4. Pressure/energy equation:

Starting from the energy equation (2.20) derive the pressure equation (2.21). For simplicity assume a scalar pressure, i.e., $\underline{\underline{\Pi}} = \underline{\underline{1}}p$, and zero source terms $Q^\rho = 0$, $\mathbf{Q}^p = 0$, and $Q^E = 0$. (Hint: Use the momentum and continuity equations to eliminate time derivatives of density and velocity in the energy equations).

Solution:

Equations:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot \rho \mathbf{u} + Q^\rho \\
\frac{\partial \rho \mathbf{u}}{\partial t} &= -\nabla \cdot (\rho \mathbf{u}\mathbf{u}) - \nabla \cdot \underline{\underline{\Pi}} + n\mathbf{F} + \mathbf{u}Q^\rho + \mathbf{Q}^p \\
\frac{\partial}{\partial t}\left(\frac{1}{\gamma-1}p + \frac{1}{2}\rho u^2\right) &= -\nabla \cdot \left(\frac{1}{2}\rho u^2 \mathbf{u} + \frac{1}{\gamma-1}p\mathbf{u} + \mathbf{u}\cdot\underline{\underline{\Pi}} + \mathbf{L}\right) \\
&\quad + n\mathbf{u}\cdot\mathbf{F} + \frac{1}{2}u^2 Q^\rho + \mathbf{u}\cdot\mathbf{Q}^p + Q^E
\end{aligned}$$

Bulk flow energy density: $\frac{1}{2}\rho \mathbf{u}^2$

Internal energy with $\gamma = 5/3$: $p/(\gamma-1)$

The internal energy equation is already available. For the bulk flow we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial \rho \mathbf{u}^2}{\partial t} &= \frac{1}{2}\rho^2\mathbf{u}^2\frac{\partial \rho^{-1}}{\partial t} + \frac{1}{2\rho}2\rho\mathbf{u}\cdot\frac{\partial \rho\mathbf{u}}{\partial t} \\
&= -\frac{1}{2}\mathbf{u}^2\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\frac{\partial \rho\mathbf{u}}{\partial t} \\
&= \frac{1}{2}\mathbf{u}^2\nabla\cdot\rho\mathbf{u} - \frac{1}{2}\mathbf{u}^2 Q^\rho \\
&\quad + \mathbf{u}\cdot\left[-\nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \nabla\cdot\underline{\underline{\Pi}} + n\mathbf{F} + \mathbf{u}Q^\rho + \mathbf{Q}^p\right] \\
&= \frac{1}{2}\mathbf{u}^2\nabla\cdot\rho\mathbf{u} + \mathbf{u}\cdot\left[-\mathbf{u}\nabla\cdot(\rho\mathbf{u}) - \rho\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\underline{\underline{\Pi}} + n\mathbf{F} + \frac{1}{2}\mathbf{u}Q^\rho + \mathbf{Q}^p\right] \\
&= -\frac{1}{2}\mathbf{u}^2\nabla\cdot\rho\mathbf{u} - \frac{1}{2}\rho\mathbf{u}\nabla\mathbf{u}^2 + \mathbf{u}\cdot\left[-\nabla\cdot\underline{\underline{\Pi}} + n\mathbf{F} + \frac{1}{2}\mathbf{u}Q^\rho + \mathbf{Q}^p\right] \\
&= -\frac{1}{2}\nabla\cdot\left[\left(\rho\mathbf{u}^2\right)\mathbf{u}\right] + \mathbf{u}\cdot\left[-\nabla\cdot\underline{\underline{\Pi}} + n\mathbf{F} + \frac{1}{2}\mathbf{u}Q^\rho + \mathbf{Q}^p\right]
\end{aligned}$$

Subtracting the bulk flow from the total kinetic energy yields

$$\begin{aligned}
\frac{1}{\gamma-1}\frac{\partial p}{\partial t} &= -\nabla\cdot\left(\frac{1}{2}\rho u^2\mathbf{u} + \frac{1}{\gamma-1}p\mathbf{u} + \mathbf{u}\cdot\underline{\underline{\Pi}} + \mathbf{L}\right) + n\mathbf{u}\cdot\mathbf{F} + \frac{1}{2}u^2 Q^\rho + \mathbf{u}\cdot\mathbf{Q}^p + Q^E \\
&\quad + \frac{1}{2}\nabla\cdot\left[\left(\rho\mathbf{u}^2\right)\mathbf{u}\right] - \mathbf{u}\cdot\left[-\nabla\cdot\underline{\underline{\Pi}} + n\mathbf{F} + \frac{1}{2}\mathbf{u}Q^\rho + \mathbf{Q}^p\right] \\
&= -\frac{1}{\gamma-1}\nabla\cdot p\mathbf{u} - \left(\underline{\underline{\Pi}}\cdot\nabla\right)\cdot\mathbf{u} - \nabla\cdot\mathbf{L} + Q^E
\end{aligned}$$

5. Heat conduction: Starting from the pressure equation

$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p+\nabla\cdot p\mathbf{u}\right)=-(\underline{\Pi}\cdot\nabla)\cdot\mathbf{u}-\nabla\cdot\mathbf{L} \tag{1}$$

derive an equation for the temperature T using an isotropic pressure (the viscosity $\underline{\mathbf{w}}=0$), $p=nkT$, and the heat flux is $\mathbf{L}=-\kappa\nabla T$. Note, to eliminate the time derivative of n use the continuity equation. Show that for vanishing velocity $\mathbf{u}=0$ the equation becomes a diffusion equation for temperature.

Solution:

Starting from the pressure equation

$$\frac{1}{\gamma-1}\left(\frac{\partial}{\partial t}p+\nabla\cdot p\mathbf{u}\right)=-(\underline{\Pi}\cdot\nabla)\cdot\mathbf{u}-\nabla\cdot\mathbf{L} \tag{2}$$

derive an equation for the temperature T using an isotropic pressure (the viscosity $\underline{\mathbf{w}}=0$), $p=nkT$, and the heat flux is $\mathbf{L}=-\kappa\nabla T$. Note, to eliminate the time derivative of n use the continuity equation. Show that for vanishing velocity $\mathbf{u}=0$ the equation becomes the heat conduction equation.

With $p=nkT$, $\underline{\underline{\Pi}}=p\underline{\underline{1}}$, and $\mathbf{L}=-\kappa\nabla T$ the pressure equation becomes

$$\frac{1}{\gamma-1}\left(\frac{\partial nkT}{\partial t}+\nabla\cdot(nkT\mathbf{u})\right)=-nkT\nabla\cdot\mathbf{u}+\nabla\cdot(\kappa\nabla T). \tag{3}$$

Note that $p=p(\mathbf{r},t)$, $n=n(\mathbf{r},t)$, $T=T(\mathbf{r},t)$, and $\mathbf{u}=\mathbf{u}(\mathbf{r},t)$. With

$$\begin{aligned}\frac{\partial nkT}{\partial t} &= nk\frac{\partial T}{\partial t}+kT\frac{\partial n}{\partial t}\\ \nabla\cdot(nkT\mathbf{u}) &= nk\mathbf{u}\cdot\nabla T+kT\nabla\cdot(n\mathbf{u}).\end{aligned}$$

Using the continuity equation $\partial n/\partial t+\nabla\cdot(n\mathbf{u})=0$ the sum of the last terms of the above equations is zero. Thus equation (3) becomes

$$\frac{1}{\gamma-1}\left(\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T\right)=-T\nabla\cdot\mathbf{u}+\frac{\kappa}{nk}\nabla^2 T. \tag{4}$$

Another way to write the temperature equation with the total derivative along the fluid velocity $\mathbf{u}$ is

$$\frac{1}{\gamma-1}\frac{dT}{dt}=-T\nabla\cdot\mathbf{u}+\frac{\kappa}{nk}\nabla^2 T.$$

For the case of negligible velocity $\mathbf{u}\approx 0$ the temperature (4) equation becomes the heat conduction equation:

$$\frac{\partial T}{\partial t}=\frac{(\gamma-1)\kappa}{nk}\nabla^2 T$$

with the heat conduction coefficient $\alpha=\frac{(\gamma-1)\kappa}{nk}$.

6. Simple hyperbolic equation:

(a) Show that the second order PDE $\partial^2 u/\partial x \partial t = 0$ is hyperbolic.

(b) Demonstrate that the system

$$\frac{\partial u}{\partial t} - v = 0 \qquad \frac{\partial v}{\partial x} = 0$$

is equivalent.

(c) Use the method for multiple 1st order PDE's to demonstrate that it is hyperbolic and show that the characteristics are the x and t axes.

Solution:

(a) Show that the second order PDE $\partial^2 u/\partial x \partial t = 0$ is hyperbolic. From class

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \tag{5}$$

with $A,..G = const$. This equation is called $\left\{ \begin{array}{ll} elliptic\, for & B^2 - 4AC < 0 \\ parabolic\, for & B^2 - 4AC = 0 \\ hyperbolic\, for & B^2 - 4AC > 0 \end{array} \right.$

=> $A, C = 0$ and $B = 1$ such that $B^2 - 4AC = 1$ implying a hyperbolic PDE.

(b) Demonstrate that the system

$$\frac{\partial u}{\partial t} - v = 0 \qquad \frac{\partial v}{\partial x} = 0$$

is equivalent. Taking the x derivative of the first equation yields

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} = 0$$

(c) Use the method for multiple 1st order PDE's to demonstrate that it is hyperbolic and show that the characteristics are the x and t axes. Using the example from class

$$A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial u}{\partial y} + A_{12}\frac{\partial v}{\partial x} + B_{12}\frac{\partial v}{\partial y} = 0 \tag{6}$$

$$A_{21}\frac{\partial u}{\partial x} + B_{21}\frac{\partial u}{\partial y} + A_{22}\frac{\partial v}{\partial x} + B_{22}\frac{\partial v}{\partial y} = 0 \tag{7}$$

and identifying x with x and t with y in the example we obtain

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

yielding the condition for the characteristics as $\det\left(\underline{\underline{\mathbf{A}}}^t dt - \underline{\underline{\mathbf{B}}}^t dx\right) = 0$ or explicit:

$$\det\begin{pmatrix} -dx & 0 \\ 0 & dt \end{pmatrix} = -dxdt = 0$$

which has the real solutions (characteristics) $dx/dt = 0$ or $x = \text{const}$ and $dt/dx = 0$ or $t = \text{const}$.

7. Time dependent flow:

The equations for one-dimensional isentropic inviscid flow are

$$\frac{\partial \rho}{\partial t} + \frac{u\partial \rho}{\partial x} + \frac{\rho \partial u}{\partial x} = 0 \tag{8}$$

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{9}$$

$$p = k\rho^{\gamma}$$

reduce this system of three dependent variables to two by eliminating p. Show that the system is hyperbolic and that the characteristics are given by $dx/dt = u \pm a$ with the sound speed $a^2 = \gamma p/\rho$.

Solution:

with $p = k\rho^{\gamma}$ and $a^2 = \gamma p/\rho$ equation (9) becomes

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \tag{10}$$

Comparison of equations (8) and (10) with the system of 2 equations for 2 dependent and two independent variables yields the following matrix elements:

$$\begin{array}{ccccc} \frac{\partial \rho}{\partial t} & +u\frac{\partial \rho}{\partial x} & & +\rho\frac{\partial u}{\partial x} & = 0 \\ A_{11} & B_{11} & A_{12} & B_{12} & \\ & \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} & +\frac{\partial u}{\partial t} & +u\frac{\partial u}{\partial x} & = 0 \\ A_{21} & B_{21} & A_{22} & B_{22} & \end{array}$$

Which gives the matrices:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad and \quad \underline{\underline{B}} = \begin{pmatrix} u & \rho \\ \frac{a^2}{\rho} & u \end{pmatrix}$$

The equation $(\underline{\underline{A}}dx - \underline{\underline{B}}dt) \cdot \mathbf{L} = 0$ has nontrivial solutions for

$$det \begin{pmatrix} \frac{dx}{dt} - u & -\rho \\ -\frac{a^2}{\rho} & \frac{dx}{dt} - u \end{pmatrix} = \left(\frac{dx}{dt} - u\right)^2 - a^2 = 0$$

with the two real solutions for the characteristics:

$$\frac{dx}{dt} = u \pm a$$

The system is hyperbolic because both solutions are real.