac{2}{\Delta x_j \Delta x_{j+1}} \\ c &= \frac{2}{\Delta x_{j+1} \left(\Delta x_j + \Delta x_{j+1} \right)} \end{aligned}$$

which yields the approximation:

$$\frac{2}{\Delta x_j \left(\Delta x_j + \Delta x_{j+1} \right)} \left[\frac{1}{r} \left(f_{j+1} - f_j \right) - \left(f_j - f_{j-1} \right) \right] = \left. \frac{d^2 f}{dx^2} \right|_j - \frac{1}{3} \Delta x_j (r-1) \left. \frac{d^3 f}{dx^3} \right|_j + ..$$

These result illustrate how one can construct appropriate difference schemes to aproximate derivatives on a non-uniform grid. Note that different from the uniform case the 3 point approximation of the 2nd derivative is not 0. However, if the grid varies smoothly $r_j = \Delta x_{j+1} / \Delta x_j \approx 1$ or $r - 1 \ll 1$.

The 2nd derivative operator computed using the Taylor expansion is actually identical to the on for linear finite elements. Note however the the 1st derivative for linear FEM is

$$\frac{1}{\Delta x_j + \Delta x_{j+1}} \left[f_{j+1} - f_{j-1} \right] = \left. \frac{df}{dx} \right|_j + \frac{1}{2} \Delta x_j (r-1) \left. \frac{d^2 f}{dx^2} \right|_j + ..$$

i.e., it is on 1st order accurate. However, in the case of the 2nd derivative approximation the error is $\sim \Delta x_j (r-1)$ and for $r - 1 \ll 1$ becomes much small than 1st order.

Basic construction of a nonuniform grid

Consider a function $h(j)$ which generates the grid, i.e. $x_j = h(j)$. Thus

$$x_j - x_{j-1} = h(j) - h(j-1) \approx \frac{dh}{dj}$$

The grid has to satisfy the conditions $h(1) = x_{min}$ and $h(n_x) = x_{max}$.

a) Linear grid:

$$\widetilde{x}(j) = x_{min} + \frac{x_{max} - x_{min}}{n_x - 1} (j - 1)$$

b) Nonuniform grid:

$$
\begin{aligned}
x_j &= \widetilde{x}(j) + \widetilde{h}(\widetilde{x}(j)) \\
with \quad & \widetilde{h}(\widetilde{x}_1) = 0 \\
& \widetilde{h}(\widetilde{x}_{nx}) = 0 \\
& d\widetilde{h}/d\widetilde{x} > -1
\end{aligned}
$$

where $\widetilde{h}$ can be any function subject to the above constraints, for instance $\widetilde{h}(\widetilde{x}) = \lambda \sin n\pi x$. Not necessary but helpful is the additional property of $d\widetilde{h}/d\widetilde{x} = 0$ at the boundaries. This simplifies boundary conditions that use specific symmetries.

- Possible to fit any number of grid points.
- The second derivative of $\widetilde{h}$ provides the variation of the grid spacing.
- Controll certain properties of the grid such as a prescribed resolution εat x_{min}.

As outlined before one can use different difference approximation for the grid. Specifically $f = f(x(i))$ such that

$$
\frac{df}{dj} = \frac{df}{dx}\frac{dx}{dj} \quad or \quad \frac{df}{dx} = \left(\frac{dx}{dj}\right)^{-1} \frac{df}{dj} = \left(\frac{dx}{dj}\right)^{-1} \frac{f_{j+1} - f_{j-1}}{2}
$$

11.2 Transformations and transformation parameters

11.3 Generalized coordinate structure of typical equations

11.4 Grid generation