8. Pressure/energy equation:

Consider the pressure equation for isotropic pressure in the absence of heat conduction

$$\frac{1}{\gamma-1}\left(\frac{\partial p}{\partial t}+\nabla\cdot p\mathbf{u}\right)=-p\nabla\cdot\mathbf{u}+Q^E$$

Assuming $h=p/\rho^\gamma$, demonstrate that the pressure equation combined with the continuity equation can be written as

$$\frac{\partial h}{\partial t}+\mathbf{u}\cdot\nabla h=R$$

and compute R as a function of Q^E.

Solution:

Assuming a function $h=p\rho^{-\gamma}$, demonstrate that the pressure equation combined with the continuity equation can be written as

$$\frac{\partial h}{\partial t}+\mathbf{u}\cdot\nabla h=R$$

Using the continuity

$$\frac{\partial \rho}{\partial t}+\mathbf{u}\cdot\nabla\rho=-\rho\nabla\cdot\mathbf{u}$$

and the pressure equations

$$\frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p=-\gamma p\nabla\cdot\mathbf{u}+(\gamma-1)\,Q^E$$

we obtain

$$\begin{aligned}\frac{dh}{dt} &= \rho^{-\gamma}\frac{dp}{dt}-\gamma p\rho^{-\gamma-1}\frac{d\rho}{dt}\\ &= -\gamma p\rho^{-\gamma}\nabla\cdot\mathbf{u}+\gamma p\rho^{-\gamma}\nabla\cdot\mathbf{u}+(\gamma-1)\,\rho^{-\gamma}Q^E\\ &= (\gamma-1)\,Q^E/\rho^\gamma\end{aligned}$$

Note that for resistive heating $Q^E=\eta j^2\geq 0$ such that $dh/dt\geq 0$. The quantity h is a measure of fluid entropy and increases only in the presence of nonadiabatic - for instance resistive heating.

9. a) Demonstrate that the two-dimensional compressible irrotational steady state flow

$$\begin{aligned}\left(\frac{u^2}{a^2}-1\right)\frac{\partial u}{\partial x}+\left(\frac{uv}{a^2}\right)\frac{\partial u}{\partial y}+\left(\frac{uv}{a^2}\right)\frac{\partial v}{\partial x}+\left(\frac{v^2}{a^2}-1\right)\frac{\partial v}{\partial y} &= 0\\ -\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} &= 0\end{aligned}$$

are equivalent to the equations

$$-\frac{\gamma p}{\rho}\nabla\cdot\mathbf{u}+\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u}) = 0 \quad (1)$$

$$\nabla\times\mathbf{u} = 0 \quad (2)$$

for the velocity $\mathbf{u}=(u,v)$ with $a^2=\gamma p/\rho$.

b) Derive equation (1) using the steady state continuity (2.18), momentum (2.19), and pressure equation (2.20) for isotropic pressure $\underline{\underline{\Pi}}=p\underline{\underline{1}}$ (no source terms, no external force terms, and no heat conduction. Hint: Use the continuity and momentum equations to replace the $\nabla\rho$ and ∇p terms in the pressure equation).

Solution:

(a) Eq. (1), 1st term:

$$-\frac{\gamma p}{\rho}\nabla\cdot\mathbf{u}=-a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Eq. (1), 2nd term:

$$\begin{aligned}\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u}) &= u\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u+v\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v\\ &= u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}\end{aligned}$$

The sum of the two terms and division by a^2 yields the first of the two equations.

2nd equation: The z component of the curl yields

$$\mathbf{e}_z\cdot\nabla\times\mathbf{u}=-\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

(b) Derivation of equation (1) from:

$$\begin{aligned}0 &= -\nabla\cdot(\rho\mathbf{u})\\ 0 &= -\nabla\cdot(\rho\mathbf{u}\mathbf{u})-\nabla p\\ 0 &= -\nabla\cdot\left(\frac{1}{2}\rho u^2\mathbf{u}+\frac{\gamma}{\gamma-1}p\mathbf{u}\right)\end{aligned}$$

yield

$$\begin{aligned}0 &= -\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}-\mathbf{u}\nabla\cdot(\rho\mathbf{u})-\nabla p\\ 0 &= -\frac{1}{2}\rho\mathbf{u}\cdot\nabla u^2-\frac{1}{2}u^2\nabla\cdot(\rho\mathbf{u})-\frac{\gamma}{\gamma-1}p\nabla\cdot\mathbf{u}-\frac{\gamma}{\gamma-1}\mathbf{u}\cdot\nabla p\end{aligned}$$

With $\nabla\cdot(\rho\mathbf{u})=0$ we obtain

$$\begin{aligned} 0 &= -\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}-\nabla p \\ 0 &= -\frac{1}{2}\rho\mathbf{u}\cdot\nabla u^2-\frac{\gamma}{\gamma-1}p\nabla\cdot\mathbf{u}-\frac{\gamma}{\gamma-1}\mathbf{u}\cdot\nabla p \end{aligned}$$

substituting ∇p

$$-\frac{1}{2}\rho\mathbf{u}\cdot\nabla u^2-\frac{\gamma}{\gamma-1}p\nabla\cdot\mathbf{u}+\frac{\gamma}{\gamma-1}\mathbf{u}\cdot\left(\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\right)=0$$

With $\nabla u^2=2\mathbf{u}\times(\nabla\times\mathbf{u})+2\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}$ we obtain

$$-\frac{\gamma p}{\rho}\nabla\cdot\mathbf{u}+\mathbf{u}\cdot\left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\right)=0$$

10. Read section 3.5 of the manuscript, derive the Fourier transform of the Navier Stokes equations, and show that the determinant for the Fourier transform matrix is

$$(\sigma_x^2+\sigma_y^2)\left[i(u\sigma_x+v\sigma_y)+\frac{1}{R_e}(\sigma_x^2+\sigma_y^2)\right]=0$$

Solution:

Define the Fourier transform as

$$\widetilde{u}(\sigma_x,\sigma_y)=\int_{\infty}^{\infty}\int_{\infty}^{\infty}u(x,y)\exp(-i\sigma_x x)\exp(-i\sigma_y y)\,dx\,dy$$

or symbolic as $\widetilde{u}=Fu$ with the property that

$$i\sigma_x\widetilde{u}=F\frac{\partial u}{\partial x}=\widetilde{\frac{\partial u}{\partial x}},\quad i\sigma_y\widetilde{u}=F\frac{\partial u}{\partial y}=\widetilde{\frac{\partial u}{\partial y}}$$

Steady Navier-Stokes equations

$$u_x+v_y = 0 \tag{3}$$

$$uu_x+vu_y+p_x-\frac{1}{R_e}(u_{xx}+u_{yy}) = 0 \tag{4}$$

$$uv_x+vv_y+p_y-\frac{1}{R_e}(v_{xx}+v_{yy}) = 0 \tag{5}$$

Fourier transform:

$$i\sigma_x\widetilde{u}+i\sigma_y\widetilde{v} = 0 \tag{6}$$

$$i\sigma_x u\widetilde{u}+i\sigma_y v\widetilde{u}+i\sigma_x\widetilde{p}-\frac{1}{R_e}\left((i\sigma_x)^2\widetilde{u}+(i\sigma_y)^2\widetilde{u}\right) = 0 \tag{7}$$

$$i\sigma_x u\widetilde{v}+i\sigma_y v\widetilde{v}+i\sigma_y\widetilde{p}-\frac{1}{R_e}\left((i\sigma_x)^2\widetilde{v}+(i\sigma_y)^2\widetilde{v}\right) = 0 \tag{8}$$

In matrix form:

$$\begin{bmatrix} i\sigma_x & i\sigma_y & 0 \\ i(u\sigma_x+v\sigma_y)+\frac{1}{R_e}(\sigma_x^2+\sigma_y^2) & 0 & i\sigma_x \\ 0 & i(u\sigma_x+v\sigma_y)+\frac{1}{R_e}(\sigma_x^2+\sigma_y^2) & i\sigma_y \end{bmatrix}\begin{bmatrix}\widetilde{u}\\ \widetilde{v}\\ \widetilde{p}\end{bmatrix}=0$$

which yields from $det[..]=0$ the equation

$$(\sigma_x^2+\sigma_y^2)\left[i(u\sigma_x+v\sigma_y)+\frac{1}{R_e}(\sigma_x^2+\sigma_y^2)\right]=0$$

11. General Technique - 2nd Derivative Approximation

a) Use the general technique to determine the coefficients a to c and the leading error term in the following expression

$$\frac{d^2f}{dx^2} = af_{i-2} + bf_{i-1} + cf_i$$

b) Do the same for the expression

$$\frac{d^2f}{dx^2} = af_{i-1} + bf_i + cf_{i+1} + df_{i+2} + ef_{i+3}$$

Solution:

(a) Taylor expansion of

$$\begin{aligned}\frac{d^2f}{dx^2} \approx af_{i-2} + bf_{i-1} + cf_i &= (a+b+c)\,f_i + (-2a-b)\,\Delta x\,\frac{df}{dx}\bigg|_i \\ &\quad + (4a+b)\,\frac{\Delta x^2}{2}\,\frac{d^2f}{dx^2}\bigg|_i + (-8a-b)\,\frac{\Delta x^3}{6}\,\frac{d^3f}{dx^3}\bigg|_i \\ &\quad + (16a-b)\,\frac{\Delta x^4}{24}\,\frac{d^4f}{dx^4}\bigg|_i + ..\end{aligned}$$

Conditions for a, b, c, and d:

$$\begin{aligned} a+b+c &= 0 \\ -2a-b &= 0 \\ 4a+b &= 2/\Delta x^2 \end{aligned}$$

Solution

$$\begin{aligned} a &= 1/\Delta x^2 \\ b &= -2/\Delta x^2 \\ c &= 1/\Delta x^2 \end{aligned}$$

such that

$$\frac{d^2f}{dx^2} = \frac{f_{i-2} - 2f_{i-1} + f_i}{\Delta x^2} - \Delta x\,\frac{d^3f}{dx^3}\bigg| + O\left(\Delta x^2\right)$$

(b) Taylor expansion of

$$\begin{aligned}\frac{d^2f}{dx^2} &\approx af_{i-1} + bf_i + cf_{i+1} + df_{i+2} + ef_{i+3} \\ &= (a+b+c+d+e)\,f_i + (-a+c+2d+3e)\,\Delta x\,\frac{df}{dx}\bigg|_i \\ &\quad + (a+c+4d+9e)\,\frac{\Delta x^2}{2}\,\frac{d^2f}{dx^2}\bigg|_i + (-a+c+8d+27e)\,\frac{\Delta x^3}{6}\,\frac{d^3f}{dx^3}\bigg|_i \\ &\quad + (a+c+16d+81e)\,\frac{\Delta x^4}{24}\,\frac{d^4f}{dx^4}\bigg|_i + (-a+c+32d+243e)\,\frac{\Delta x^5}{120}\,\frac{d^5f}{dx^5}\bigg|_i\end{aligned}$$

Conditions for a, b, c, and d:

$$\begin{aligned} a+b+c+d+e &= 0 \\ -a+c+2d+3e &= 0 \\ a+c+4d+9e &= 2/\Delta x^2 \\ -a+c+8d+27e &= 0 \\ a+c+16d+81e &= 0 \end{aligned}$$

eliminating a from (2) to (5) (by adding 2 and 3, 3and 4, 4 and 5):

$$\begin{aligned} 2c+6d+12e &= 2/\Delta x^2 \\ 2c+12d+36e &= 2/\Delta x^2 \\ 2c+24d+108e &= 0 \end{aligned}$$

eliminating c (by subtracting 1from 2 and from 3) in the above equations:

$$\begin{aligned} 6d+24e &= 0 \\ 9d+48e &= -/\Delta x^2 \end{aligned}$$

Solution:

$$\begin{aligned} e &= -1/12\Delta x^2 \\ d &= 4/12\Delta x^2 \\ c &= 6/12\Delta x^2 \\ a &= 11/12\Delta x^2 \\ b &= -20/12\Delta x^2 \end{aligned}$$

and

$$-a+c+32d+243e = -120/12\Delta x^2$$

such that

$$\frac{d^2 f}{dx^2} = \frac{11 f_{i-1} - 20 f_i + 6 f_{i+1} + 4 f_{i+2} - f_i}{12\Delta x^2} - \frac{\Delta x^3}{12}\left.\frac{d^5 f}{dx^5}\right| + O\left(\Delta x^4\right)$$

The order of the error for the 3 point approximation in part (a) is linear in Δx and for the 5 point approximation in part (b) it is Δx^3. In general for the nth derivative at least n+1 grid points are needed for the approximation. For an m point approximation the error is usually of order Δx^{m-n}.