26. Stability of the linear FEM for the diffusion equation

Using linear finite elements on a uniform grid result in the following expression for the one-dimensional diffusion equation

$$M_x \Delta T_j^{n+1} - \alpha \Delta t L_{xx} \left[(1-\beta) T_j^n + \beta T_j^{n+1} \right] = 0$$

with the mass operator $M_x = (1/6, 2/3, 1/6)$, the second derivative operator $L_{xx} = (1, -2, 1)/\Delta_x^2$, and $\Delta T_j^{n+1} = T_j^{n+1} - T_j^n$. Consider $0 \leq \beta \leq 1$.

(a) Derive the discretized equation and show that the amplification factor for the von Neumann stability analysis is

$$g = \frac{\left(\frac{2}{3} - 2s(1-\beta)\right) + 2\left(\frac{1}{6} + s(1-\beta)\right)\cos(k\Delta x)}{\left(\frac{2}{3} + 2s\beta\right) + 2\left(\frac{1}{6} - s\beta\right)\cos(k\Delta x)}$$

(b) Determine the stability properties for the parameter s.

(Hint: It can be helpful to distinguish the cases $\beta > 1/2$ and $\beta < 1/2$)

Solution:

(a) Discretized equation:

$$\begin{aligned}
M_x \Delta T_j^{n+1} &= \alpha \Delta t L_{xx} \left[(1-\beta) T_j^n + \beta T_j^{n+1} \right] \\
\frac{1}{6}\Delta T_{j-1}^{n+1} + \frac{2}{3}\Delta T_j^{n+1} + \frac{1}{6}\Delta T_{j+1}^{n+1} &= s(1-\beta)\left(T_{j-1}^n - 2T_j^n + T_{j+1}^n\right) \\
&\quad + s\beta\left(T_{j-1}^{n+1} - 2T_j^{n+1} + T_{j+1}^{n+1}\right) \\
\left(\frac{2}{3} + 2s\beta\right) T_j^{n+1} + \left(\frac{1}{6} - s\beta\right)\left(T_{j-1}^{n+1} + T_{j+1}^{n+1}\right) &= \left(\frac{2}{3} - 2s(1-\beta)\right) T_j^n \\
&\quad + \left(\frac{1}{6} + s(1-\beta)\right)\left(T_{j-1}^n + T_{j+1}^n\right)
\end{aligned}$$

Substituting $T_j^n = \widetilde{T} \exp\left[i(\omega t_n + kx_j)\right]$ with the relations $g = T_j^{n+1}/T_j^n$ for the amplification factor, $\exp(ik\Delta x) = T_{j+1}^n / T_j^n$ and $\left(T_{j-1}^n + T_{j+1}^n\right)/T_j^n = (\exp(-ik\Delta x) + \exp(ik\Delta x)) = 2\cos(k\Delta x)$ we divide the equation by T_j^n:

$$g\left[\left(\frac{2}{3} + 2s\beta\right) + 2\left(\frac{1}{6} - s\beta\right)\cos(k\Delta x)\right] = \left(\frac{2}{3} - 2s(1-\beta)\right) + 2\left(\frac{1}{6} + s(1-\beta)\right)\cos(k\Delta x)$$

or

$$g = \frac{\left(\frac{2}{3} - 2s(1-\beta)\right) + 2\left(\frac{1}{6} + s(1-\beta)\right)\cos(k\Delta x)}{\left(\frac{2}{3} + 2s\beta\right) + 2\left(\frac{1}{6} - s\beta\right)\cos(k\Delta x)}$$

(b) Stability $-1 \leq g \leq 1$:

(1) Case $-1 \leq g$:

$$
\begin{aligned}
-1 &\leq \frac{\left(\frac{2}{3}-2s\left(1-\beta\right)\right)+2\left(\frac{1}{6}+s\left(1-\beta\right)\right)\cos\left(k\Delta x\right)}{\left(\frac{2}{3}+2s\beta\right)-+2\left(\frac{1}{6}-s\beta\right)\cos\left(k\Delta x\right)} \\
-\left(\frac{1}{3}+s\beta\right)-\left(\frac{1}{6}-s\beta\right)\cos\left(k\Delta x\right) &\leq \left(\frac{1}{3}-s+s\beta\right)+\left(\frac{1}{6}+s-s\beta\right)\cos\left(k\Delta x\right) \\
-\left(\frac{2}{3}+2s\beta-s\right) &\leq \left(\frac{1}{3}+s-2s\beta\right)\cos\left(k\Delta x\right) \\
-\frac{2}{3}+s\left(1-2\beta\right) &\leq \left(\frac{1}{3}+s\left(1-2\beta\right)\right)\cos\left(k\Delta x\right)
\end{aligned}
$$

i) Consider first $1/3+s\left(1-2\beta\right) \geq 0$ or $s\left(1-2\beta\right) \geq -1/3$ which is always satisfied for $\beta \leq 1/2$ (For $\beta > 1/2$ this implies $s\left(2\beta-1\right) \leq 1/3$).

$$
\begin{aligned}
-\frac{2}{3}+s\left(1-2\beta\right) &\leq \left(\frac{1}{3}+s\left(1-2\beta\right)\right)\cos\left(k\Delta x\right) \leq -\frac{1}{3}-s\left(1-2\beta\right) \\
s\left(1-2\beta\right) &\leq 1/6 \\
with \qquad s &\leq \frac{1}{6\left(1-2\beta\right)} \qquad for \qquad \beta < 1/2
\end{aligned}
$$

and no stability limit for $\beta \geq 1/2$ (but the condition to evaluate the inequality requires $s\left(2\beta-1\right) \leq 1/3!$).

ii) Now consider $1/3+s\left(1-2\beta\right) < 0$ or $s\left(1-2\beta\right) < -1/3$ which requires $\beta \geq 1/2$ and implies $s\left(2\beta-1\right) > 1/3$

$$
-\frac{2}{3}+s\left(1-2\beta\right) \leq \left(\frac{1}{3}+s\left(1-2\beta\right)\right)\cos\left(k\Delta x\right) \leq \frac{1}{3}+s\left(1-2\beta\right)
$$

This is always satisfied and required the range $s\left(2\beta-1\right) > 1/3$.

Combining the results from (i) and (ii) the inequality $-1 \leq g$ is satisfied for all s if $\beta \geq 1/2$ and requires $s \leq 1/6\left(1-2\beta\right)$ for $\beta < 1/2$.

(2) Case $g \leq 1$

$$
\begin{aligned}
\left(\frac{1}{3}-s\left(1-\beta\right)\right)+\left(\frac{1}{6}+s\left(1-\beta\right)\right)\cos\left(k\Delta x\right) &\leq \left(\frac{1}{3}+s\beta\right)+\left(\frac{1}{6}-s\beta\right)\cos\left(k\Delta x\right) \\
-s+s\cos\left(k\Delta x\right) &\leq 0 \\
-s\left(1-\cos\left(k\Delta x\right)\right) &\leq 0
\end{aligned}
$$

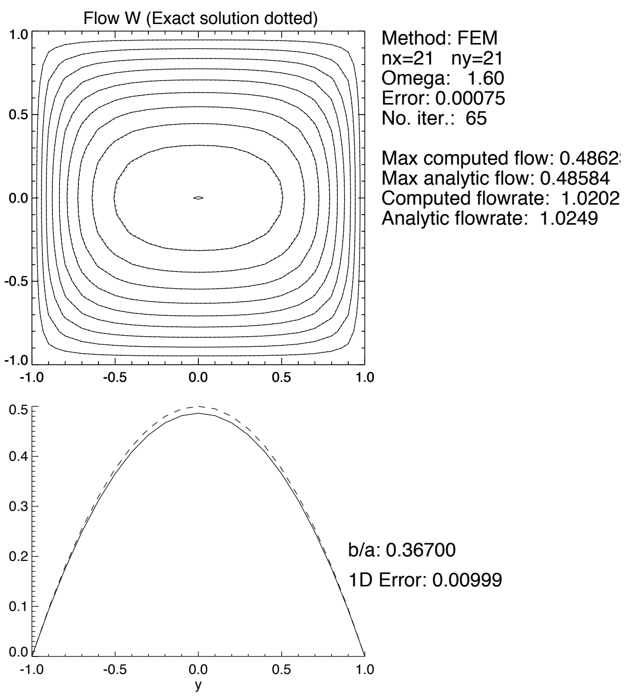
This is satisfied for all s (note s must be positive) such that no additional stability limits result.

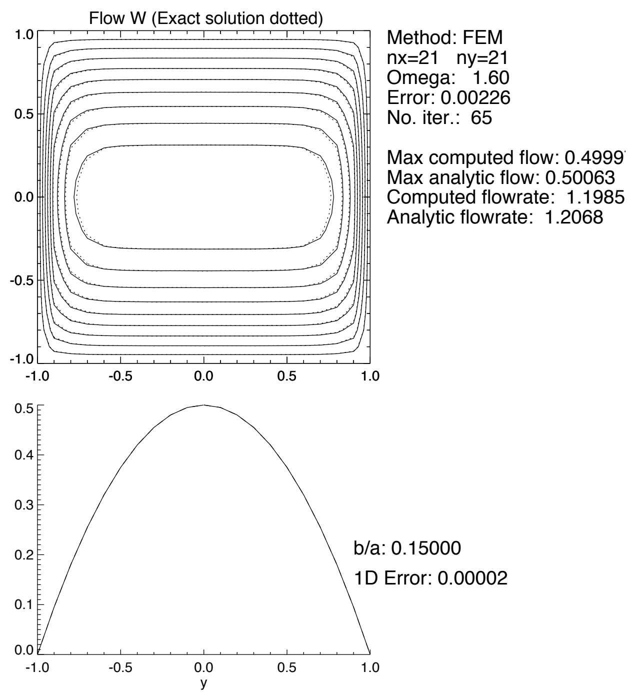
In summary the scheme is unconditionally stable for $\beta \geq 1/2$ and requires $s \leq 1/6\left(1-2\beta\right)$ for $\beta < 1/2$.

27. Obtain solutions using the program duct on an 11×11 grid for decreasing values of b/a until the centerline solution across the smaller dimension is within 1% rms of the one-dimensional parabolic profile $u = u_0(1 - y^2)$ with $u_0 = 0.5$.

Solution:

Problem 24 demonstrates that the centerline flow for $b/a = 1$ is about 0.29, i.e., different from 0.5 as expected for the 1D solution. Following are the results for $b/a = 0.374$ and 0.15. The dashed lines give the parabolic profile.





The 1% rms error of the one-dimensional parabolic is reached for the left plot. It demonstrates a considerable distortion from the solution for $b/a = 1$. The right plots show a solution for even an smaller value of $b/a = 0.15$. In this solution it is rather obvious that the solution for the most part appears one-dimensional except close to the boundaries in x. (Note: the plots are actually for a 21^2 grid in which case the 1% error is obtained for $b/a = 0.367$)

Background: Smaller values of b/a imply that the x derivative term becomes less important, i.e., the solution should become more one-dimensional. The original differential equation was the same in x and y only the aspect ratio of the computational domain (width in x and width y) is changed. The smaller dimension (y) starts to dominate the solution because the boundaries in y are much closed to the interior domain than the boundaries in x.

28. Viscous flow in a rectangular duct is governed by $(b/a)^2\,\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 + 1 = 0$ subject to the boundary conditions $w = 0$ at $x = \pm 1$, $y = \pm 1$. The exact solution for this problem is given by

$$w = \left(\frac{8}{\pi^2}\right)^2 \sum_{i=1,3,5..}^{L} \sum_{j=1,3,5..}^{L} \left[\frac{(-1)^{(i+j)/2-1}}{ij\left((ib/a)^2 + j^2\right)} \cos(0.5i\pi x)\cos(0.5j\pi y)\right]$$

with sufficiently large L. As an approximate solution, choose

$$w = \sum_{j=1}^{N} a_j (1-x^2)^j (1-y^2)^j.$$

Obtain approximate solutions using the subdomain method with $N = 1$ and 2 (use the domains $|x|$, $|y| \le 1$ and $|x|$, $|y| \le a$ with $a = \sqrt{1/2}$). Compare these with the exact solution $L = 21$. Comment your results.

Solution:

Trial functions:

$$w = \sum_{j=1}^{N} a_j (1-x^2)^j (1-y^2)^j$$

For $N = 1$: $w = a_1(1-x^2)(1-y^2)$

$$\frac{\partial^2 w}{\partial x^2} = -2a_1(1-y^2) \qquad \frac{\partial^2 w}{\partial y^2} = -2a_1(1-x^2)$$

Residual: $R_1 = -2a_1\left[r_a(1-y^2) + (1-x^2)\right] + 1$

For $N = 2$**:** $w = a_1(1-x^2)(1-y^2) + a_2(1-x^2)^2(1-y^2)^2$

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= -2a_1(1-y^2) - 4a_2(1-3x^2)(1-y^2)^2 \\ \frac{\partial^2 w}{\partial y^2} &= -2a_1(1-x^2) - 4a_2(1-x^2)^2(1-3y^2)\end{aligned}$$

Residual:

$$R_2 = -2a_1\left[r_a(1-y^2) + (1-x^2)\right] - 4a_2\left[r_a(1-3x^2)(1-y^2)^2 + (1-x^2)^2(1-3y^2)\right] + 1$$

Subdomain Method Method:

Weight functions for the residual integral are step functions which are equal to 1 in the chosen subdomain and 0 otherwise. For N=1we choose the entire domain:

i) Integral for $N = 1$**:**

$$\int_{-a}^{a}\int_{-a}^{a} \left\{-2a_1\left[r_a(1-y^2) + (1-x^2)\right] + 1\right\} dx\,dy = 0$$

or

$$
\begin{aligned}
2a_1 \int_{-a}^{a}\int_{-a}^{a} \left[r_a(1-y^2)+(1-x^2)\right] dxdy &= \int_{-a}^{a}\int_{-a}^{a} dxdy \\
2a_1 2a2a\left[1-\frac{a^2}{3}\right](r_a+1) &= 4a^2
\end{aligned}
$$

With the integrals below this yields: $2a_1\,(r_a I_1 + I_1) = 4$

$$
= > \qquad a_1 = \frac{1}{2\left(1-\frac{a^2}{3}\right)}(r_a+1)^{-1} \tag{1}
$$

$$
or \qquad a_1 = \frac{3}{4}(r_a+1)^{-1} \qquad \text{for} \qquad a=1 \tag{2}
$$

$$
or \qquad a_1 = \frac{3}{5}(r_a+1)^{-1} \qquad \text{for} \qquad a=\sqrt{\frac{1}{2}} \tag{3}
$$

ii) Integrals for $N=2$: The choice of the subdomains is not fixed such that the result is not unique. Here we choose a square with $|x|$, $|y| \le a$ keeping a variable for the moment (later choosing $a=1$ and $1/\sqrt{2}$) :

Domain a:

$$
\begin{aligned}
&2a_1 \int_{-a}^{a}\int_{-a}^{a} \left[r_a(1-y^2)+(1-x^2)\right] dxdy \\
&+4a_2 \int_{-a}^{a}\int_{-a}^{a} \left[r_a(1-3x^2)(1-y^2)^2+(1-x^2)^2(1-3y^2)\right] dxdy = \int_{-a}^{a}\int_{-a}^{a} dxdy
\end{aligned}
$$

or

$$
\begin{aligned}
&2a_1\left[r_a 2a2a\left(1-\frac{a^2}{3}\right)+2a2a\left(1-\frac{a^2}{3}\right)\right]+ \\
&+4a_2\left[r_a 2a(1-a^2)2a\left(1-\frac{2}{3}a^2+\frac{1}{5}a^4\right)+2a(1-a^2)2a\left(1-\frac{2}{3}a^2+\frac{1}{5}a^4\right)\right] = 4a^2 \\
&=> \qquad \left(1-\frac{a^2}{3}\right)a_1+2\left(1-a^2\right)\left(1-\frac{2}{3}a^2+\frac{1}{5}a^4\right)a_2 = \frac{1}{2}(r_a+1)^{-1}
\end{aligned} \tag{4}
$$

We obtain two equations to determine a_1 and a_2 from (4) for two different choices of a. Using $a=1$ yields again

$$
a_1 = \frac{3}{4}(r_a+1)^{-1}
$$

and substitution in a_2:

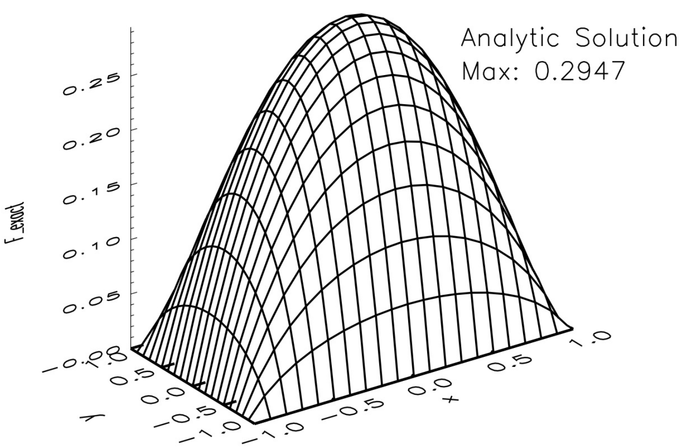
$$
\begin{aligned}
a_2 &= \frac{1}{2(1-a^2)}\left(1-\frac{2}{3}a^2+\frac{1}{5}a^4\right)^{-1}\left[\frac{1}{2}(r_a+1)^{-1}-\left(1-\frac{a^2}{3}\right)a_1\right] \\
&= \frac{1}{2(1-a^2)}\left(1-\frac{2}{3}a^2+\frac{1}{5}a^4\right)^{-1}\frac{1}{4}\left(a^2-1\right)(r_a+1)^{-1} \\
&= -\frac{1}{8}\left(1-\frac{2}{3}a^2+\frac{1}{5}a^4\right)^{-1}(r_a+1)^{-1}
\end{aligned}
$$

With $a = 1/\sqrt{2}$ the equations for a_1 and a_2 are

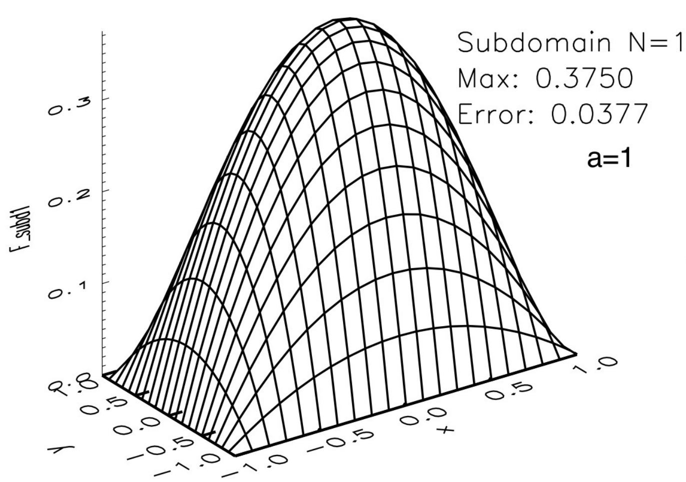
$$a_1 = \frac{3}{4}(r_a+1)^{-1} \quad (5)$$
$$a_2 = -\frac{15}{86}(r_a+1)^{-1} \quad (6)$$

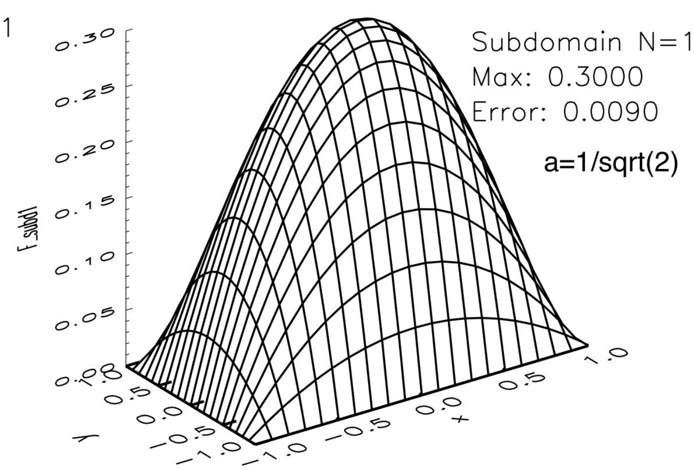
Graphics for the various methods:

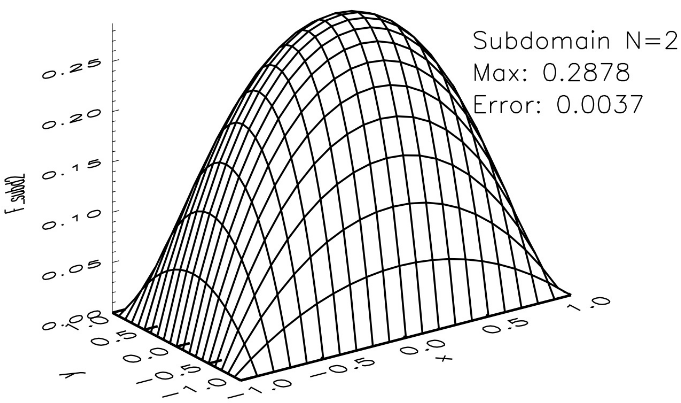
Analytic solution:



Subdomain method:







Integrals used in the computation of the residual integrals:

$$I_1 = \int_{-a}^{a} (1-x^2)dx = 2a\left[1-\frac{a^2}{3}\right]$$

$$I_2 = \int_{-a}^{a} (1-x^2)^2 dx = \int_{-a}^{a} (1-2x^2+x^4)dx = 2a\left[1-\frac{2}{3}a^2+\frac{a^4}{5}\right]$$

$$I_3 = \int_{-a}^{a} (1-3x^2)dx = = 2a\left[1-a^2\right]$$