18. Stability analysis:

For the diffusion equation it seems rather appropriate to use a scheme

$$\frac{T_j^{n+1} - T_j^{n-1}}{2\Delta t} - \frac{\alpha(T_{j-1} - 2T_j + T_{j+1})^n}{\Delta x^2} = 0$$

because the centered time should give higher accuracy for the time derivative. Test the stability of this method. Is it worth an attempt to implement it?

Solution: With $T_j^{n+1}/T_j^n = \exp[i\omega\Delta t] = g$ and $T_{j+1}^n = T_j^n \exp[ik\Delta x]$ we obtain

$$\frac{1}{2\Delta t}g - \frac{1}{2\Delta t}\frac{1}{g} - \frac{\alpha}{\Delta x^2}\left(\exp[-ik\Delta x] - 2 + \exp[ik\Delta x]\right) = 0$$

Multiplying with $2\Delta t$, using the definition for g, and noting that
$\cos(k\Delta x) = 0.5\left(\exp[-ik\Delta x] + \exp[ik\Delta x]\right)$ we obtain

$$g - \frac{1}{g} - 4s\left(\cos k\Delta x - 1\right) = 0$$

With the relation $1 - \cos(k\Delta x) = 2\sin^2(k\Delta x/2)$ and multiplying with g we obtain

$$g^2 + 8sg\sin^2(k\Delta x/2) - 1 = 0$$

With $h = 4s\sin^2(k\Delta x/2)$

$$(g+h) = 1 + h^2$$

or

$$g_\pm = -h \pm \sqrt{1 + h^2}$$

Since h is positive, the solution $g_- = -h - \sqrt{1+h^2} < -1$ for $s \neq 0$. **Therefore the proposed scheme is always unstable.**

19. Stability test

Use the program from problem 16 to test the stability of the scheme in Problem 13 as a function of s for different values of the parameters d. Does the result for $d = 0$ agree with your analytic stability limit in Problem 15? How does the stability limit vary for different values of d including the particular choice of $d = 1 - \frac{1}{12s}$.

Use the program from problem 15 to test the stability of the scheme in Problem 12 as a function of s for different values of the parameters d. Does the result for $d = 0$ agree with your analytic stability limit in Problem 14? How does the stability limit vary for different values of d including the particular choice of $d = 1 - \frac{1}{12s}$.
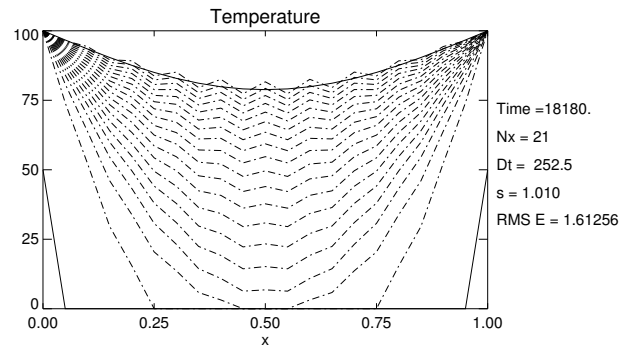
Solution:

The modified scheme is stable in the range of $s \in [0, 1]$ for $d = 0$.
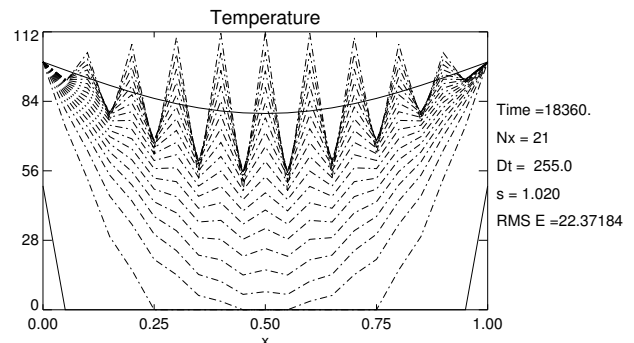
Stability limit through numerical simulation: The modified three-level code runs stable up to $s = 1.00$. The following shows the result of the run with $s = 1.00$, $nx = 21$, $t_{max} = 18000$, $n_{out} = 4$.
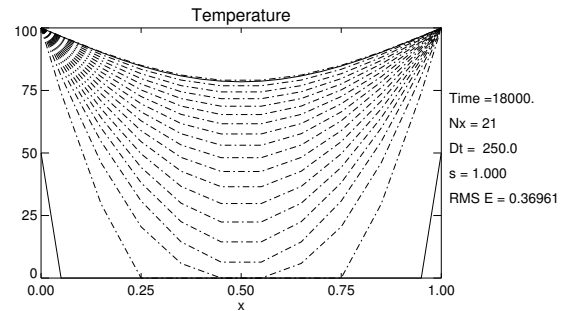
Increasing s to $s = 1.01$ ($t_{max} = 18180$) yields some small oscillations:

These become more pronounced for $s = 1.02$ ($t_{max} = 18360$).

The oscillations are on the grid scale. This is explained by the stability condition $s \sin^2(k\Delta x/2) < 1$.

With a wavelength λ (in units of the grid spacing), the wavenumber becomes $k = 2\pi/\lambda$ such that the argument in the sin function is $\pi/2$ for $\lambda = 2\Delta x$. Other wavelength can be stable because the sin function is smaller than 1 such that $s \sin^2(k\Delta x/2) < 1$ can be satisfied for longer waves and the chosen s values.

For larger values of d the stability limit for s decreases. For instance for the choice of $d = 0.5$ the stability limit is $s = 0.52$. For the choice of $d = 1 - \frac{1}{12s}$ the stability limit is $s = 0.4$.

20. Weighted residual method:

The equation $d^2y/dx^2 + y = \left(1 - (5\pi/6)^2\right)\sin(5\pi x/6)$, subject to the boundary conditions $y(0) = 0$ and $y(1) = 0.5$, is to be solved in the domain $x \in [0,1]$ using the following methods of the weighted residuals

(a) Collocation,

(b) Galerkin.

Assume the approximate solution to be $y = a_1(x - x^3) + a_2(x^2 - x^3) + 0.5x^3$ with the base functions $\Phi_1 = (x - x^3)$ and $\Phi_2 = (x^2 - x^3)$.

Compare the computational solution with the exact solution $y = \sin(5\pi x/6)$.

Solution: Residual:

$$
\begin{aligned}
R &= \frac{d^2y}{dx^2} + y - \left(1 - (5\pi/6)^2\right)\sin(5\pi x/6) \\
&= -6a_1 x + 2a_2 - 6a_2 x + 3x + a_1(x - x^3) + a_2(x^2 - x^3) + 0.5x^3 \\
&\quad - \left(1 - (5\pi/6)^2\right)\sin(5\pi x/6) \\
&= -a_1(5x + x^3) + a_2(2 - 6x + x^2 - x^3) + 3x + 0.5x^3 \\
&\quad - \left(1 - (5\pi/6)^2\right)\sin(5\pi x/6)
\end{aligned}
$$

In the following we will need the integrals with $\alpha = 5\pi/6$

$$
\begin{aligned}
I_0 = \int_0^1 \sin(\alpha x)\,dx &= \left. -\frac{1}{\alpha}\cos(\alpha x)\right|_0^1 = \frac{1}{\alpha}(1 - \cos(\alpha)) \\
I_1 = \int_0^1 x\sin(\alpha x)\,dx &= \left.-\frac{x}{\alpha}\cos(\alpha x)\right|_0^1 + \left.\frac{1}{\alpha^2}\sin(\alpha x)\right|_0^1 \\
&= -\frac{1}{\alpha}\cos(\alpha) + \frac{1}{\alpha^2}\sin(\alpha) \\
I_2 = \int_0^1 x^2 \sin(\alpha x)\,dx &= \left.-\frac{x^2}{\alpha}\cos(\alpha x)\right|_0^1 + \left.\frac{2x}{\alpha^2}\sin(\alpha x)\right|_0^1 + \left.\frac{2}{\alpha^3}\cos(\alpha x)\right|_0^1 \\
&= -\frac{1}{\alpha}\cos(\alpha) + \frac{2}{\alpha^2}\sin(\alpha) + \frac{2}{\alpha^3}(\cos(\alpha) - 1) \\
I_3 = \int_0^1 x^3 \sin(\alpha x)\,dx &= \left.-\frac{x^3}{\alpha}\cos(\alpha x)\right|_0^1 + \left.\frac{3x^2}{\alpha^2}\sin(\alpha x)\right|_0^1 + \left.\frac{6x}{\alpha^3}\cos(\alpha x)\right|_0^1 - \left.\frac{6}{\alpha^4}\sin(\alpha x)\right|_0^1 \\
&= -\frac{1}{\alpha}\cos(\alpha) + \frac{3}{\alpha^2}\sin(\alpha) + \frac{6}{\alpha^3}\cos(\alpha) - \frac{6}{\alpha^4}\sin(\alpha)
\end{aligned}
$$

(a) Collocation method with $x_1 = 0.5$ and $x_2 = 1.0$

Weighted residual R_1 with $\alpha = 5\pi/6$:

$$
\begin{aligned}
R_1 &= \int_0^1 \delta(x - x_1)\left[-a_1(5x + x^3) + a_2(2 - 6x + x^2 - x^3) + 3x + 0.5x^3\right]dx \\
&\quad - \int_0^1 \delta(x - x_1)\left[(1 - \alpha^2)\sin(\alpha x)\right]dx \\
&= -a_1\left(\frac{5}{2} + \frac{1}{8}\right) + a_2\left(2 - \frac{6}{2} + \frac{1}{4} - \frac{1}{8}\right) + \left(\frac{3}{2} + \frac{1}{16}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \left(1-\alpha^2\right)\sin\left(\alpha/2\right)\\
&= -a_1\frac{21}{8} - a_2\frac{7}{8} + \frac{25}{16} - \left(1-\alpha^2\right)\sin\left(\alpha/2\right)
\end{aligned}
$$

Weighted residual R_2:

$$
\begin{aligned}
R_2 &= \int_0^1 \delta\left(x-x_2\right)\left[-a_1\left(5x+x^3\right)+a_2\left(2-6x+x^2-x^3\right)+3x+0.5x^3\right]dx\\
&\quad - \int_0^1 \delta\left(x-x_2\right)\left[\left(1-\alpha^2\right)\sin\left(\alpha x\right)\right]dx\\
&= -a_1\left(5+1\right)+a_2\left(2-6+1-1\right)+\left(3+\frac{1}{2}\right)\\
&\quad - \left(1-\alpha^2\right)\sin\left(\alpha\right)\\
&= -a_1 6 - a_2 4 + \frac{7}{2} - \left(1-\alpha^2\right)\sin\left(\alpha\right)
\end{aligned}
$$

This yields a system of equations of the form

$$
\begin{aligned}
c_{11}a_1 + c_{12}a_2 &= d_1\\
c_{21}a_1 + c_{22}a_2 &= d_2
\end{aligned}
$$

General Solution:

$$
a_1 = \left[\frac{c_{11}}{c_{12}} - \frac{c_{21}}{c_{22}}\right]^{-1}\left[\frac{d_1}{c_{12}} - \frac{d_2}{c_{22}}\right] \qquad a_2 = \left[\frac{c_{12}}{c_{11}} - \frac{c_{22}}{c_{21}}\right]^{-1}\left[\frac{d_1}{c_{11}} - \frac{d_2}{c_{21}}\right]
$$

Evaluating the coefficients yields

$$
\begin{pmatrix} -2.625 & -0.875 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -7.217 \\ -6.427 \end{pmatrix}
$$

with the solution $a_1 = 4.427$, $a_2 = -5.034$.

Collocation method with $x_1 = 1/3$ and $x_2 = 2/3$

Weighted residual R_1 with $\alpha = 5\pi/6$:

$$
\begin{aligned}
R_1 &= \int_0^1 \delta\left(x-x_1\right)\left[-a_1\left(5x+x^3\right)+a_2\left(2-6x+x^2-x^3\right)+3x+0.5x^3\right]dx\\
&\quad - \int_0^1 \delta\left(x-x_1\right)\left[\left(1-\alpha^2\right)\sin\left(\alpha x\right)\right]dx\\
&= -a_1\left(\frac{5}{3}+\frac{1}{27}\right)+a_2\left(2-\frac{6}{3}+\frac{1}{9}-\frac{1}{27}\right)+\left(\frac{3}{3}+\frac{1}{54}\right)\\
&\quad - \left(1-\alpha^2\right)\sin\left(\alpha/3\right)\\
&= -a_1\frac{46}{27} + a_2\frac{2}{27} + \frac{55}{54} - \left(1-\alpha^2\right)\sin\left(\alpha/3\right)
\end{aligned}
$$

Weighted residual R_2:

$$R_2 = \int_0^1 \delta(x-x_2)\left[-a_1(5x+x^3)+a_2(2-6x+x^2-x^3)+3x+0.5x^3\right]dx$$

$$-\int_0^1 \delta(x-x_2)\left[(1-\alpha^2)\sin(\alpha x)\right]dx$$

$$= -a_1\left(\frac{10}{3}+\frac{8}{27}\right)+a_2\left(2-\frac{12}{3}+\frac{4}{9}-\frac{8}{27}\right)+\left(\frac{6}{3}+\frac{4}{27}\right)$$

$$-(1-\alpha^2)\sin(2\alpha/3)$$

$$= -a_1\frac{98}{27}-a_2\frac{50}{27}+\frac{58}{27}-(1-\alpha^2)\sin(2\alpha/3)$$

This yields a system of equations of the form

$$c_{11}a_1+c_{12}a_2 = d_1$$
$$c_{21}a_1+c_{22}a_2 = d_2$$

Evaluating the coefficients yields

$$\begin{pmatrix} -1.704 & 0.0741 \\ -3.630 & -1.852 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -5.503 \\ -7.913 \end{pmatrix}$$

with the solution $a_1 = 3.1475$, $a_2 = -1.896$.

(b) Galerkin method: with $\phi_1 = (x-x^3)$ and $\phi_2 = (x^2-x^3)$
Weighted residual R_1:

$$R_1 = \int_0^1 (x-x^3)\left[-a_1(5x+x^3)+a_2(2-6x+x^2-x^3)+3x+0.5x^3\right]dx$$

$$-\int_0^1 (x-x^3)\left[\left(1-(5\pi/6)^2\right)\sin(5\pi x/6)\right]dx$$

$$= \int_0^1 \left[-a_1(5x^2+x^4)+a_2(2x-6x^2+x^3-x^4)+3x^2+0.5x^4\right]dx$$

$$+\int_0^1 \left[a_1(5x^4+x^6)-a_2(2x^3-6x^4+x^5-x^6)-3x^4-0.5x^6\right]dx$$

$$-\left(1-(5\pi/6)^2\right)\int_0^1 (x-x^3)\sin(5\pi x/6)\,dx$$

$$= -a_1\left(\frac{5}{3}+\frac{1}{5}\right)+a_2\left(\frac{2}{2}-\frac{6}{3}+\frac{1}{4}-\frac{1}{5}\right)+\frac{3}{3}+\frac{1}{10}$$

$$+a_1\left(\frac{5}{5}+\frac{1}{7}\right)-a_2\left(\frac{2}{4}-\frac{6}{5}+\frac{1}{6}-\frac{1}{7}\right)-\frac{3}{5}-\frac{1}{14}$$

$$-\left(1-(5\pi/6)^2\right)\int_0^1 (x-x^3)\sin(5\pi x/6)\,dx$$

With these integrals the integral over the sin in the residual yield

$$R_{1\sin} = -(1-\alpha^2)\int_0^1 (x-x^3)\sin(\alpha x)\,dx$$

$$= -(1-\alpha^2)\left(-\frac{1}{\alpha}\cos{(\alpha)}+\frac{1}{\alpha^2}\sin{(\alpha)}\right.$$

$$\left.+\frac{1}{\alpha}\cos{(\alpha)}-\frac{3}{\alpha^2}\sin{(\alpha)}-\frac{6}{\alpha^3}\cos{(\alpha)}+\frac{6}{\alpha^4}\sin{(\alpha)}\right)$$

$$= -(1-\alpha^2)\left(-\frac{2}{\alpha^2}\sin{(\alpha)}-\frac{6}{\alpha^3}\cos{(\alpha)}+\frac{6}{\alpha^4}\sin{(\alpha)}\right)$$

and the residual R_1 with $\alpha = 5\pi/6$ is

$$
\begin{aligned}
R_1 &= -a_1\left(\frac{5}{3}+\frac{1}{5}\right)+a_2\left(\frac{2}{2}-\frac{6}{3}+\frac{1}{4}-\frac{1}{5}\right)+\frac{3}{3}+\frac{1}{10}\\
&\quad +a_1\left(\frac{5}{5}+\frac{1}{7}\right)-a_2\left(\frac{2}{4}-\frac{6}{5}+\frac{1}{6}-\frac{1}{7}\right)-\frac{3}{5}-\frac{1}{14}\\
&\quad +2\frac{1-\alpha^2}{\alpha^2}\left(\sin{(\alpha)}+\frac{3}{\alpha}\cos{(\alpha)}-\frac{3}{\alpha^2}\sin{(\alpha)}\right)\\
&= -a_1\left(\frac{2}{3}+\frac{2}{35}\right)+a_2\left(-\frac{1}{4}-\frac{1}{6}+\frac{1}{7}\right)+\frac{2}{5}+\frac{1}{35}\\
&\quad +2\frac{1-\alpha^2}{\alpha^2}\left(\sin{(\alpha)}+\frac{3}{\alpha}\cos{(\alpha)}-\frac{3}{\alpha^2}\sin{(\alpha)}\right)
\end{aligned}
$$

Weighted residual R_2:

$$
\begin{aligned}
R_2 &= \int_0^1 (x^2-x^3)\left[-a_1(5x+x^3)+a_2(2-6x+x^2-x^3)+3x+0.5x^3\right]dx\\
&\quad -\int_0^1 (x^2-x^3)\left[\left(1-(5\pi/6)^2\right)\sin{(5\pi x/6)}\right]dx\\
&= \int_0^1\left[-a_1(5x^3+x^5)+a_2(2x^2-6x^3+x^4-x^5)+3x^3+0.5x^5\right]dx\\
&\quad +\int_0^1\left[a_1(5x^4+x^6)-a_2(2x^3-6x^4+x^5-x^6)-3x^4-0.5x^6\right]dx\\
&\quad -\left(1-(5\pi/6)^2\right)\int_0^1 (x^2-x^3)\sin{(5\pi x/6)}\,dx\\
&= -a_1\left(\frac{5}{4}+\frac{1}{6}\right)+a_2\left(\frac{2}{3}-\frac{6}{4}+\frac{1}{5}-\frac{1}{6}\right)+\frac{3}{4}+\frac{1}{12}\\
&\quad +a_1\left(\frac{5}{5}+\frac{1}{7}\right)-a_2\left(\frac{2}{4}-\frac{6}{5}+\frac{1}{6}-\frac{1}{7}\right)-\frac{3}{5}-\frac{1}{14}\\
&\quad -\left(1-(5\pi/6)^2\right)\int_0^1 (x^2-x^3)\sin{(5\pi x/6)}\,dx
\end{aligned}
$$

The integrals over the sin in the residual yield

$$
\begin{aligned}
R_{2\sin} &= -(1-\alpha^2)\int_0^1 (x^2-x^3)\sin{(\alpha x)}\,dx\\
&= -(1-\alpha^2)\left(-\frac{1}{\alpha}\cos{(\alpha)}+\frac{2}{\alpha^2}\sin{(\alpha)}+\frac{2}{\alpha^3}(\cos{(\alpha x)}-1)\right.
\end{aligned}
$$

$$+\frac{1}{\alpha}\cos\left(\alpha\right)-\frac{3}{\alpha^2}\sin\left(\alpha\right)-\frac{6}{\alpha^3}\cos\left(\alpha\right)+\frac{6}{\alpha^4}\sin\left(\alpha\right)\Bigg)$$

$$=-\left(1-\alpha^2\right)\left(-\frac{1}{\alpha^2}\sin\left(\alpha\right)-\frac{4}{\alpha^3}\cos\left(\alpha\right)-\frac{2}{\alpha^3}+\frac{6}{\alpha^4}\sin\left(\alpha\right)\right)$$

and the residual R_2 is

$$
\begin{aligned}
R_2 \;=\; & -a_1\left(\frac{5}{4}+\frac{1}{6}\right)+a_2\left(\frac{2}{3}-\frac{6}{4}+\frac{1}{5}-\frac{1}{6}\right)+\frac{3}{4}+\frac{1}{12}\\
& +a_1\left(\frac{5}{5}+\frac{1}{7}\right)-a_2\left(\frac{2}{4}-\frac{6}{5}+\frac{1}{6}-\frac{1}{7}\right)-\frac{3}{5}-\frac{1}{14}\\
& -\left(1-\alpha^2\right)\left(-\frac{1}{\alpha^2}\sin\left(\alpha\right)-\frac{4}{\alpha^3}\cos\left(\alpha\right)-\frac{2}{\alpha^3}+\frac{6}{\alpha^4}\sin\left(\alpha\right)\right)\\
=\;& -a_1\left(\frac{1}{4}+\frac{1}{42}\right)+a_2\left(\frac{1}{3}-1+\frac{2}{5}+\frac{1}{7}\right)+\frac{3}{20}+\frac{1}{84}\\
& +\frac{1-\alpha^2}{\alpha^2}\left(\sin\left(\alpha\right)+\frac{4}{\alpha}\cos\left(\alpha\right)+\frac{2}{\alpha}-\frac{6}{\alpha^2}\sin\left(\alpha\right)\right)
\end{aligned}
$$

This yields a system of equations of the form

$$
\begin{aligned}
c_{11}a_1+c_{12}a_2 &= d_1\\
c_{21}a_1+c_{22}a_2 &= d_2
\end{aligned}
$$

Evaluating the coefficients yields

$$
\begin{pmatrix} -0.724 & -0.274 \\ -0.274 & -0.124 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} -1.644 \\ -0.586 \end{pmatrix}
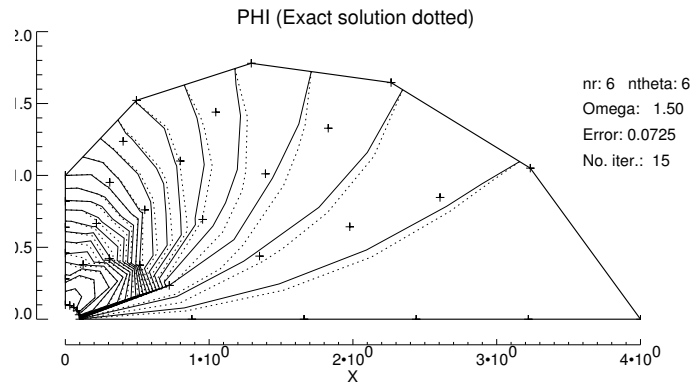$$

with the solution $a_1=2.9450$, $a_2=-1.7818$.

The plot shows the results of the different WR methods where C1 represents the collocation method for $x = [0.5, 1]$, C2 is collocation for $x = [1/3, 2/3]$, S and LS represent the subdomain and the least squares (not computed in this homework) methods, and G indicates the Galerkin method. The solid line represents the exact solution. Clearly the Galerkin method appears superior in this comparison and the collocation method with well chosen collocation point isn't too bad either.
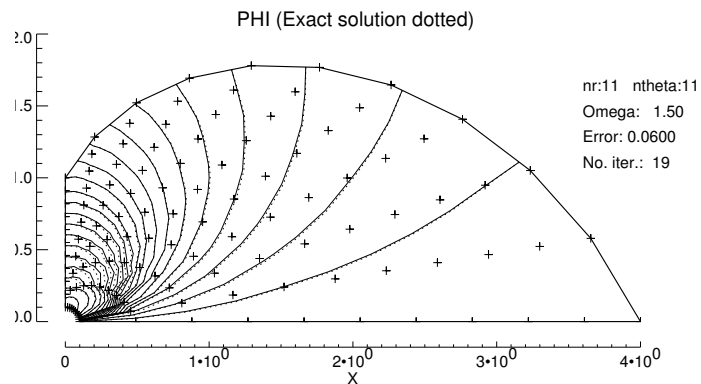
Weighted Residual Approximations
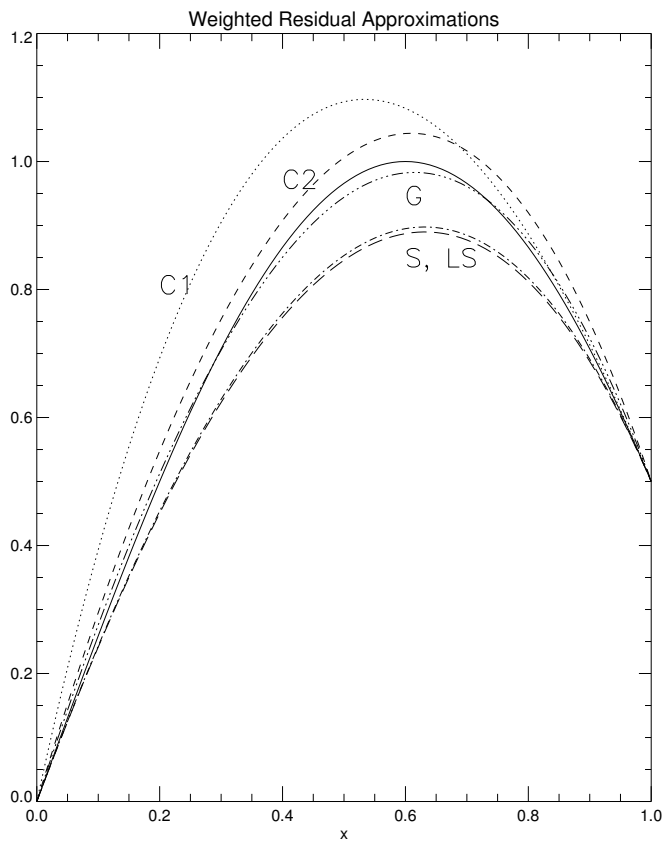
21. Fivol and Laplaces equation:

Obtain solutions to Laplace's equation in the region introduced with the program fivol.f with this program for the following parameters:
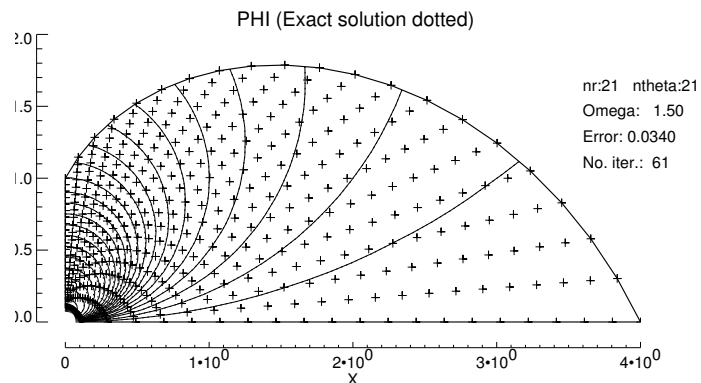
(a) $r_W = 0.1$, $r_X = 4.00$, $r_Y = 1.00$, $r_Z = 0.10$, $\theta_{WX=0}$, and $\theta_{ZY} = 90$, $\lambda = 1.5$.: $J_{max} = 6, K_{max} = 6$:
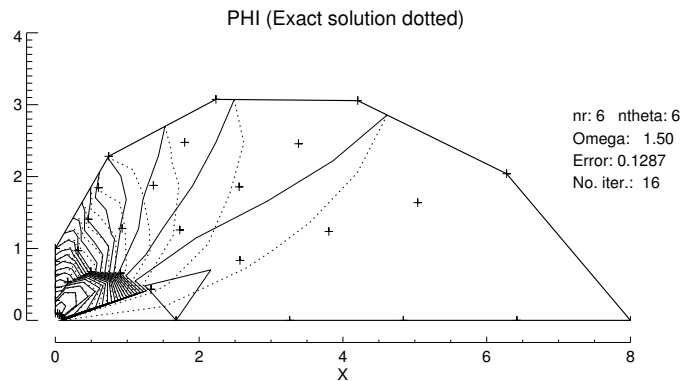

PHI (Exact solution dotted)

nr: 6   ntheta: 6
Omega: 1.50
Error: 0.0725
No. iter.: 15

$J_{max} = 11, K_{max} = 11$:

PHI (Exact solution dotted)

nr:11   ntheta:11
Omega: 1.50
Error: 0.0600
No. iter.: 19

$J_{max} = 21, K_{max} = 21$:

PHI (Exact solution dotted)

nr:21   ntheta:21
Omega: 1.50
Error: 0.0340
No. iter.: 61
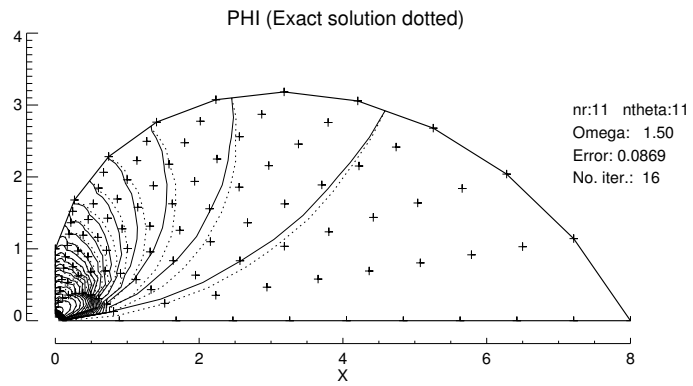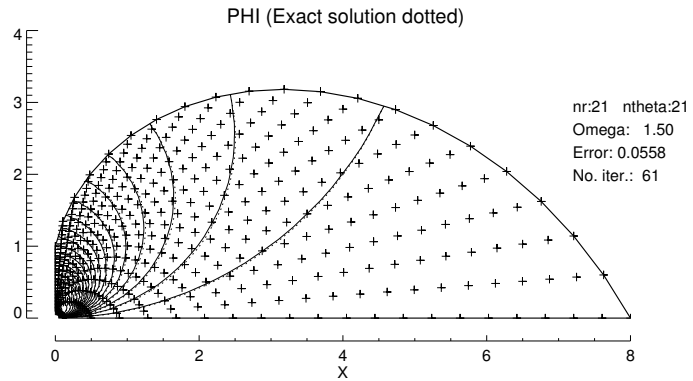
(b) $r_W = 0.1$, $r_X = 8.00$, $r_Y = 1.00$, $r_Z = 0.10$, $\theta_{WX=0}$, and $\theta_{ZY} = 90$. $J_{max} = 6, K_{max} = 6$:

PHI (Exact solution dotted)

nr: 6   ntheta: 6
Omega: 1.50
Error: 0.1287
No. iter.: 16

PHI (Exact solution dotted)

$J_{max} = 11, K_{max} = 11$:

nr:11   ntheta:11
Omega: 1.50
Error: 0.0869
No. iter.: 16

PHI (Exact solution dotted)

$J_{max} = 21, K_{max} = 21$:

nr:21   ntheta:21
Omega: 1.50
Error: 0.0558
No. iter.: 61

Results for the accuracy and number of iterations for the reference cases ($r_X = 1$):

Grid	$||\phi - \overline{\phi}||_{rms}$	No of iterations for convergence
6×6	0.1326	15
11×11	0.0471	19
21×21	0.0138	51

Results for the accuracy and number of iterations for this homework:

Grid	$||\phi_{(a)} - \overline{\phi}||_{rms}$	No of iter. in (a)	$||\phi_{(b)} - \overline{\phi}||_{rms}$	No of iter. in (b)
6×6	0.0725	15	0.1287	16
11×11	0.0600	19	0.0869	16
21×21	0.0340	61	0.0558	61

The distorted domains show in general a slightly error and for the higher resolution also a larger no of iterations for convergence.