30. Obtain solutions using the program duct ($b/a = 1$) for a 21^2, 41^2, 81^2, 161^2, and 321^2 grid both for the finite element and the finite difference method.

a) For each grid determine through iteration the optimum value of the iteration (SOR) parameter λ_{opt}, and the corresponding number of iterations N_{iter}. Help: You need to determine to evaluate λ_{opt} to an accuracy of $\sim 10^{-3}$ (or better for higher grid numbers) and examine if you can use the value of $2-\lambda_{opt}$ as a predictor for the next higher grid number.

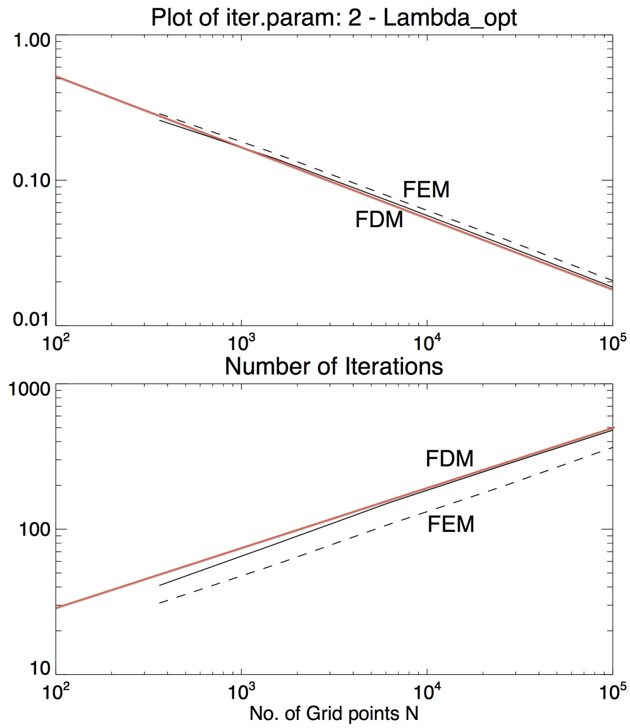
b) Plot $\ln N_{iter}$ versus $\ln N$ (with $N = N_x \cdot N_y$) for the fde and fem methods. What asymptotic ($N \to \infty$) scaling do you obtain for N_{iter} for the respective discretization?

c) Plot $\ln(2-\lambda_{opt})$ versus $\ln N$for the fde and fem methods. What asymptotic ($N \to \infty$) scaling do you obtain for the value of $2-\lambda_{opt}$.

a) The following table shows total number of grid points, best iteration paramater λ_{opt}, the value of $2-\lambda_{opt}$, the number of iterations, and the predicitor for the best valueλ for 4 times the number of grid points based on the current value of $2-\lambda_{opt}$. The actual value of λ_{opt} is found by iteration of λ with an accuracy of 10^{-3} (or some times better). The table illustrates that the value of $2-\lambda_{opt}$ is approximately cut in half by quadrupling the number of grid points, i.e., $2-\lambda_{opt,N} \sim 1/\sqrt{N}$. Therefore a value of $\lambda_{opt,N} \approx 2-0.5\cdot\left(2-\lambda_{opt,N-1}\right)$ can be used to narrow the range of iterations to find $\lambda_{opt,N}$. The table further illustrates that the number of iteration increase by somewhat less than a factor of 2 for about 4 times the number of grid points. This behavior is about the same for the finite element and finite difference methods with the finite difference where the number of iterations for the FDM is about a 3rd larger than for the FEM.

	N	λ_{opt}	$2-\lambda_{\mathrm{opt}}$	N_{iter}	Predictor for λ_{opt}
fdm0021	361	1.74100	0.25900	41	1.87050
fdm0041	1521	1.85900	0.14100	79	1.92950
fdm0081	6241	1.92800	0.07200	152	1.96400
fdm0161	25281	1.96360	0.03640	273	1.98180
fdm0321	101761	1.98175	0.01825	483	1.99088
fem0021	361	1.71300	0.28700	31	1.85650
fem0041	1521	1.84700	0.15300	57	1.92350
fem0081	6241	1.92250	0.07750	108	1.96125
fem0161	25281	1.96000	0.04000	196	1.98000
fem0321	101761	1.97975	0.02025	368	1.98988

b and c) Plots of $\ln N_{iter}$ (b) and $\ln(2-\lambda_{opt})$ (c) versus $\ln N$.



The slope of the double log plot $\ln N_{iter}$ vs $\ln N$ is between 0.41 and 0.43 (the smaller value is obtained for large values of N). Within the accuracy the slope is identical for the finite difference and finite element methods (where the number of iterations is always a bit larger for the finite difference method). This implies that the number of iterations scales as

$$N_{iter} \sim N^{0.41 \pm 0.01}$$

such that the overall computational effort for SOR in the duct problem scales approximately as $N^{1.41 \pm 0.01}$.

The slope for the parameter $2 - \lambda_{opt}$ is -0.49 ± 0.01 or

$$2 - \lambda_{opt} \sim N^{-0.49 \pm 0.01}$$

Here the smaller absolute value of -0.48 corresponds to the overall average slope and while for large values of N the slope approaches -0.5 or $2 - \lambda_{opt} \sim 1/\sqrt{N}$ confirming the expectation from the table.

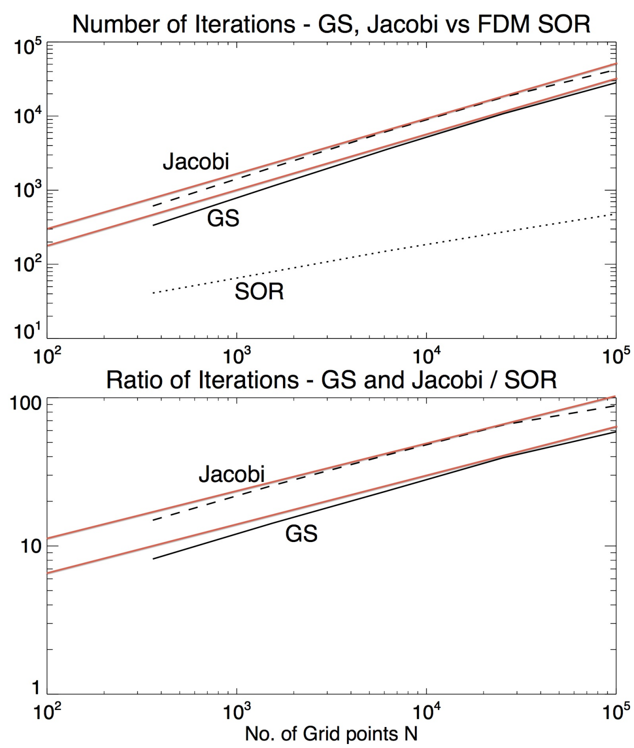
It appears that this allows a rather good prediction or estimate for the optimum value of λ for large values of N.

31. Use the same grid sequence (as in problem 30) for the program duct with finite differences and determine the number of iterations for Jacobi and Gauss-Seidel (GS) iterations. Plot the number of iterations $\ln N_{iter}$ versus the number of grid points ($\ln N$). What scaling do you obtain now for N_{iter} and how does the numerical effort for the two methods compare to the SOR method in problem 30 (How do $N_{iter,GS}/N_{iter,SOR}$ and $N_{iter,J}/N_{iter,SOR}$ scale with the total number of grid points)?

The following table shows the number of iterations for Gauss-Seidel (GS) and Jacobi (FDM) iteration in comparison to the best SOR results for finite differences. The number of iterations for the Jacobi and GS methods is about 1 to 2 orders of magnitude larger than for the best SOR iteration.

	GS FDM N_{iter}	Jacobi FDM N_{iter}	FDM SOR
361	335	612	41
1521	1119	2010	79
6241	3582	6263	152
25281	10745	17894	273
101761	28628	42873	483

Plots of $\ln N_{iter}$ versus $\ln N$ for the Jacobi and Gauss-Seidel method in comparison to the FDM SOR result) and of the ratios of $N_{iter,Jacobi}/N_{iter,SOR}$ and $N_{iter,GS}/N_{iter,SOR}$ vs $\ln N$.



The slope of the Jacobi and Gauss-Seidel iteration (upper plot) is 0.74 ± 0.03 yielding

$$N_{iter,Jacobi}, N_{iter,GS} \sim N^{0.74\pm0.03}$$

where the slopes of the Jacobi and GS are approximately the same with a small offset. The lower plot shows the ratios $N_{iter,Jacobi}/N_{iter,SOR}$ and $N_{iter,GS}/N_{iter,SOR}$ vs number of grid points (not expected in this homework). These plots show a slope of 0.32 ± 0.02 yielding

$$N_{iter,Jacobi}/N_{iter,SOR}, N_{iter,GS}/N_{iter,SOR} \sim N^{0.32\pm0.02}$$