26. Stability of the linear FEM for the diffusion equation

Using linear finite elements on a uniform grid result in the following expression for the one-dimensional diffusion equation

$$M_x \Delta T_j^{n+1} - \alpha \Delta t L_{xx} \left[(1-\beta)\, T_j^n + \beta T_j^{n+1} \right] = 0$$

with the mass operator $M_x = (1/6, 2/3, 1/6)$, the second derivative operator $L_{xx} = (1, -2, 1)/\Delta_x^2$, and $\Delta T_j^{n+1} = T_j^{n+1} - T_j^n$. Consider $0 \le \beta \le 1$.

(a) Derive the discretized equation and show that the amplification factor for the von Neumann stability analysis is

$$g = \frac{\left(\frac{2}{3} - 2s\,(1-\beta)\right) + 2\left(\frac{1}{6} + s\,(1-\beta)\right)\cos(k\Delta x)}{\left(\frac{2}{3} + 2s\beta\right) + 2\left(\frac{1}{6} - s\beta\right)\cos(k\Delta x)}$$

(b) Determine the stability properties for the parameter s.

(Hint: It can be helpful to distinguish the cases $\beta > 1/2$ and $\beta < 1/2$)

27. Obtain solutions using the program duct on an 11×11 grid for decreasing values of b/a until the centerline solution across the smaller dimension is within 1% rms of the one-dimensional parabolic profile $u = u_0(1-y^2)$ with $u_0 = 0.5$.

28. Viscous flow in a rectangular duct is governed by $(b/a)^2\, \partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 + 1 = 0$ subject to the boundary conditions $w = 0$ at $x = \pm 1$, $y = \pm 1$. The exact solution for this problem is given by

$$w = \left(\frac{8}{\pi^2}\right)^2 \sum_{i=1,3,5..}^{L} \sum_{j=1,3,5..}^{L} \left[\frac{(-1)^{(i+j)/2-1}}{ij\,((ib/a)^2 + j^2)} \cos(0.5 i\pi x) \cos(0.5 j\pi y) \right]$$

with sufficiently large L. As an approximate solution, choose

$$w = \sum_{j=1}^{N} a_j (1-x^2)^j (1-y^2)^j.$$

Obtain approximate solutions using the subdomain method with $N = 1$ and 2 (use the domains $|x|$, $|y| \le 1$ and $|x|$, $|y| \le a$ with $a = \sqrt{1/2}$). Compare these with the exact solution $L = 21$. Comment your results.

29. Report on the progress of your work on the project.

Please turn in the solutions to the homework on Monday, 4/1/2013