Chapter 3

Classification of PDE's and Related Properties

3.1 Linear Second Order PDE's in two Independent Variables

The most general form of a linear, second order PDE in two independent variables x, y and the dependent variable $u(x,y)$ is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \tag{3.1}$$

with $A,..G = const$. This equation is called $\begin{cases} elliptic\, for & B^2 - 4AC < 0 \\ parabolic\, for & B^2 - 4AC = 0 \\ hyperbolic\, for & B^2 - 4AC > 0 \end{cases}$

There are many linear equations which can be classified with this scheme. For instance the heat conduction equation falls into this category. Another such example are the steady Euler equations for irrotational flow ($\mathbf{u} = \nabla\Phi$) which are of the form

$$(1 - M^2)\frac{\partial^2 \Phi}{\partial s^2} + \frac{\partial^2 \Phi}{\partial n^2} = 0 \tag{3.2}$$

where $M = u^2/c_s^2$ is the so-called Mach number with c_s equal to the speed of sound, s is the coordinate along a stream line and n is the coordinate perpendicular to a stream line. Evaluating the determinant $B^2 - 4AC$ shows that the equation is hyperbolic for supersonic ($M > 1$) flow, parabolic if the flow speed is exactly the sound speed, and elliptic for sub-sonic flow. One also concludes that the problem is always elliptic if the flow is incompressible because in that case the sound speed is infinite such that $M = 0$ always.

In most cases fluid equations with an explicit time dependence are of the hyperbolic type. They may be parabolic if dissipation is present, e.g., the heat conduction equation. However, it should be kept in mind that the present example is rather simple in that it assumes only a single 2nd order linear differential equation with constant coefficients. It is easy to see that realistic problems are more complex and in particular may have coefficients which depend on the independent variables. Realistic problems are also nonlinear which can pose problems for a unique classification. This can make the classification more difficult and in fact the character of a differential PDE for a given system may depend on the location in the system.

Exercise: Demonstrate that the time dependent diffusion equation $\partial T/\partial t = \mu \partial^2 T/\partial x^2$, is parabolic.

Exercise: Show that the time independent diffusion equation in two dimensions with $\mu_1 \partial^2 T/\partial x^2 + \mu_2 \partial^2 T/\partial y^2$ with $\mu_1, \mu_2 > 0$ is elliptic.

3.2 Well Posed Problems:

A mathematical problem is called well posed if

- the solution exists
- the solution is unique
- the solution depends continuously on the auxiliary data (e.g., boundary and initial conditions)

In the same terms one can define a numerical problem as well posed through existence, uniqueness, and the continuous dependence on initial and boundary conditions.

In most cases in particular for numerical modeling the existence of a solution does not pose a problem. If a solution is found whether this is an analytical or a numerical solution one should conclude that a solution exists. Existence may become an issue if for instance an iteration does not lead to convergence. In such cases one possible cause could be the nonexistence of a solution.

Uniqueness can be a serious problem. Non-uniqueness of solutions occurs for instance if the boundary conditions do not match the type or order of the set of differential equations. Non-unique solutions are also common for nonlinear problems similar to the case of algebraic equations. In such cases the numerical model usually will provide only one of several solutions. Typical situations for non unique solutions are phase transitions, bifurcations in nonlinear dynamics, or the transition from sub-sonic to supersonic flow. To deal with non unique solutions can be rather tedious. The solution that a numerical model realizes depends on the method. Certain branches of the solution may have particular properties which can in cases be considered as a constraint to the method. In general non unique solutions require careful consideration and analysis.

The continuous dependence of the solution on auxiliary data is frequently related to the uniqueness of the solution. In numerical models this can pose a problem because a non continuous dependence of the solution on boundary or initial conditions implies that only small errors in those will cause large changes in the solution. Since a numerical method is not arbitrarily exact it is not clear which of the possible solutions is actually assumed. A typical situation of this type occurs in chaotic systems even if the governing equations are deterministic (deterministic chaos). This is a situation where an arbitrarily small change of the initial conditions will lead to a large change of the solution within a finite amount of time. An example is the the location and momentum of balls in pool. Position and velocity of a ball are exactly determined after any collision. However, a small variation of the location or velocity of the incoming ball is amplified after a collision. After about 7 such collisions the final location and velocity of a ball on a pool table is arbitrary because the initial conditions can only be determined subject to quantum rules (Heisenberg's uncertainty relation).

3.3 Boundary and Initial Conditions

There are three types of boundary conditions. Defining a domain R, its boundary ∂R, the coordinates n and s normal (outward) and along the boundary, and functions f, g on the boundary, the three boundary conditions are

- Dirichlet conditions with $u = f$ on ∂R.
- Neumann conditions with $\partial u/\partial n = f$ or $\partial u/\partial s = g$ on ∂R.
- Mixed (Robin) conditions $\partial u/\partial n + ku = f$, $k > 0$, on ∂R.

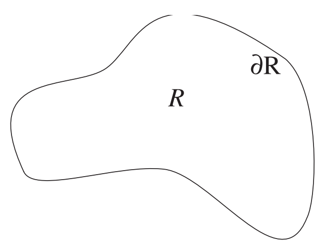

Figure 3.1: Simulation domain R with a boundary ∂R.

Dirichlet conditions can only be applied if the solution is known on the boundary and if f is analytic. These are frequently used for the flow (velocity) into a domain. More common are Neumann conditions.

3.4 Classification Through Characteristics

3.4.1 First order equation

Characteristics are curves in the space of the independent variables along which the governing PDE has only total differentials. Let us consider the first order equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = c \tag{3.3}$$

and seek solutions of the form $u = u(x(t), t)$. The total derivative of u is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} \tag{3.4}$$

This allows us to identify $dx/dt = b/a$ and $du/dt = c/a$ with the solutions

$$\begin{aligned} x &= x_0 + \frac{b}{a}t \\ u &= u_0 + \frac{c}{a}t \end{aligned}$$

for the initial condition $u(x_0, 0) = u_0$. Figure 3.2 illustrates that characteristics are lines with constant slope in the x, t plane. The concept becomes very obvious if $c = 0$. In this case $u = const$ on the characteristics. For non-constant c u grows linear in time along a characteristic line. For $t = 0$ one can define a profile $u(x, 0) = f(x)$. At any later time u can be constructed according to the solution on the characteristic curve.

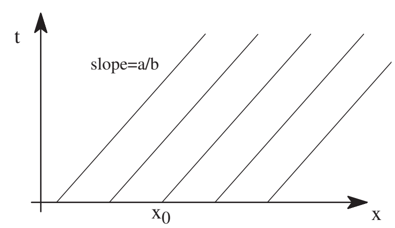

Figure 3.2: Illustration of the characteristics for (3.3)

It is worth noting that the partial derivative of (3.3) with respect to t gives

$$a\frac{\partial^2 u}{\partial t^2} - \frac{b^2}{a}\frac{\partial^2 u}{\partial x^2} = 0$$

implying that (3.3) is hyperbolic. It should be noted the the above equation has to roots such that differentiating (3.3) generated a new artificial solution in addition to the original solution.

3.4.2 Second order equations

Consider the PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + H = 0 \tag{3.5}$$

As above we attempt to solve this equation through a total differential i.e.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Thus we define $P = \frac{\partial u}{\partial x}$ and $Q = \frac{\partial u}{\partial y}$ and determine the total differentials

$$\begin{aligned} dP &= Rdx + Sdy \\ dQ &= Sdx + Tdy \end{aligned}$$

- From the definition of P, Q we determine: $R = \frac{\partial^2 u}{\partial x^2}, \quad S = \frac{\partial^2 u}{\partial x \partial y}, \quad T = \frac{\partial^2 u}{\partial y^2}$.
- Evaluating $A\frac{dP}{dx} + C\frac{dQ}{dy}$ yields: $A\frac{\partial^2 u}{\partial x^2} + \left(A\frac{dy}{dx} + C\frac{dx}{dy}\right)\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$
- Comparison with (3.5) gives: $\left(A\frac{dy}{dx} + C\frac{dx}{dy}\right) - B = 0$ or

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$$

with the solutions

$$\frac{dy}{dx} = \frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC} \tag{3.6}$$

The discriminant $B^2 - 4AC$ determines the number of real solutions. The solutions of dy/dx are called characteristics. Comparison with the earlier definition for the classification leads to a unique identification of the type of PDE with the number of real characteristic solutions:

$$\begin{cases} Hyperbolic\,PDE: & B^2 - 4AC > 0, \quad 2\,real\,solutions \\ Parabolic\,PDE: & B^2 - 4AC = 0, \quad 1\,real\,solution \\ Elliptic\,PDE: & B^2 - 4AC < 0, \quad 0\,real\,solution \end{cases}$$

The constant H can be considered in the solution for the total differentials.

For the case of a time dependent compressible fluid without dissipation the governing set of PDE's is hyperbolic with two real characteristics. In this case the characteristics are the lines in space time along which information propagates (sound waves). Considering the changes at a fixed location in a system over a small time interval implies that any changes are originating from the immediate vicinity of this point. In contrast any small change of the boundary conditions for an elliptic PDE effect the entire system.

Equation (3.6) provides the curves along which one has to propagate the values of P and Q to find a solution to the PDE. This solution can be constructed from any boundary along which P, Q, and u are specified. To do so one has to solve the equations

$$\begin{aligned} du &= Pdx + Qdy \\ A\frac{dP}{dx} + C\frac{dQ}{dy} + H &= 0 \end{aligned}$$

or as finite differences

$$\begin{aligned} \Delta u &= P\Delta x + Q\Delta y \\ A\frac{\Delta P}{\Delta x} + C\frac{\Delta Q}{\Delta y} + H &= 0 \end{aligned}$$

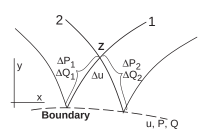

Figure 3.3: Illustration of the construction of the solution to a simple hyperbolic PDE using the characteristics.

For any intersection point Z of two characteristics close to a line where P, Q, and u are specified, the above equations provide two equations for ΔP and ΔQ from the boundary to the intersection. Thus it is possible to compute the values of P and Q at the point of the intersection. Using P and Q one can compute Δu and thus u at the point of the intersection. This method can be employed to construct the solution u to a hyperbolic PDE.

Exercise: Compute the characteristics of the wave equation $a\frac{\partial^2 u}{\partial t^2} - \frac{b^2}{a}\frac{\partial^2 u}{\partial x^2} = 0$ and show that there is an additional characteristic to the first order equation (3.3).

3.4.3 Coordinate transformations

An important question is whether a coordinate transformation can change the character of a differential equation. Obviously this should not be the case because the type of a PDE should be a characteristic property of a PDE with real implications (i.e., determine the type of solutions) and thus the type of a PDE ought to be invariant against coordinate transformation.

Assuming a coordinate transformation $\xi(x,y)$, $\eta(x,y)$ and $\widetilde{u}(\xi(x,y),\eta(x,y)) = u(x,y)$ equation (3.5) is transformed into

$$A'\frac{\partial^2\widetilde{u}}{\partial\xi^2} + B'\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta} + C'\frac{\partial^2\widetilde{u}}{\partial\eta^2} + H' = 0 \tag{3.7}$$

with

$$\begin{aligned} A' &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ B' &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ C' &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \end{aligned}$$

Evaluating the discriminant yields: $(B')^2 - 4A'C' = J^2[B^2 - 4AC]$

where J is the Jacobian of the transformation: $J = \xi_x\eta_y - \xi_y\eta_x$

In this formulation we have introduced the notation $\xi_x \equiv \partial\xi/\partial x$, $\xi_y \equiv \partial\xi/\partial x$, etc.

Since the Jacobian squared is always a positive number, the transformation does not alter the sign of the discriminant.

Exercise: Derive the transformed equation and coefficients A', B', and C' for the linear transformation $\xi = \alpha_1 x + \beta_1 y$ and $\xi = \alpha_2 x + \beta_2 y$. Show that the type of the PDE has not changed.

Exercise: Derive the transformed equation and coefficients A', B', and C' for a general transformation. Show that the type of the PDE has not changed.

3.4.4 Multidimensional equations

Multiple independent variables

Let us consider now a single dependent variable u but m independent variables. Similar to (3.5) the corresponding second order differential equation is

$$\sum_{j=1}^{m}\sum_{k=1}^{m} a_{jk}\frac{\partial^2 u}{\partial x_j\partial x_k} + H = 0 \tag{3.8}$$

In this multidimensional case the character of the differential equation is determined by the eigenvalues (EV) of the matrix $\underline{\underline{\mathbf{A}}} = (a_{jk})$.

The equation is $\begin{cases} parabolic & if\, any\, EV\, is\, 0 \\ elliptic & if\, all\, EVs\, are \neq 0\, and\, of\, the\, same\, sign \\ hyperbolic & if\, all\, EVs\, are \neq 0\, and\, all\, but\, one\, have\, the\, same\, sign \end{cases}$

Note that there are cases (e.g., if all Eigenvalues (EV's) are nonzero but several have a different sign) where a classification is not straightforward.

Systems of equations - two dependent variables

Thus far we have considered only a single dependent variable. Frequently this is not the case such that one should consider methods for classification which can be applied to systems of PDE's. Let us first consider the case of

$$A_{11}\frac{\partial u}{\partial x}+B_{11}\frac{\partial u}{\partial y}+A_{12}\frac{\partial v}{\partial x}+B_{12}\frac{\partial v}{\partial y} = 0 \tag{3.9}$$

$$A_{21}\frac{\partial u}{\partial x}+B_{21}\frac{\partial u}{\partial y}+A_{22}\frac{\partial v}{\partial x}+B_{22}\frac{\partial v}{\partial y} = 0 \tag{3.10}$$

Note that a single second order equation can be brought into a system of linear equations. In a compact form these equations can be expressed as

$$\begin{aligned} \underline{\underline{\mathbf{A}}}\cdot\frac{\partial \mathbf{q}}{\partial x}+\underline{\underline{\mathbf{B}}}\cdot\frac{\partial \mathbf{q}}{\partial y} &= 0 \\ with \quad \mathbf{q} &= \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

We seek a solution again in terms of a total differential of the form
$m_1 du + m_2 dv = m_1\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+m_2\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)$.
Multiplying (3.9) with L_1 and (3.10) with L_2 we obtain for the coefficients of the derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$:

$$\begin{aligned} L_1A_{11}+L_2A_{21} &= m_1 dx \\ L_1B_{11}+L_2B_{21} &= m_1 dy \\ L_1A_{12}+L_2A_{22} &= m_2 dx \\ L_1B_{12}+L_2B_{22} &= m_2 dy \end{aligned}$$

Eliminating the rhs terms by multiplying these equation with dx or dy we obtain:

$$\begin{aligned} (L_1A_{11}+L_2A_{21})dy &= (L_1B_{11}+L_2B_{21})dx \\ (L_1A_{12}+L_2A_{22})dy &= (L_1B_{12}+L_2B_{22})dx \end{aligned}$$

or in vector form

$$\left(\underline{\underline{\mathbf{A}}}^t dy-\underline{\underline{\mathbf{B}}}^t dx\right)\cdot\mathbf{L}=0 \tag{3.11}$$

with the transposed matrices $\underline{\underline{\mathbf{A}}}^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$ and $\underline{\underline{\mathbf{B}}}^t = \begin{pmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{pmatrix}$. A nontrivial solution to (3.11) requires that the determinant of the coefficient matrix is 0, i.e., $det\left(\underline{\underline{\mathbf{A}}}^t dy - \underline{\underline{\mathbf{B}}}^t dx\right) = 0$. As in the previous second order PDE case this yields a quadratic equation for dy/dx of the form

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0 \tag{3.12}$$

where the number and type of solutions is determined by the discriminant $D = B^2 - 4AC$ with the result

$$\begin{cases} D > 0 & hyperbolic \\ D = 0 & parabolic \\ D < 0 & elliptic \end{cases}$$

Exercise: Show that

$$\begin{aligned} \det\left(\underline{\underline{\mathbf{A}}}^t \frac{dy}{dx} - \underline{\underline{\mathbf{B}}}^t\right) &= (A_{11}A_{22} - A_{12}A_{21})\left(\frac{dy}{dx}\right)^2 \\ &\quad - (A_{11}B_{22} - A_{21}B_{12} + A_{22}B_{11} - A_{12}B_{21})\left(\frac{dy}{dx}\right) \\ &\quad + (B_{11}B_{22} - B_{12}B_{21}) \end{aligned}$$

Exercise: Demonstrate that the discriminant for the above expression is

$$D = (A_{11}B_{22} - A_{21}B_{12} + A_{22}B_{11} - A_{12}B_{21})^2 - 4(A_{11}A_{22} - A_{12}A_{21})(B_{11}B_{22} - B_{12}B_{21})$$

Exercise: Show that $det\left(\underline{\underline{\mathbf{A}}}dy - \underline{\underline{\mathbf{B}}}dx\right) = 0$ yields the same solution as $det\left(\underline{\underline{\mathbf{A}}}^t dy - \underline{\underline{\mathbf{B}}}^t dx\right) = 0$.

Example: Let us consider the example of two-dimensional steady compressible potential flow. Potential flow implies that $\nabla \times \mathbf{u} = 0$ for the velocity component $\mathbf{u} = (u, v)$. The equations are

$$\left(\frac{u^2}{a^2} - 1\right)\frac{\partial u}{\partial x} + \left(\frac{uv}{a^2}\right)\frac{\partial u}{\partial y} + \left(\frac{uv}{a^2}\right)\frac{\partial v}{\partial x} + \left(\frac{v^2}{a^2} - 1\right)\frac{\partial v}{\partial y} = 0 \tag{3.13}$$

$$-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{3.14}$$

Exercise: Derive the above equations from (2.18), (2.19), and (2.20).

Evaluating the matrices in (3.11) and the determinant yields the coefficients $A = \frac{u^2}{a^2} - 1$, $B = 2\frac{uv}{a^2}$, and $C = \frac{v^2}{a^2} - 1$. The discriminant for these is $D = 4\left(\frac{u^2+v^2}{a^2} - 1\right)$. Here the Mach number is $M = \frac{u^2+v^2}{a^2}$. Thus the solution is hyperbolic for $M > 0$, and it is elliptic for $M < 0$.

Systems of n first order partial differential equations

In general the approach used in the case with 2 dependent variables can be applied to n equations resulting in n×n matrices. The equation to be solved assumes the form

$$\left(\underline{\underline{\mathbf{A}}}^t \frac{dy}{dx} - \underline{\underline{\mathbf{B}}}^t\right) \cdot \mathbf{L} = 0 \tag{3.15}$$

for a vector $\mathbf{L}$ of dimension n, and $\underline{\underline{\mathbf{A}}}$, $\underline{\underline{\mathbf{B}}}$ are n x n matrices. The determinant leads to an nth order equation for dy/dx. In this case the type of differential equation is

$$\left\{ \begin{array}{lcc} hyperbolic & \text{for n real roots} & \\ parabolic & f\text{or 1 to n} - \text{1 real roots and} & \text{no complex roots} \\ elliptic & \text{for no real roots} & \end{array} \right. \tag{3.16}$$

c) More than two independent variables:

For the most general case one has to consider more than two independent variables. For instance a steady state three-dimensional problem requires 3 variables x, y, and z. In the time dependent case the system is four-dimensional. In the case of 3 independent variables the governing equations are

$$\underline{\underline{\mathbf{A}}} \cdot \frac{\partial \mathbf{q}}{\partial x} + \underline{\underline{\mathbf{B}}} \cdot \frac{\partial \mathbf{q}}{\partial y} + \underline{\underline{\mathbf{C}}} \cdot \frac{\partial \mathbf{q}}{\partial z} = 0$$

with the vector $\mathbf{q} = (u, v, w, ..)$. The solution to the characteristics now requires surfaces rather than curves. For a three-dimensional problem (3 independent variables) this implies two dimensional surfaces. For the general case of m independent variables the characteristic surfaces ore m-1 dimensional. At any point these surfaces can be described by the normal direction in m dimensions. The solution for the characteristics is now given by

$$det\left(\underline{\underline{\mathbf{A}}}^t \lambda_x + \underline{\underline{\mathbf{B}}}^t \lambda_y + \underline{\underline{\mathbf{C}}}^t \lambda_z\right) = 0$$

where $\lambda = (\lambda_x, \lambda_y, \lambda_z)$ gives the direction normal to the characteristic surface.

Example: Transform a higher order equation into a system of first order PDE's. Consider the two-dimensional incompressible Navier-Stokes equations. Note that incompressibility requires $\nabla \cdot \mathbf{u} = 0$. The equations are given by

$$u_x + v_y = 0 \tag{3.17}$$

$$uu_x + vu_y + p_x - \frac{1}{R_e}(u_{xx} + u_{yy}) = 0 \tag{3.18}$$

$$uv_x + vv_y + p_y - \frac{1}{R_e}(v_{xx} + v_{yy}) = 0 \tag{3.19}$$

Here we use the notation $u_x \equiv \partial u/\partial x$. The variables are the two components of the velocity $\mathbf{u} = (u,v)$ and the pressure p and R_e is the Reynolds number. Introducing auxiliary variables $R = v_x$, $S = v_y$, and $T = u_y$ the Navier-Stokes equations can be expressed as

$$\begin{array}{cccccccc}
u & v & R & S & T & p & & \\
u_y & & & & & & = & T \\
u_x & +v_y & & & & & = & 0 \\
 & & -R_y & +S_x & & & = & 0 \\
 & & & S_y & +T_x & & = & 0 \\
 & & & S_x/R_e & -T_y/R_e & +p_x & = & uS - vT \\
 & & -R_x/R_e & -S_y/R_e & & +p_y & = & uR - vS
\end{array}$$

where the top row marks the variable used in the corresponding column. Note that there is no additional auxiliary variable for u_x because $u_x = -v_y = -S$. Formally one can now replace x derivatives with a multiplier λ_x, y derivative with a multiplier λ_y, and so on. The results in the following determinant

$$det \begin{pmatrix}
\lambda_y & 0 & 0 & 0 & 0 & 0 \\
\lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_y & -\lambda_x & 0 & 0 \\
0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\
0 & 0 & 0 & \frac{1}{R_e}\lambda_x & -\frac{1}{R_e}\lambda_y & \lambda_x \\
0 & 0 & -\frac{1}{R_e}\lambda_x & -\frac{1}{R_e}\lambda_y & 0 & \lambda_y
\end{pmatrix} = 0$$

leading to

$$\frac{1}{R_e}\lambda_y^2\left(\lambda_x^2+\lambda_y^2\right)^2 = 0$$

The vector $\underline{\Lambda} = (\lambda_x, \lambda_y)$ determines the direction normal to the characteristic such that only the ratio of $R = \lambda_y/\lambda_x$ is required to determine the type of PDE which yields $R^2\left(R^2+1\right)^2 = 0$ with the roots $R = 0$ and four complex roots with $R^2 = -1$. This is clearly not hyperbolic nor parabolic. Although it has a root at $R = 0$ all other roots are complex such that one would probably call this system elliptic even though it does not fit exactly the criteria of the (3.16).

3.5 Classification Through Fourier Analysis

To illustrate the approach using Fourier analysis let us consider the homogeneous and linear PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + H = 0 \tag{3.20}$$

A frequently used method to solve PDE's is by Fourier analysis where a solution is expanded in a Fourier series. In two dimensions this takes the form

$$u = \frac{1}{4\pi^2} \sum_j \sum_k \widetilde{u}_{jk} \exp\left[i(\sigma_x)_j x\right] \exp\left[i(\sigma_y)_k y\right] \tag{3.21}$$

Substitution in the equation (3.20) yields $A\sigma_x^2 + B\sigma_x\sigma_y + C\sigma_y^2 = 0$ or

$$A(\sigma_x/\sigma_y)^2 + B\sigma_x/\sigma_y + C = 0$$

which gives the same discriminant as the characteristic equation for this PDE. In many cases with non-periodic boundary conditions the use of a Fourier transform may be appropriate.

Her we define the Fourier transform as

$$\widetilde{u}(\sigma_x, \sigma_y) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} u(x,y) \exp(-i\sigma_x x) \exp(-i\sigma_y y)\, dx\, dy$$

or symbolic as $\widetilde{u} = Fu$. It can easily be shown that

$$i\sigma_x = F\tfrac{\partial u}{\partial x}, \quad i\sigma_y = F\tfrac{\partial u}{\partial y}$$

Taking the Fourier transform of (3.20) results in $\left[A(i\sigma_x)^2 + B(i\sigma_x)(i\sigma_y) + C(i\sigma_y)^2\right]\widetilde{u} = 0$, i.e., again the same equation as for the characteristics.

An advantage of the Fourier transform is that it can be directly applied to a system of PDE's. For the case of the steady incompressible Navier-Stokes equations (3.17) to (3.19) the Fourier transform approach yields

$$\begin{bmatrix} i\sigma_x & i\sigma_y & 0 \\ i(u\sigma_x + v\sigma_y) + \frac{1}{R_e}(\sigma_x^2 + \sigma_y^2) & 0 & i\sigma_x \\ 0 & i(u\sigma_x + v\sigma_y) + \frac{1}{R_e}(\sigma_x^2 + \sigma_y^2) & i\sigma_y \end{bmatrix} \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{p} \end{bmatrix} = 0$$

which yields from $det[..] = 0$ the equation

$$(\sigma_x^2 + \sigma_y^2)\left[i(u\sigma_x + v\sigma_y) + \frac{1}{R_e}(\sigma_x^2 + \sigma_y^2)\right] = 0$$

The first coefficient implies again the only complex solutions for σ_x/σ_y exist and that the system therefore must be elliptic.

3.6 Hyperbolic Partial Differential Equations

3.6.1 Examples and general properties

The solution of the characteristic equation (3.12) for the compressible steady convection problem (3.13) and (3.14) yields the characteristic curves

$$\frac{dy}{dx} = \left[\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2} - 1}\right]\left[\left(\frac{u}{a}\right)^2 - 1\right]^{-1} .$$

As noted before the PDE's for this problem are hyperbolic if the Mach number $M = \frac{u^2+v^2}{a^2} > 1$. This examples shows that the characteristics can depend on the local solution and are in general curved lines. Straight characteristic are obtained if the coefficient A, B, and C in the characteristic equation are constants. A common example of this type of a hyperbolic partial differential equation is the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{3.22}$$

Assuming a domain from $x = 0$ to $x = 1$, initial conditions

$$u(x,0) = \sin(\pi x), \quad \frac{\partial u}{\partial t}(x,0) = 0$$

and boundary conditions

$$u(0,t) = u(1,t) = 0$$

the solution of the wave equation is

$$u(x,t) = \sin(\pi x)\cos(\pi t) = \frac{1}{2}\left[\sin\pi(x-t) + \sin\pi(x+t)\right] \tag{3.23}$$

A typical property of hyperbolic PDE's is the propagation of information through a system.In the above solution two sine waves are traveling into the positive and negative directions. The superposition of these yields a standing oscillating wave. More insight into this property is shed by the linear transformations

$$\xi = x + t, \quad \eta = x - t \ .$$

This transforms the wave equation (3.22) into the form

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = 0 \tag{3.24}$$

which has the general solution

$$\widetilde{u}(\xi\,\eta) = f(\xi) + g(\eta) \tag{3.25}$$

Initial conditions:

$$u(x,0) = S(x), \quad \frac{\partial u}{\partial t}(x,0) = T(x)$$

With $u(x,0) = \widetilde{u}(\xi(x,0), \eta(x,0)) = f(\xi|_{t=0}) + g(\eta|_{t=0}) = f(x) + g(x) = S(x)$
and $\frac{\partial u}{\partial t} = \frac{\partial \widetilde{u}(\xi\,\eta)}{\partial t} = \frac{\partial \widetilde{u}}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \widetilde{u}}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial f}{\partial \xi}|_{t=0} - \frac{\partial g}{\partial \eta}|_{t=0} = \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x} = T(x)$
one obtains

$$f(x) = \tfrac{1}{2}\left(S(x) + \int^x T(x')dx'\right) + C, \quad g(x) = \tfrac{1}{2}\left(S(x) - \int^x T(x')dx'\right) + D$$

However, since $f = f(\xi)$ and $g = g(\eta)$ we can just replace x in the above expressions with ξ and η in the above expressions to arrive at the general solution $u(x,t) = f(\xi) + g(\eta)$ or

$$u(x,t) = \frac{1}{2}\left(S(x+t) + S(x-t) + \int_{x-t}^{x+t} T(x')dx'\right) \tag{3.26}$$

3.6.2 Role of Characteristics and Boundary/Initial Conditions:

Most common examples of hyperbolic equations are the various time dependent forms of the fluid equations without dissipation (heat conduction, viscosity, resistivity). In those examples the physical role of characteristics is the transport of information. This is achieved by wave, for instance sound waves which carry this information along the characteristic lines or surfaces. It should be noted that more complex systems may have more than just the compressional sound wave solutions. For instance, motion of elastic media like the earth can involve shear motion as well. Plasmas typically have at least three different types of wave. The number of initial conditions has to correspond to the number of unknowns or waves. For the present purpose we assume to deal with an ordinary wave equations which requires to specify two initial conditions (i.e., the dependent variable and its derivative).

The example of the one-dimensional equation in Figure 3.2 demonstrates already the propagation along characteristics. For the second order equation any point source along x will influence its vicinity along the corresponding characteristics as demonstrated in Figure 3.4.

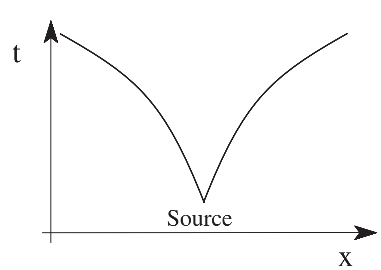

Figure 3.4: Illustration of characteristics branching from a point source. For the wave equation (3.22) the characteristics are straight lines.

This illustrates that any point in space is the origin of two characteristics. At any point in space it takes two characteristics to define a solution uniquely. This has important implications for boundary and initial conditions.

Figure 3.5 illustrates various situations for which a solution is uniquely defined.

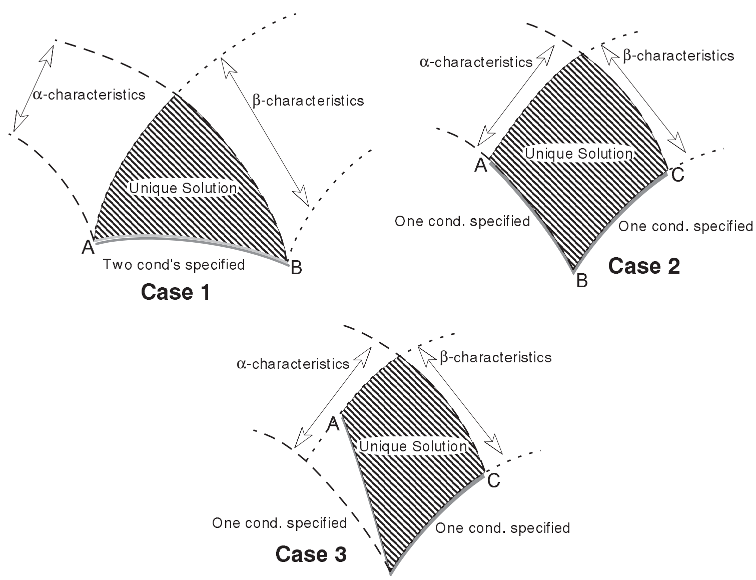

Figure 3.5: Illustration of 3 cases of boundary conditions which yield unique solutions in the shaded domains.

In case 1 both conditions are specified along the line AB. Thus any point along AB is the source of two characteristics. The region in which both the α-characteristics and the β-characteristics overlap has a unique solution. In case 2 a boundary condition is defined along AB which is also a characteristic of type α and another boundary condition is defined along BC which is also a characteristic. Obviously the characteristics originating from these lines overlap in the shaded region. Finally in case 3 a boundary condition is defined along BC which is as well a characteristic curve. The other boundary condition is defined a long a non-characteristic line AB. However, since AB maps continuously and uniquely along the characteristics of type alpha the solution is uniquely defined by one boundary condition provided along AB and the other along BC.

It is worth mentioning that time can be considered a just as a coordinate such that the distinction between boundary and initial conditions is not necessary. With this in mind the line AB of case 3 can for instance also represent the time or the x axis. If it represents the time axis the condition would become the more traditional boundary condition. If AB represents the x axis (at $t = 0$) the condition becomes the traditional initial condition.

For the example of steady supersonic potential flow characteristic cannot be interpreted in the physical sense of information flow because time is not a variable.

3.7 Parabolic Partial Differential Equations

Parabolic equations are typical for dissipative processes. Classical examples are heat conduction and diffusion.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

For the initial condition

$$u(x,0) = \sin \pi x$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0$$

the solution is

$$u(x,t) = \sin(\pi x) \exp\left(-\pi^2 t\right)$$

Characteristics: An interpretation in terms of characteristics is useless because the is only one characteristic. Any local perturbation influences the entire domain, however, with decreasing magnitude with increasing distance from the source of the perturbation.

Initial Conditions: Typical are Dirichlet conditions.

Boundary Conditions: Mixed conditions are possible but Neumann conditions must agree with the initial conditions. Dirichlet conditions are easier.

3.8 Elliptic Partial Differential Equations

Elliptic PDE's are typical for steady state problems or equilibria (note sub-sonic steady flow problem). They also occur when potentials need to be determined by the corresponding equation such as the electrostatic potential in Poisson's equation. Another example is the gravitational potential. The typical homogeneous elliptic equation is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

with boundary conditions

$$\Phi(x,0) = \sin(\pi x), \quad \Phi(x,1) = \sin(\pi x) \exp(-\pi), \quad \Phi(0,y) = \Phi(1,y) = 0$$

the solution is

$$\Phi(x,1) = \sin(\pi x)\exp(-\pi y)$$

Characteristics: There are no real characteristics.

Maximum minimum principle: For an elliptic equation of type (3.1) with constant coefficients A, .., G, both the maximum and the minimum are assumed on the boundary ∂R.

Any disturbance in the interior influence the entire computational domain.

Boundary Conditions can be Dirichlet, Neumann or mixed. If $f(s) = \frac{\partial \Phi}{\partial n}$is prescribed it should be noted that $\oint_{\partial R} f ds = -\int_R \nabla^2 \Phi dV$.